DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CRETE

Knossou Ave., P.C. 714 09, Heraklion, Crete, Greece,
Tel.: +30 2810 - 393800, Fax: +30 2810 – 393804, e-mail: hatzikon@edu.uoc.gr
Web-site: http://www.edu.uoc.gr/~hatzikon

Orthogonal Arrays: Construction Methodology and Application on the Design of Fractional Factorials

Panagiotis E. Chatzikonstantis
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Supervisor

Nikolaos Christakis
Visiting Assistant Professor

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University of Crete
School of Science and Engineering
Department of Mathematics

Abstract

The purpose of this work is to introduce the concept of Orthogonal Arrays and their utilization in the design of Fractional Factorial experiments. A brief introduction on basic elements of the theory of finite and Galois fields is given, along with some key aspects of OA theory. Then, OA construction methodology follows and various examples of how to construct different “statistically equivalent” OAs are presented. The importance of OAs in the design of experiments is discussed, in order to determine the minimum number of experiments out of which meaningful information about the ablution of a system/process may be extracted. Then, a method is described for the construction of OAs over a finite field $\mathbb{F}_2$ (two-level OAs) and conclusions will be drawn on the derivation of a unified methodology for constructing multi-factor, multi-leveled OAs, which is something that is of great importance in engineering, when the impact of various control factors on any given experiment needs to be assessed.

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Panagiotis E. Chatzikonstantis

To the memory of
my Grandfather Panagiotis E. Chatzikonstantis

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Chapter 1 – Introduction

Introduction

Orthogonal Arrays (OAs) are very important mathematical arrangements not only because they combine essential information and knowledge from various fields of mathematics, such as Geometry, Field Theory, Combinatorics, Statistics etc. but also because they can be utilised in a number of fields in engineering for the optimisation of experiment design.

OAs were first introduced by C.R.Rao in a series of papers during the 1940's (K.R.Nair; C.R.Rao (1948), "Confounding in Asymmetrical Factorial Experiment"), where certain combinatorial arrangements were presented. However, the idea of utilising such arrangements for the optimisation of engineering processes belongs to Genichi Taguchi (G.Taguchi, "On Robust Technology Development"). The terminology "Robust Engineering" originates from the work of Taguchi and is used to denote the methodology for the designing and manufacturing stages of a product in order to increase its life and utility. During the initial stages of production there are experiments which need to be performed in order to assess the effect of various factors on the product. The results of these experiments with the variation of a number of factors are stored in Arrays. When only a few factors are tested, the resulting arrays are small. What happens though, when the number of factors to be examined is large, as well as the levels of variation of these factors? Consider for example a K-factor experiment where each factor takes values from an S-value set. Then, in order to account for all possible combinations an array of S^k different trials needs to be set up. Theoretically, an array with S^k rows and K columns may be easily constructed and stored. In practice though, when resources are (i.e. time and money) are limited only a small number of experiments is possible. This means that only a few, carefully selected experiments may be performed in order to assess the impact of the most important factors on the product. The question posed is how do we select these experiments in order to obtain meaningful results? In other words, how do we go from a Complete Factorial experiment to a statistically equivalent Fractional Factorial experiment? The answer to this question is that we do so by using Orthogonal Arrays.

The aim of this work is to introduce Orthogonal Arrays and demonstrate through the use of different construction techniques, how a smaller OA may be generated from a Complete experiment array of S^k rows and K columns. No generalised OA construction methodology exists and conclusions will be drawn on the capabilities of developing such a scheme.

In Chapter 2, an introduction to OAs and some basic theorems regarding their properties will be presented.

In Chapter 3, basic ideas from Finite and Galois Fields theory are given and some important methodologies and new alternate schemes for OA construction based on Field theory principles are presented.

Chapter 4 is dedicated to the utilisation of OAs in the design of experiments and construction strategies based on Linear Code Theory.

Finally, Chapter 5 summarises the main points of this report and suggests directions for further research on construction methodologies for OAs and their utilisation in the design of experiments.

Chapter 2 – An Introduction to Orthogonal Arrays

2.1 Introduction to Arrays

An array A(N, k) of k columns and N rows, is a range of numbers from a set Σ entered in a d-dimensional vector-space in a particular parataxis. The most common type is the 2-level rectangular array, which consists of rows and columns ($N \times k$). An array is needed when an experiment is taking place in order to store the resulting data (which consist of numbers), then to compare the outcome in each trial and choose the set of values which provide the optimum result. It is important to say that an array differs from a mathematical matrix and the properties of matrices cannot be applied on arrays. The differences will be given in a following section.

Assume there is an array $N \times k$ with six levels (0,1,2,3,4,5) taken from an experiment for which results were extracted (Figure 1). In each trial, each factor is assigned to a number which represents the condition of the factor (i.e. on/off if the factor is a light bulb (2-level factor) or good/fair/bad (three level factor) if the factor is the quality of a received signal etc.). In this hypothetical array, the number of factors (i.e. columns) is k, the number of trials (i.e. rows) is N and the levels are six.

	FACTORS					Results
	A	B	C	…	k	
Trial No.						
1	3	2	5	…	2	24
2	1	5	4	…	5	32
3	2	3	2	…	0	34
4	1	1	3	…	0	22
5	5	5	4	…	1	15
6	5	2	4	…	3	11
7	2	0	3	…	3	53
8	0	5	4	…	2	21
.					.	
.					.	
.					.	
N	5	3	0	…	2	13

Figure 1 "A six-level Array of N-trials and k-factors"

There is also a column named *Results* in which the resulting data of each trial is stored and when the array is completed, all trials considered have been the results of the trials are compared with one another. When the comparison finishes, the optimum solution to the problem has been found. An array like the on described is depicted above in Figure 1.

2.1.1 Brief Report in Utilization of Arrays

When scientific research is in progress, experimentation is a very important tool almost in all scientific areas and especially in engineering and statistical research. This is because experimentation is the key to acquire new knowledge. An experiment consists of trying the impact of many factors on a process, and the experimentalists-modellers during their experiments use trials as rows and factors as columns on paper, so that blocks are formed. The data-outcomes of the experiments are stored in these blocks, as well as the results from each trial. Arrays are used in Complete Factorial (CF) experiments.

2.2 Differences between Arrays and Matrices

2.2.1 Definition of a Matrix

A *matrix* is the essential tool to work and represent linear transformations in a concise and useful way. For every linear transformation there exists and corresponds only one matrix and every matrix corresponds to a unique linear transformation.

Assume there is a system of equations which represent a linear transformation:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ x_m' &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned}$$

This can be depicted by a matrix equation:

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_m' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where a_{ij} are called *matrix elements.*

Example No1

A 4×4 matrix with elements a_{ij}, where $i = 1,\cdots,4$ and $j = 1,\cdots,4$ is denoted by $A^{4\times4}$. Such a matrix is the following:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

2.2.2 Comparison of Arrays and Matrices

It is now clear that arrays differ from matrices for a main reason, which is the way they are used. Matrices are used by mathematicians in order to depict linear transformations whereas arrays are used in order to store data, compare the outcomes and find the optimal solution.

2.3 Orthogonal Arrays

An Orthogonal Array OA(N,k,s,2) of strength two is an $N \times k$ array with numbers from a Σ set of $s \geq 2$ elements , with the property that in any two rows each ordered pair of numbers from Σ occurs exactly λ times. The set Σ is assumed to be the set of integers $0,1,2,\cdots,s-1$.

For convenience an orthogonal array is symbolized as OA(N,k,s,t) where:

N = Number of the trials of an experiment,

k = Number of constraints (factors),

s = levels of the array (number of the numbers that can be used in an array),

t = strength of the array (number of rows that is needed to compare to prove the orthogonality),

λ = index of OA (the number of appearances of the combinations).

