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Master Thesis

On the study of Metastable Patterns for the One-Dimensional Cahn-Hilliard Equation

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ΣΧΟΛΗ ΘΕΤΙΚΩΝ ΚΑΙ ΤΕΧΝΟΛΟΓΙΚΩΝ ΕΠΙΣΤΗΜΩΝ
ΜΕΤΑΠΤΥΧΙΑΚΟ ΠΡΟΓΡΑΜΜΑ ¨ΜΑΘΗΜΑΤΙΚΑ ΚΑΙ ΕΦΑΡΜΟΓΕΣ ΤΟΥΣ’ ΤΩΝ ΤΜΗΜΑΤΩΝ ΜΑΘΗΜΑΤΙΚΩΝ ΚΑΙ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ

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Μελέτη Μεταευσταθών Προτύπων για τη Μονοδιάστατη Εξίσωση Cahn-Hilliard

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Abstract

In this thesis the results of P.W. Bates and J. Xun regarding the metastable patterns for the Cahn-Hilliard equation are reviewed. The main tool that is used for studying the dynamics of the patterns is the Invariant Manifold. The Invariant manifold, which is firstly constructed and suggested by J. Carr and R.L. Pego