Example No2

An orthogonal array of N = 4, k = 2, s = 2, t = 2 which is symbolized OA(4,2,2,2) is depicted below:

		FACTORS	
		A	B
Trial No.			
1		0	0
2		0	1
3		1	0
4		1	1

Table 1 "OA(4,2,2,2)"

In the table above it can be seen that trial No1 corresponds to the (0,0) pair, trial No2 corresponds to the (0,1) pair and following this pattern trial No3 to (1,0) and trial No4 to (1,1). So, all different combinations of 2^2 have been taken exactly once. This is why this array is orthogonal of strength 2 (two columns compared each time), with 2 levels (the data equal to 0 or 1), 2 factors (A and B), and 4 Number of Trials. The index of this OA is 1, because each possible combination occurs once.

Below the OA(4,2,2,2) is shown reversed.

		Trial Number			
		1	2	3	4
Factors					
A		0	0	1	1
B		0	1	0	1

Table 2 "Reversed OA(4,2,2,2)"

Reversed OAs are used in order to reduce on paper the space they consume, especially when the number of trials is large. The comparison of "Table 1" and "Table 2" shows that the *OA(4,2,2,2) reversed* consumes less space than the *OA(4,2,2,2).* It can be easily assumed that an OA of many trials is going to be written on paper in its reversed form for convenience and less space consumption. In this dissertation both forms are will be used, depending on the size of each orthogonal array.

Example No3

Another OA is the OA(8,4,2,3) which is depicted below:

	FACTORS			
	A	B	C	D
Trial No.				
1	0	0	0	0
2	0	0	1	1
3	0	1	0	1
4	0	1	1	0
5	1	0	0	1
6	1	0	1	0
7	1	1	0	0
8	1	1	1	1

Table 3 "OA(8,4,2,3) N = 8, k = 4, s = 2, t = 3, index unity"

Example No4

A *mixed* or *asymmetrical* orthogonal array is an array like the one below:

			FACTORS			
		A	B	C	D	E
Trial No.						
1		0	0	0	0	0
2		0	1	1	1	1
3		1	0	1	0	1
4		1	1	0	1	0
5		2	0	0	1	1
6		2	1	1	0	0
7		3	0	1	1	0
8		3	1	0	0	1

Table 4 "Mixed Orthogonal Array"

It can be observed that there are factors which do not have the same number of levels as the others. In the above example factor A has 4-levels instead of 2 that every other factor has. This is why this orthogonal array is termed mixed or asymmetrical. However, the property of orthogonality is conserved because in any two columns possible pairs occur with the same frequency. Moreover, consider taking column A (factor A) with any other column; then every pair that comes up occurs once, which means that the index is unity. On the other hand, if every other column is compared with a random column (except the first) the resulting pairs occur twice and as a result it can be deduced that a mixed orthogonal array does not have fixed index.

2.4 Basic Properties of OAs

1. If A_i with i = 1,…, r is an OA(N_i, k, s, t_i) then the array A which can be obtained from the juxtaposition of the r arrays $A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_r \end{bmatrix}$ is also an OA, where $N = N_1 + N_2 + ... + N_r$ and $t \geq \min\{t_1, t_2, ..., t_r\}$.
2. $N = \lambda s^t \Leftrightarrow \lambda = \frac{N}{s^t}$.
3. Any orthogonal array of strength t is also an orthogonal array of strength t' where $0 \leq t' \leq t$.
4. A permutation of the trials or factors of an orthogonal array results in an orthogonal array.
5. A permutation of the levels of any factor in an orthogonal array results in an orthogonal array.
6. Any $N \times k'$ with $k' < k$ sub-array of an OA(N,k,s,t) is an OA(N, k',s, t'), where $t' = \min\{k', t\}$.
7. If there are trials of an OA(N,k,s,t) that begin with 0 (or any other number) and the corresponding column is omitted then the resulting array is OA($\frac{N}{s}$, $k-1$, s, $t-1$).

The OA(N,k,s,t) is depicted below:

0 $\vdots$ 0	OA($\frac{N}{s}$, $k-1$,s, $t-1$)
1 $\vdots$ 1	
$\vdots$ $\vdots$ $\vdots$	

8. Assume that C is the set of all possible trials that could have occurred in a particular orthogonal array A. For $c \in C$ let f_c be the frequency of c in A. Then, the array which contains trial c with frequency $f - f_c$ for every $c \in C$ is the set-theoretic complement or the complement of A. The complement of OA(N,k,s,t) is the OA($fs^k - N$, k, s, t).

9. Suppose $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ is an OA(N,k,s,t) when A_1 is an OA(N_1 , k, s, t_1) and A_2 is OA(N_2 , k, s, t_2). Then A_2 is an OA($N - N_2$, k, s, t_2) with $t_2 \geq \min\{t, t_1\}$.

2.5 Important Definitions

Definition 1 (D1)

Two orthogonal arrays are said to be *isomorphic* if one can be obtained from the other by a sequence of permutation of the columns, the rows and the levels of each factor.

Definition 2 (D2)

Two orthogonal arrays are said to be *statistically equivalent* if one can be obtained from the other by a sequence of permutations of the trials (rows).

Example No5

Isomorphic and Statistically Equivalent arrays

Assume that the orthogonal array OA(8, 4, 2, 3) is taken and name it *original OA(8,4,2,3)* :

	FACTORS			
	A	B	C	D
Trial No.				
1	0	0	0	0
2	0	0	1	1
3	0	1	0	1
4	0	1	1	0
5	1	0	0	1
6	1	0	1	0
7	1	1	0	0
8	1	1	1	1

Then the:

	FACTORS			
	C	A	B	D
Trial No.				
3	0	0	1	1
1	0	0	0	0
8	1	1	1	1
5	0	1	0	1
4	1	0	1	0
6	1	1	0	0
2	1	0	0	1
7	0	1	1	0

is the isomorphic orthogonal array of the original one but because of the column permutations it is not statistically equivalent to it.

However, the:

	FACTORS			
	A	B	C	D
Trial No.				
3	0	1	0	1
1	0	0	0	0
8	1	1	1	1
5	1	0	0	1
4	0	1	1	0
6	1	0	1	0
2	0	0	1	1
7	1	1	0	0

Is statistically equivalent to the original orthogonal array and is also isomorphic with it.

2.6 Important Theorems

Theorem 1 (Plackett and Burman's inequality)

For any orthogonal array ($N = \lambda s^2$, k, s, 2) of strength 2 the number of constraints (factors) k satisfies the inequality:

$$k \leq \left[\frac{\lambda s^2 - 1}{s - 1} \right] \qquad \text{(i)}$$

(The proof of this theorem is given in Appendix A)

Theorem 2 (Rao's inequality)

For any orthogonal array (λs^3, k, s, 3) of strength 3 the number of constraints k satisfies the inequality:

$$k \leq \left[\frac{\lambda s^2 - 1}{s - 1} \right] + 1 \qquad \text{(ii)}$$

Theorem 3 (Relation of a matrix to an OA)

An $N \times k$ matrix A with entries from a finite field $\mathbb{F}_s = \{0, 1, 2, \cdots, s-1\}$ is an $OA(N, k, s, t)$ if and only if:

$$\sum_{u = row\, of\, A} \zeta^{uv^T} = 0 \qquad \text{(iii)}$$

For all k-tuples v over $\{0, 1, 2, \cdots, s-1\}$ with w non-zero entries, for all w in the range $1 \le w \le t$, where $\zeta = e^{\frac{2\pi i}{s}}$ and uv^T is evaluated modulo s.

(The proof of this theorem is given in Appendix B)

Theorem 4 (Special Case of Theorem Γ)

When $s = 2$, ζ reduces to -1 ($\zeta = e^{\frac{2\pi i}{2}} = e^{\pi i} = -1$), so an $N \times k$ matrix A with $0, 1$ entries is an orthogonal array $OA(N, k, 2, t)$ if and only if :

$$\sum_{u = row\, of\, A} (-1)^{uv^T} = 0 \qquad \text{(iv)}$$

For all $0, 1$ vectors v containing w 1's, for all w in the range $1 \le w \le t$, where the sum is over all rows $u\ of\ A$.

Example No6

Application Of Theorem 4

Let D be the $N \times k$ matrix:

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Then, according to the Theorem 4 :

$u:[0,0,0],[1,1,0],[0,1,1],[1,0,1]$ then in order to find the 3-tuples of v it must be considered that $1 \le w \le 2$ because $t=2$ where $w = non-zero\ entries$ so:

v: $[1,1,0],[0,1,1],[1,0,1]$, then uv^T :

a) $$[0,0,0]\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}=0,\ [0,0,0]\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}=0,\ [0,0,0]\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}=0$$

b) $$[1,1,0]\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}=2,\ [1,1,0]\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}=1,\ [1,1,0]\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}=1$$

c) $$[0,1,1]\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}=1,\ [0,1,1]\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}=2,\ [0,1,1]\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}=1$$

d) $$[1,0,1]\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}=1,\ [1,0,1]\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}=1,\ [1,0,1]\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}=2$$

Then, through the formula $$\sum_{u=row\,of\ D}(-1)^{uv^T}=$$

$$\begin{aligned}&(-1)^0+(-1)^0+(-1)^0+\\&(-1)^2+(-1)^1+(-1)^1+\\&(-1)^1+(-1)^2+(-1)^1+\\&(-1)^1+(-1)^1+(-1)^2=\end{aligned}$$

$$=1+1+1+1+(-1)+(-1)+(-1)+1+(-1)+(-1)+(-1)+1=\ 6+(-6)=0$$

Which according to the Theorem 4 shows that:

$$\begin{matrix}0&0&0\\1&1&0\\0&1&1\\1&0&1\end{matrix}$$

is also an $OA(4,3,2,2)$.

In contrast…

If D were:

$$D=\begin{pmatrix}0&0&0\\1&1&0\\0&1&1\\1&0&1\end{pmatrix}$$

Then for the application of Theorem 4:

$u:[0,0,0],[1,1,1],[0,1,1],[1,0,1]$ and $v:[0,1,1],[1,0,1]$ so all the possible combinations uv^T will be:

a) $$[0,0,0]\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}=0,\ [0,0,0]\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}=0$$

b) $$[1,1,1]\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}=2,\ [1,1,1]\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}=2$$

c) $$[0,1,1]\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}=2,\ [0,1,1]\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}=1$$

d) $$[1,0,1]\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}=1,\ [1,0,1]\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}=2$$

$(-1)^0+(-1)^0+(-1)^2+(-1)^2+(-1)^2+(-1)^1+(-1)^1+(-1)^2=$
$=1+1+1+1+1+(-1)+(-1)+1=\ 6-2=4\neq 0$ ⇨ D is not an OA.

Chapter 3 – Construction of OAs

3.1 Rings, Fields and Galois Fields

Before proceeding to describe construction methodologies of OAs, it is useful to present at this stage some key elements from the theory of fields, rings and *Galois Fields*. The definition of a Field and a Ring follow:

Definition 3.1.1 A *Field* is a permutational *Ring* of division

Definition 3.1.2 A *Ring* $<R,+,\cdot>$ is a set R with two binary operations (+ and ·) which are called addition and multiplication denoted in R. The following properties must hold:

R_1: $<R,+>$ must be an *abelian group.*

R_2: In group $<R,\cdot>$, if a, b, c $\in R$ then $a\cdot(b\cdot c)=(a\cdot b)\cdot c$ (associative property in multiplication).

R_3: For every a, b, c $\in R$, multiplication distributes over addition $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$ and $(a+b)\cdot c=(a\cdot c)+(b\cdot c)$ (distributive property).

3.1.3 Definition of Galois Field

Galois Field is a field with a finite field order i.e finite number of elements. The order of a Galois Field (GF) is always a prime number or a power of a prime number. For each prime power, there exists exactly one $GF(p^n)$. $GF(p)$ is called the prime field of order p (or finite field $\mathbb{F}_p$), and is the field of residue classes modulo p, where the p elements are denoted as $0,1,2,\cdots,p-1$. In $GF(p)$ $a=b$ is equivalent to $a\equiv b \bmod p$.

In order to construct an orthogonal array the theory of Galois Fields has to be used. It will be shown how the extension from a Finite Field of a prime number p to a Galois Field of p^n occurs, where $n \in \mathbb{N}$. This extension is used by R.C.Bose and K.A.Bush in order to construct OAs. This will be showed by various examples that follow.

Example No7

Let $\mathbb{F}_3 = \{0, 1, 2\}$ be a finite field and $f(x) = x^2 + 2x + 2$ be an irreducible polynomial. For the construction of $GF(3^2)$, the methodology is as follows:

Assume $a \in \mathbb{F}_3$ to be a solution of $f(a) = a^2 + 2a + 2 = 0$, then $a^2 = -(2a + 2) \Rightarrow$ $a^2 \equiv -(2a + 2) \bmod 3 \Rightarrow a^2 \equiv -2(a + 1) \bmod 3 \Rightarrow$

$$a^2 \equiv a + 1 \tag{7.1}$$

The Galois Field $GF(9)$ will include the elements of $\mathbb{F}_3$ and we must determine the six other elements in order to complete the $GF(3^2)$. Until now: $GF(9) = \{0, 1, 2, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots\}$.

The next element is going to be "α". The rest of the elements will be determined via all additive and multiplicative combinations of the existing elements (as long as they do not already exist), in this way the next element is going to be a added to the second element, therefore $a + 1$, the next element is going to be a added to the third element, therefore $a + 2$. So, until now:

$$GF(9) = \{0, 1, 2, a, a + 1, a + 2, \ldots, \ldots, \ldots\} \tag{7.2}$$

Now, in order to complete $GF(3^2)$ the multiplicative combinations have to be determined. So, element a multiplied with the first element of (7.2) gives 0 which is already in the (7.2) set, a multiplied with second element gives a which already exists too, but a multiplied with 2 gives $2a$ which goes to the (7.2) as the 7th element. The additive combinations now of $2a$ with its preceding elements should be considered, i.e. $2a + 0$ (which already exists), $2a + 1$, $2a + 2$.

This completes $GF(3^2) = GF(9) = \{0, 1, 2, a, a + 1, a + 2, 2a, 2a + 1, 2a + 2\}$ and in this way we have demonstrated the extension from $\mathbb{F}_3$ to $GF(3^2)$.

Example No8

An extension/construction of $GF(2^3)$ from a finite field $\mathbb{F}_2 = \{0, 1\}$ through the irreducible polynomial $f(x) = x^3 + x + 1$ is going to be demonstrated in this example. It is important to say that $f(x)$ being irreducible means that the elements of $\mathbb{F}_2$ are not solutions to $f(x) = 0$.

Assuming $a \in \mathbb{F}_2$ such that $f(a) = a^3 + a + 1 = 0$, then $a^3 = -(a+1)$ and because we work over $\mathbb{F}_2$ $a^3 \equiv -(a+1) \bmod 2 \Rightarrow$

$$a^3 \equiv a + 1 \tag{8.1}$$

Through this equality and using method described previously (example No7) with the addition and multiplication properties the constructed Galois Field is $GF(2^3) = GF(8) = \{0, 1, a, a+1, a^2, a^2+1, a^2+a, a^2+a+1\}$.

Elements 0 and 1 come from $\mathbb{F}_2$. Then a is the 3rd element, $a+1$ (addition) is the 4th element, $a \cdot a = a^2$ (multiplication) is the 5th element, a^2+1 (addition) is the 6th element. a^2+a (addition) is the 7th element and finally a^2+a+1 (addition) is the 8th element.

Example No9

In this report we are mostly concerned with OAs of level 2. As a consequence it is of great interest to see extensions of Galois Fields with $p = 2$.

Let us now see the extension of $\mathbb{F}_2$ to $GF(2^4)$ through the irreducible polynomial $f(x) = x^4 + x + 1$.

Assuming $a \in \mathbb{F}_2$ such that $f(a) = a^4 + a + 1 = 0$, then $a^4 = -(a+1)$ $\Rightarrow a^4 \equiv -(a+1) \bmod 2 \Rightarrow a^4 \equiv a + 1$ and therefore

$$a^4 = a + 1 \tag{9.1}$$

Hence, $GF(2^4)$ is:

$$\begin{aligned} GF(2^4) = GF(16) = \{&0, 1, a, a+1, a^2, a^2+1, a^2+a, a^2+a+1, a^3, a^3+1, a^3+a, \\ &a^3+a+1, a^3+a^2, a^3+a^2+1, a^3+a^2+a, a^3+a^2+a+1\} \end{aligned}$$

Until the 8^{th} element the construction is similar to the previous example. The 9^{th} element results from the multiplication of a^2 with a and the rest of the elements result from the addition of a^3 with its preceding elements.

3.2 Bush's Construction

Theorem 3.2.1

If $s \geq 2$ is a prime power then an $OA(s^t, s+1, s, t)$ of index unity exists whenever $s \geq t-1 \geq 0$.

On the analysis of Theorem 3.2.1

First an $s^t \times s$ array is going to be constructed, the columns of which are labeled by the elements of $GF(s)$ with $s \in \mathbb{F}_s = \{0, 1, \ldots, s-1\}$ and rows are labeled by s^t polynomials over $GF(s)$ of degree at most $t-1$. These polynomials are denoted:

$$\varphi_1(x), \varphi_2(x), \cdots, \varphi_{s^t}(x)$$

In order to take an $s^t \times s+1$ array another factor has to be added to the $s^t \times s$ array. This factor/column consists of the coefficient of X^{t-1} in every j-row of $\varphi_1(x), \varphi_2(x), \cdots, \varphi_{s^t}(x)$ polynomials with $j = 1, \ldots, s^t$.

Therefore:

	0	1	…	$s-1$	$coef\ \varphi_j(X^{t-1})$
$\varphi_1(x)$	$\varphi_1(0)$	$\varphi_1(1)$	…	$\varphi_1(s-1)$	$coef\ \varphi_1(X^{t-1})$
$\varphi_2(x)$	$\varphi_2(0)$	$\varphi_2(1)$	…	$\varphi_2(s-1)$	$coef\ \varphi_2(X^{t-1})$
$\vdots$	$\vdots$	$\vdots$	…	$\vdots$	$\vdots$
$\varphi_{s^t}(x)$	$\varphi_{s^t}(0)$	$\varphi_{s^t}(1)$	…	$\varphi_{s^t}(s-1)$	$coef\ \varphi_{s^t}(X^{t-1})$

Figure 2 "Bush's Construction Scheme using Theorem 3.2.1"

On the application of Theorem 3.2.1

Let us construct an orthogonal array of 4 trials, 3 factors, 2 levels 2 strength and index unity. Then Theorem 3.2.1 may be used. The first column is going to consist of the elements of $GF(2^2)$. By applying the described methodology the following array is extracted:

	0	1	$coef\ \varphi_j(X)$
$\varphi_1(x)=0$	0	0	0
$\varphi_2(x)=1$	1	1	0
$\varphi_3(x)=x$	0	1	1
$\varphi_4(x)=x+1$	1	0	1

It is obvious then that an $OA(4,3,2,2)$ has been constructed. This method is introduced by Bush and is known as Bush's method of constructing an OA.

Theorem 3.2.2

If $s=2^m$, $m\geq 1$, $t=3$, then there exists an $OA(s^3, s+2, s, t)$.

On the analysis of Theorem 3.2.2

The analysis of the theorem 3.2.2 is similar to the analysis of theorem 3.2.1. The only difference is that when the two same columns are constructed (because $k=s+2$) the first one consists of the coefficient of X^{t-1} for every j-row of $\varphi_1(x), \varphi_2(x), \cdots, \varphi_{s^t}(x)$

polynomials with $j=1,\ldots,s^t$, and the second column consists of the coefficient X^{t-2} for every j-row of $\varphi_1(x),\varphi_2(x),\cdots,\varphi_{s^t}(x)$ polynomials with $j=1,\ldots,s^t$.

Therefore:

	0	1	$\ldots$	$s-1$	$coef\ \varphi_j(X^{t-1})$	$coef\ \varphi_j(X^{t-2})$
$\varphi_1(x)$	$\varphi_1(0)$	$\varphi_1(1)$	$\ldots$	$\varphi_1(s-1)$	$coef\ \varphi_1(X^{t-1})$	$coef\ \varphi_1(X^{t-2})$
$\varphi_2(x)$	$\varphi_2(0)$	$\varphi_2(1)$	$\ldots$	$\varphi_2(s-1)$	$coef\ \varphi_2(X^{t-1})$	$coef\ \varphi_2(X^{t-2})$
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$
$\varphi_{s^t}(x)$	$\varphi_{s^t}(0)$	$\varphi_{s^t}(1)$	$\ldots$	$\varphi_{s^t}(s-1)$	$coef\ \varphi_{s^t}(X^{t-1})$	$coef\ \varphi_{s^t}(X^{t-2})$

Figure 3 "Bush's Construction using Theorem 3.2.2"

<u>On the application of Theorem 3.2.2</u>

Let us now construct an $OA(8,4,2,3)$. Obviously it can be related to the Theorem 3.2.2 because $OA(8,4,2,3)=OA(2^3,2+2,2,3)$ so such an orthogonal array may be constructed. Following the previous analysis an $OA(8,4,2,3)$ is extracted:

	0	1	$coef\ X^2$	$coef\ X$
0	0	0	0	0
1	1	1	0	0
x	0	1	0	1
$x+1$	1	0	0	1
x^2	0	1	1	0
x^2+1	1	0	1	0
x^2+x	0	0	1	1
x^2+x+1	1	1	1	1

.

3.3 Bose & Bush Construction Method (Method of Differences)

3.3.1 Method of Differences Theorem

Let M be a module (additive group) consisting of s elements, $e_0, e_1, \ldots, e_{s-1}$. Suppose it is possible to find a scheme of r rows, with elements belonging to M

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rn} \end{matrix}$$

such that among the differences of the corresponding elements of any two rows, each element of M occurs exactly λ times ($n = \lambda s$); then the method of constructing a completely resolvable orthogonal array $(\lambda s^2, r, s, 2)$ of strength 2 is as follows:

Write down the addition table of M. Then replace each element in the scheme by the row of the addition table corresponding to the element (using only the suffixes if the set Σ is taken as $0, 1, \ldots, s-1$). This gives the completely resolvable array $(\lambda s^2, r, s, 2)$. A new row can be added to obtain an array $(\lambda s^2, r+1, s, 2)$ of $r+1$ constraints (factors).

<u>The construction</u>

Now let us demonstrate how this method works through the construction of OA(18, 7, 3, 2)

Assume that there is an addition table named A such as:

$$\begin{array}{c|ccc} + & e_0 & e_1 & e_2 \\ \hline e_0 & e_0 & e_1 & e_2 \\ e_1 & e_1 & e_2 & e_0 \\ e_2 & e_2 & e_0 & e_1 \end{array} \qquad \text{(v)}$$

Where $e_0 = 0$, $e_1 = 1$, $e_2 = 2$.

First the OA(18, 6, 3, 2) is going to be constructed.

A six-rowed scheme is constructed by trial:

$$\begin{matrix} e_0 & e_0 & e_0 & e_0 & e_0 & e_0 \\ e_0 & e_0 & e_1 & e_2 & e_1 & e_2 \\ e_0 & e_1 & e_0 & e_2 & e_2 & e_1 \\ e_0 & e_2 & e_2 & e_0 & e_1 & e_1 \\ e_0 & e_1 & e_2 & e_1 & e_0 & e_2 \\ e_0 & e_2 & e_1 & e_1 & e_2 & e_0 \end{matrix} \qquad \text{(vi)}$$

It can be seen that the first column, the first row and the diagonal consist of the element e_0. Also, the 2nd row and the 2nd column are the same; the 3rd column and the 3rd row are the same etc. It can be easily seen that there is symmetry over and under the diagonal of scheme (vi). Also notice that by taking differences of the corresponding elements in any two rows, each of the three e_i with $i = 0,1,2$ occurs twice. The aim of these observations is to clarify the "by trial" construction of scheme (vi). This will become clear in the following section.

From scheme (v) assume that every element from the first column corresponds to the sequence of numbers on the right:

$$\begin{matrix} e_0 & \to & 0,1,2 \\ e_1 & \to & 1,2,0 \\ e_2 & \to & 2,0,1 \end{matrix} \qquad \text{(vii)}$$

In order to have the OA(18, 6, 3, 2) each element in scheme (vi) is replaced by each sequence of numbers as shown in (vii). So, the resulting array is:

$$\begin{matrix} 0&1&2&0&1&2&0&1&2&0&1&2&0&1&2&0&1&2 \\ 0&1&2&0&1&2&1&2&0&2&0&1&1&2&0&2&0&1 \\ 0&1&2&1&2&0&0&1&2&2&0&1&2&0&1&1&2&0 \\ 0&1&2&2&0&1&2&0&1&0&1&2&1&2&0&1&2&0 \\ 0&1&2&1&2&0&2&0&1&1&2&0&0&1&2&2&0&1 \\ 0&1&2&2&0&1&1&2&0&1&2&0&2&0&1&0&1&2 \end{matrix} \qquad \text{(viii)}$$

Note that this array is written in "reversed" form, where trials are represented in the columns, and the factors are represented in the rows. The level is 3, the strength is 2 and index is 2. Note that $k \le \left[\frac{2(3)^2-1}{3-1}\right] = \left[\frac{17}{2}\right] = [8.5] = 8$ from Theorem 1 (Plackett and Burman's inequality), which means that the factors can be up to 8 in order not to lose the orthogonality.

The array (viii) is not yet complete, though it is orthogonal. The construction of OA(18, 7, 3, 2) is required and this far we have described the construction of OA(18, 6, 3, 2). The final step requires the addition of a final row under the six first rows (to add one more factor). This row consists of 6 zeros, 6 ones and 6 twos. The idea is that all the elements 0, 1, 2 should be equally placed in that row and because the 7th row has 18 blocks so $\frac{18}{3} = 6$ blocks correspond to each element. If first the 6 zeros, then the 6 ones and last the 6 twos are placed, then the resulting array is:

$$\begin{array}{cccccccccccccccccc} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 & 0 & 0 & 1 & 2 & 2 & 0 & 1 & 2 & 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 & 1 & 2 & 0 & 1 & 0 & 1 & 2 & 1 & 2 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 & 1 & 2 & 0 & 0 & 1 & 2 & 2 & 0 & 1 \\ 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 & 1 & 2 & 0 & 2 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \end{array} \qquad \text{(ix)}$$

which is the required orthogonal array OA(18, 7, 3, 2).

Comment

The key to the Bose&Bush construction method (Method of Differences - MoD) is to understand how scheme (vi) is formulated. It has been mentioned that scheme (vi) is constructed by trial and error, however the pattern of construction results from the fact that the constructed array is symmetric and that each element e_i should appear twice in the difference of the corresponding elements of any two rows.

3.4 The construction of Scheme (vi)

It is important to note that scheme (vi) is not constructed by randomly placing e_0, e_1, e_2. A pattern may be observed.

The first row is completed by e_0 only, thus:

$$e_0 \quad e_0 \quad e_0 \quad e_0 \quad e_0 \quad e_0$$

The same holds true for the first column and the diagonal too, so:

$$\begin{array}{cccccc} e_0 & e_0 & e_0 & e_0 & e_0 & e_0 \\ e_0 & e_0 & & & & \\ e_0 & & e_0 & & & \\ e_0 & & & e_0 & & \\ e_0 & & & & e_0 & \\ e_0 & & & & & e_0 \end{array}$$

Then, in every incomplete row the elements e_0, e_1, e_2 have to appear twice. In the second row e_0 occurs twice already, so the four remaining elements will be $2e_1$'s and 2 e_2's.The sequence is chosen to be e_1, e_2, e_1, e_2 but it could be any other sequence such as e_1, e_2, e_2, e_1 or e_1, e_1, e_2, e_2, this results from the fact that the first row consists of only e_0's, so that taking the difference between e_0 and e_1, and e_0 and e_2 in whichever order, gives the desired frequency of appearance of e_1's and e_2's. For example, $e_0 - e_1 = e_2$, $e_0 - e_2 = e_1$ if taken as row1-row2 or $e_1 - e_0 = e_1$, $e_2 - e_0 = e_2$ if taken as row2-row1. The important thing is that this sequence is going to be used as column too, because of the symmetry above and below the diagonal of the scheme. So:

$$\begin{array}{cccccc} e_0 & e_0 & e_0 & e_0 & e_0 & e_0 \\ e_0 & e_0 & e_1 & e_2 & e_1 & e_2 \\ e_0 & e_1 & e_0 & & & \\ e_0 & e_2 & & e_0 & & \\ e_0 & e_1 & & & e_0 & \\ e_0 & e_2 & & & & e_0 \end{array}$$

Now, in order to fill up the remaining blank blocks of the 3rd row it is observed that e_0 occurs twice and e_1 occurs once. So e_1 is going to be used once again and e_2 twice. The same is repeated for the 3rd column. Then:

$$\begin{matrix} e_0 & e_0 & e_0 & e_0 & e_0 & e_0 \\ e_0 & e_0 & e_1 & e_2 & e_1 & e_2 \\ e_0 & e_1 & e_0 & e_2 & e_2 & e_1 \\ e_0 & e_2 & e_2 & e_0 & & \\ e_0 & e_1 & e_2 & & e_0 & \\ e_0 & e_2 & e_1 & & & e_0 \end{matrix}$$

Now, in the 4th row the elements e_0 and e_2 already appear twice. So the only remaining possibility is e_1, e_1. The same holds true for the 4th column. So:

$$\begin{matrix} e_0 & e_0 & e_0 & e_0 & e_0 & e_0 \\ e_0 & e_0 & e_1 & e_2 & e_1 & e_2 \\ e_0 & e_1 & e_0 & e_2 & e_2 & e_1 \\ e_0 & e_2 & e_2 & e_0 & e_1 & e_1 \\ e_0 & e_1 & e_2 & e_1 & e_0 & \\ e_0 & e_2 & e_1 & e_1 & & e_0 \end{matrix}$$

Apparently, the fifth row needs to be filled with element e_2 which occurs only once. The same holds true for the 5th column again. So finally, the array:

$$\begin{matrix} e_0 & e_0 & e_0 & e_0 & e_0 & e_0 \\ e_0 & e_0 & e_1 & e_2 & e_1 & e_2 \\ e_0 & e_1 & e_0 & e_2 & e_2 & e_1 \\ e_0 & e_2 & e_2 & e_0 & e_1 & e_1 \\ e_0 & e_1 & e_2 & e_1 & e_0 & e_2 \\ e_0 & e_2 & e_1 & e_1 & e_2 & e_0 \end{matrix}$$

is obtained from the Bose&Bush construction of the differences (MoD).

Note that the condition that all elements e_i should appear twice if we take the differences of the corresponding elements of any two rows, is always satisfied.

3.5 Alternative OA Construction using B&B MoD

With the pattern described above other similar schemes may be formulated which lead to alternative OA constructing schemes like scheme (vi). 4 such schemes have been constructed in order to be presented in this report. These schemes result from the fact that different combinations of e_1's and e_2's when constructing row2 of scheme (vi) may be taken. Of course, the basic condition of MoD as mentioned in the end of the previous section should always be satisfied. In this way, the following combinations for the remaining elements of row2 have been considered: $e_1\ e_1\ e_2\ e_2$, $e_1\ e_2\ e_2\ e_1$, $e_2\ e_2\ e_1\ e_1$. Thus, the following schemes may be formulated:

$$\begin{array}{cccccc}
e_0 & e_0 & e_0 & e_0 & e_0 & e_0 \\
e_0 & e_0 & e_1 & e_1 & e_2 & e_2 \\
e_0 & e_1 & e_0 & e_2 & e_2 & e_1 \\
e_0 & e_1 & e_2 & e_0 & e_1 & e_2 \\
e_0 & e_2 & e_2 & e_1 & e_0 & e_1 \\
e_0 & e_2 & e_1 & e_2 & e_1 & e_0
\end{array}$$

"Scheme A"

$$\begin{array}{cccccc}
e_0 & e_0 & e_0 & e_0 & e_0 & e_0 \\
e_0 & e_0 & e_1 & e_1 & e_2 & e_2 \\
e_0 & e_1 & e_0 & e_2 & e_1 & e_2 \\
e_0 & e_1 & e_2 & e_0 & e_2 & e_1 \\
e_0 & e_2 & e_1 & e_2 & e_0 & e_1 \\
e_0 & e_2 & e_2 & e_1 & e_1 & e_0
\end{array}$$

"Scheme B"

$$\begin{array}{cccccc}
e_0 & e_0 & e_0 & e_0 & e_0 & e_0 \\
e_0 & e_0 & e_1 & e_2 & e_2 & e_1 \\
e_0 & e_1 & e_0 & e_2 & e_1 & e_2 \\
e_0 & e_2 & e_2 & e_0 & e_1 & e_1 \\
e_0 & e_2 & e_1 & e_1 & e_0 & e_2 \\
e_0 & e_1 & e_2 & e_1 & e_2 & e_0
\end{array}$$

"Scheme Γ"

$$\begin{array}{cccccc}
e_0 & e_0 & e_0 & e_0 & e_0 & e_0 \\
e_0 & e_0 & e_2 & e_2 & e_1 & e_1 \\
e_0 & e_2 & e_0 & e_1 & e_2 & e_1 \\
e_0 & e_2 & e_1 & e_0 & e_1 & e_2 \\
e_0 & e_1 & e_2 & e_1 & e_0 & e_2 \\
e_0 & e_1 & e_1 & e_2 & e_2 & e_0
\end{array}$$

"Scheme Δ"

Using the method of differences and the addition table (v) scheme A gives the following OA(18, 6, 3, 2):

0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
0	1	2	0	1	2	1	2	0	1	2	0	2	0	1	2	0	1
0	1	2	1	2	0	0	1	2	2	0	1	2	0	1	1	2	0
0	1	2	1	2	0	2	0	1	0	1	2	1	2	0	2	0	1
0	1	2	2	0	1	2	0	1	1	2	0	0	1	2	1	2	0
0	1	2	2	0	1	1	2	0	2	0	1	1	2	0	0	1	2

Table 5 "OA(18, 6, 3, 2) constructed from Scheme A"

This is an OA(18, 6, 3, 2) which is constructed and presented in this report using the MoD by Bose&Bush. By using the other Schemes shown above and following the same pattern different OAs may be constructed such that $N = 18$, $k = 6$, $s = 3$, $t = 2$.

Chapter 4 – Use of OAs in the Design of Experiments

4.1 Utilization and Use of OAs

Definition 3 (D3)

An experiment is named *complete factorial (CF)* when all possible trials and factor combinations are considered.

Definition 4 (D4)

An experiment is named *fractional factorial (FrF)* when specific trials are omitted in order to reduce cost and time loss.

In the following section the importance of statistically equivalent orthogonal arrays will be demonstrated, which allows statisticians to reduce the number of trials without the possibility to omit the optimum solution for the problem. Reducing the number of trials means that two very important factors such as the cost and the loss of time are reduced.

It has already been mentioned that arrays are used when an experiment (physical-numerical) is taking place. It is reminded that an experiment involves a number of trials and the influence of various factors. The factors affect the problem in a positive or in a negative way and the experimentalists-modellers will try different factor combinations in order to find the optimal combination of these factors. The number of such trials could be very large and/or each trial could cost a lot of money. So, statisticians face the problem of making fewer trials and still being able to obtain a meaningful answer from the remaining combination of factors. If this is going to be by randomly choosing trials from the total number of trials, it is highly possible for the process to fail. On the other hand, it is more effective for the experimentalists to trust a statistically equivalent array constructed from their primary array of trials and factors. This reduced array, constructed from the primary array, can be an orthogonal array and may be constructed by a sequence of permutations of the trials of the primary array and neglecting specific trials in order to go from a Complete Factorial array to a Fractional Factorial one. This is the key to reduce the number of trials without risking omitting the most important factors. This concept was first introduced by C.R.Rao (1947), who presented certain combinatorial arrangements based on Galois Fields and finite projective geometries.

4.2 Construction using linear code theory elements

Let us assume a 4 2-level factor experiment. Therefore, 2^4 are all the possible combinations between these factors which is the number of all possible trials in the complete factorial experiment.

The resulting array is:

	A	B	C	D
Trial 1	0	0	0	0
Trial 2	1	0	0	0
Trial 3	0	1	0	0
Trial 4	0	0	1	0
Trial 5	1	1	0	0
Trial 6	1	0	1	0
Trial 7	0	1	1	0
Trial 8	1	1	1	0
Trial 9	0	0	0	1
Trial 10	1	0	0	1
Trial 11	0	1	0	1
Trial 12	0	0	1	1
Trial 13	1	1	0	1
Trial 14	1	0	1	1
Trial 15	0	1	1	1
Trial 16	1	1	1	1

Table 6

"The Array of a Complete Factorial experiment when $s = 2$ ***and*** $k = 4$ ***"***

The order in which the trials are presented is called “reverse lexicographic”. The columns correspond to each factor A, B, C, D. The rows correspond to each trial. This array is used in a complete factorial experiment in order to account for all interactions between factors. In this chapter we will demonstrate how an OA(8, 4, 2, 3) may be constructed in such a way that all important interactions are taken into consideration. This methodology is used for the construction of every $OA(2^{k-n}, k, 2, R-1)$, where R: minimal distance of a linear code, n: dimension of the code, k: number of columns of the generator matrix of the code, 2: order of the field (number of levels).

The rows of the above array correspond to the codewords:

$$\begin{aligned}
(1) &= (0,0,0,0)\\
A &= (1,0,0,0)\\
B &= (0,1,0,0)\\
C &= (0,0,1,0)\\
AB &= (1,1,0,0)\\
AC &= (1,0,1,0)\\
BC &= (0,1,1,0)\\
ABC &= (1,1,1,0)\\
D &= (0,0,0,1)\\
AD &= (1,0,0,1)\\
BD &= (0,1,0,1)\\
CD &= (0,0,1,1)\\
ABD &= (1,1,0,1)\\
ACD &= (1,0,1,1)\\
BCD &= (0,1,1,1)\\
ABCD &= (1,1,1,1)
\end{aligned}$$

Where the convention has been used that each letter corresponds to the highest value of each factor.

4.2.1 Some basics of linear code theory

A linear code C over field $\mathbb{F}$ (of order s) is simply a linear subspace of $\mathbb{F}^k$ and the vectors in this subspace are called codewords. Every code is specified by an $n \times k$ generator matrix whose rows form a basis (n and k are the dimension and length respectively of C). The minimal distance R of the code is defined as the smallest number of non-zero components in any non-zero codeword

A word in the defining relation of a fraction of a 2^k factorial (we are working over $\mathbb{F}_2$) is represented by a binary vector of length k, with a 1 in the i-th coordinate if and only if the i-th factor appears in the word. In this way, if k = 4, A represents $(1,0,0,0)$, AC represent $(1,0,1,0)$, ABD $(1,0,1,1)$, etc.

Interactions between factors (multiplication of words) may be determined with the use of binary arithmetic, i.e. (AB)(BC) = AB^2C = AC since $(1,1,0,0)+(0,1,1,0)=(1,0,1,0)$. The product AC is called the generalised interaction of AB and BC.

In order to obtain from the 2^k factorial a 2^{k-n} fractional factorial, a relation linear code C should be generated, of dimension n and length k. Then, n generating vectors may be selected and corresponding words $W_1, W_2, \ldots, W_n$ obtained. Then the defining relation:

$$I_0 = \pm W_1 = \pm W_2 = \ldots = W_n = \text{generalized interactions}$$

specifies a 2^{k-n} fractional factorial. The word in the defining relation of the fraction is of length R, i.e. the shortest non-zero word. R is also known as the resolution of the fraction.

Once K, n and R have been chosen then the appropriate $n \times k$ generator matrix of the relation code C may be constructed with as many non-zero entries per row as determined by the minimal distance R. Then, the defining relation I_0 is derived from the generator matrix, through codewords of non-zero entries and of length R. Then, a matrix is constructed with the rows of (k-n) factors set independently and the rows for the rest of the factors found through the generalised interactions. These (k-n) factors are chosen such that in a row of the generator matrix they cannot be used to form a non-zero codeword. The first of the (k-n) columns is set by alternating -1 and +1, starting with -1. The second column is set by alternating groups of -1's and +1's of size $2^1 = 2$ starting with -1's. The third column is set by alternating groups of -1's and +1's of size $2^2 = 4$, again starting with -1's, and so on until all (k-n) columns have been constructed. Once the full matrix has been constructed, an OA is derived by simply substituting all -1's with 0's and +1's with 1's (i.e. the lower and higher values).

4.2.2 The application of linear code methodology to the construction of OA(16, 6, 2, 3)

Before proceeding to construct OA(8, 4, 2, 3) we will demonstrate how this method (linear code method) works through the construction of OA(16, 6, 2, 3). Here, k = 6, n = 2 , t = R-1 ⇨ R = 4. Hence we will demonstrate how from a 2^6 factorial we may go to 2^4 fractional factorial and OA(16, 6, 2, 3).

The relation code C has length 6 and a generator matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Because R = 4 has been set as its minimal distance. Its defining relation should be a codeword of non-zero entries of length 4:

$$I_0 = ABCD = ABEF = CDEF$$

The first and third codewords appear with non-zero entries in the first and second rows respectively of the generator matrix whereas in the second codeword A and B appear with non-zero entries in the first row and E and F appear with non-zero entries in the second row of the generator matrix.

In order to construct the matrix we choose 4 = (6-2) factors that cannot be used to form a non-zero codeword in a row of the generator matrix. In this example A, B, C and E from the first row are chosen (we could have also chosen A, B, C, F from 1st row, A, D, E, F from the 2nd row etc.). Then, their columns are set with alternating groups of -1's and +1's with size 2^0, 2^1, 2^2, 2^3 respectively. The columns of D and F are found through the defining relations, i.e.

$$I_0 = ABCD \Rightarrow D = ABCD^2 = ABC$$
$$I_0 = ABEF \Rightarrow F = ABEF^2 = ABE$$

and the multiplication of the corresponding elements of each row, e.g. the elements of the 1st row for D is $ABC = (-1) \cdot (-1) \cdot (-1) = -1$.

The $2^{6\text{-}2}$ fractional factorial matrix then becomes:

A	B	C	E	D = ABC	F = ABE
-1	-1	-1	-1	-1	-1
+1	-1	-1	-1	+1	+1
-1	+1	-1	-1	+1	+1
+1	+1	-1	-1	-1	-1
-1	-1	+1	-1	+1	-1
+1	-1	+1	-1	-1	+1
-1	+1	+1	-1	-1	+1
+1	+1	+1	-1	+1	-1
-1	-1	-1	+1	-1	+1
+1	-1	-1	+1	+1	-1
-1	+1	-1	+1	+1	-1
+1	+1	-1	+1	-1	+1
-1	-1	+1	+1	+1	+1
+1	-1	+1	+1	-1	-1
-1	+1	+1	+1	-1	-1
+1	+1	+1	+1	+1	+1

substituting -1 with 0 and +1 with 1, then an OA(16, 4, 2, 3) is constructed.

The OA(16, 6, 2, 3) then is:

A	B	C	D	E	F	interactions
0	0	0	0	0	0	(*)
1	0	0	1	0	1	adf
0	1	0	1	0	1	bdf
1	1	0	0	0	0	ab
0	0	1	1	0	0	cd
1	0	1	0	0	1	acf
0	1	1	0	0	1	bcf
1	1	1	1	0	0	abcd
0	0	0	0	1	1	ef
1	0	0	1	1	0	ade
0	1	0	1	1	0	bde
1	1	0	0	1	1	abef
0	0	1	1	1	1	cdef
1	0	1	0	1	0	ace
0	1	1	0	1	0	bce
1	1	1	1	1	1	abcdef

Figure 4 “OA(16, 6, 2, 3)”

In the last column, the interactions considered in this array are shown.

4.2.3 Construction of OA(8, 4, 2, 3)

In order to construct $OA_1(8,4,2,3)$ we need to generate a relation linear code C. Therefore, the generator matrix is:

$$[1, 1, 1, 1] \quad (x)$$

The defining relation should be a codeword with length R = 4 ⇨ I_0 = ABCD and the 3 = (4-1) factors that cannot be used to form a non-zero codeword, required for the construction of the 8×4 matrix are A, B and C (A, B, D or B, C, D etc. could have also been employed). Then, D may be obtained from: $D = ABCD^2 = ABC$.

Therefore:

Trials	A	B	C	D=ABC
No.1	-1	-1	-1	-1
No.2	1	-1	-1	1
No.3	-1	1	-1	1
No.4	1	1	-1	-1
No.5	-1	-1	1	1
No.6	1	-1	1	-1
No.7	-1	1	1	-1
No.8	1	1	1	1

Then, by leaving the 1's as they are (the high value) and by substituting the -1's with 0 (the low value), the next matrix is formed:

Trial No.	A	B	C	D	Interactions
1.	0	0	0	0	(*)
2.	1	0	0	1	ad
3.	0	1	0	1	bd
4.	1	1	0	0	ab
5.	0	0	1	1	cd
6.	1	0	1	0	ac
7.	0	1	1	0	bc
8.	1	1	1	1	abcd

Figure 5 "OA(8, 4, 2, 3) constructed using linear code Theory"

Where interactions between pairs of factors (2nd order interactions) have been considered. As defining relation we could have also used $I_0 = -ABCD$, so that $D = -ABCD^2 = -ABC$.

Therefore:

Trials	A	B	C	D
No1	-1	-1	-1	1
No2	1	-1	-1	-1
No3	-1	1	-1	-1
No4	1	1	-1	1
No5	-1	-1	1	-1
No6	1	-1	1	1
No7	-1	1	1	1
No8	1	1	1	-1

and again, by substituting:

Trial No.	A	B	C	D	Interactions
1.	0	0	0	1	d
2.	1	0	0	0	a
3.	0	1	0	0	b
4.	1	1	0	1	abd
5.	0	0	1	0	c
6.	1	0	1	1	acd
7.	0	1	1	1	bcd
8.	1	1	1	0	abc

is also an $OA_2(8,4,2,3)$ extracted from the Complete Factorial. In this case all main effects have been considered as well as all third order interactions between triplets of factors.

It may be observed that the combination of the two constructed OAs forms the original array of the experiment where all 16 main and higher order effects are considered. Through the construction of these arrays we have managed to go from a complete factorial to a fractional factorial experiment. Where only certain interactions are considered in such a way (i.e. by utilization of OAs) so that fewer trials are required in order to obtain a meaningful result.

Chapter 5 – Conclusions & Suggestions

5.1 Conclusions & Suggestions for Future Work

In this report we have examined in detail some basic properties of Orthogonal Arrays and various methodologies for their construction. As has been already mentioned, there is no generalised method for their (OAs) construction. An important application of OAs is in the design of experiments, where we need to reduce the number of performed experiments but in such a way that the remaining trials still yield meaningful answers. The methodology of selecting between all possible trials appropriate set of fewer trials is based on linear code theory and the construction from 2^k factorials of appropriate 2^{k-n} fractional factorials. With the use of this methodology we construct appropriate OAs which may be used in order to reduce the number of trials. Thus going from a complete factorial experiment to a fractional factorial experiment. In this way, we only perform a limited number of experiments and are still able to obtain a meaningful answer.

There are still a number of open issues for research in OAs and their application in the design of experiments, such as a unified methodology for their construction, as well as the use of linear code theory for the construction of fractional factorials over $\mathbb{F}_s$ where s prime number and $s \geq 3$.

Appendices

Appendix A

Proof of Plackett and Burman's inequality for OAs of strength 2.

Let

$$\lambda - 1 = a(s-1) + b, \qquad 0 \le b < s-1, \quad a \ge 0.$$

Therefore

$$\frac{\lambda s^2 - 1}{s-1} = \lambda s + \lambda + a + \frac{b}{s-1} \tag{a.1}$$

Suppose there exists an array with $k = \lambda s + \lambda + a + 1$. Then

$$k - 1 = s(\lambda + a) + b + 1.$$

The integer x is at our disposal. Let us choose $x = \lambda + a$; then $a = b + 1$, so that $0 < a < s$. From the following equation

$$D = (s-1)(k-a-1) + a(a+1)$$

It is clear that

$$D = (s-1)(\lambda + a) + a(a+1) > 0$$

So that

$$\frac{a(s-a)}{D} > 0. \tag{a.2}$$

The following inequality exists

$$\frac{\lambda s^2 - 1}{s-1} \ge k\left\{\frac{a(s-a)}{D} + 1\right\} \tag{a.3}$$

Hence, by (a.1), (a.2) and (a.3)

$$\frac{b}{s-1} > 1$$

This is a contradiction. Hence, the value $k = \lambda s + \lambda + a + 1$ is inadmissible and so are all the higher values.

Appendix B

Proof of Theorem 3

If A is an orthogonal array then it is easy to see that (iii) and (iv) hold. Suppose $s=t=2$ and (iv) holds. It is going to be showed in the first two columns of A all pairs $00, 01, 10, 11$ occur equally often. Let $n_{00}, \ldots, n_{11}$ be the number of occurrences of these pairs. Since the total number of runs is N and by taking v in (iv) to be respectively $010\ldots0, 100\ldots0$ and $110\ldots0$, the following equations are obtained:

$$
\begin{aligned}
n_{00}+n_{01}+n_{10}+n_{11}&=N\\
n_{00}-n_{01}+n_{10}-n_{11}&=0\\
n_{00}+n_{01}-n_{10}-n_{11}&=0\\
n_{00}-n_{01}-n_{10}+n_{11}&=0
\end{aligned}
$$

Plainly $n_{00}=n_{01}=n_{10}=n_{11}=N/4$ is a solution. Since the coefficient matrix

$$
\begin{bmatrix}
+1 & +1 & +1 & +1\\
+1 & -1 & +1 & -1\\
+1 & +1 & -1 & -1\\
+1 & -1 & -1 & +1
\end{bmatrix}
$$

is invertible, the solution is unique.

The general proof of the converse statements uses the same argument. Let $n(i_1, i_2, \ldots, i_t)$ denote the number of occurrences of the t-tuple $(i_1, i_2, \ldots, i_t)$ in the t columns under consideration, where each i_r is in the range $0 \le i_r \le s-1$. By choosing the vector v to have all possible s^t different values in these t coordinates, and to be zero elsewhere, we obtain s^t equations for the s^t unknowns $n(i_1, i_2, \ldots, i_t)$. If v is identically zero the right hand side of the equation is N, otherwise it is 0. Certainly setting all $n(i_1, i_2, \ldots, i_t)$ equal to N/s^t is a solution. The coefficient matrix is the character table of an elementary abelian group of the type s^t, which (by the orthogonality of characters) is an invertible matrix. Therefore the solution is unique. So the proof is complete.

References

1. R.C.Bose and K.A.Bush, "Orthogonal Arrays of Strength two and three"
2. R.Fletcher, "Practical Methods of Optimization"
3. John B.Fraleigh, "Introduction to Algebra"
4. Jonathan Gross, Jay Yellen, "Graph Theory and Its Applications"
5. Berton Gunter, "A Perspective on the Taguchi Methods"
6. A.S.Hedayat, N.J.A.Sloane, J.Stufken, "Orthogonal Arrays Theory and Applications"
7. A. Hedayat; J. Stufken, "On the Maximum Number of Constraints in Orthogonal Arrays"
8. Jan Van Leeuwen, "Algorithms and Complexity volume A"
9. K.R.Nair; C.R.Rao (1948), "Confounding in Asymmetrical Factorial Experiment"
10. R.L.Plackett; J.P.Burman, "The Design of Optimum Multifactorial Experiments"
11. C. R. Rao (chief ed.), "*Handbook of Statistics*", Vols 1-18. New York and Amsterdam: North Holland/Elsevier Science Publishers.
12. C. R. Rao. 2000, "Statistical proofs of some matrix inequalities. *Linear Algebra and its Applications*"
13. C. R. Rao. 1998, "A note on statistical analysis of shape through triangulation of landmarks", *Proc. National Academy of Sciences* 99: 2995-2998.
14. P.J.Ross, "Taguchi Techniques for Quality Engineering"
15. Esther Seiden, "On the Problem of Construction of Orthogonal Arrays"
16. G.Taguchi, "On Robust Technology Development"
17. http://mathworld.wolfram.com/
18. http://www.mathematics.gr/
19. http://encyclopedia.thefreedictionary.com/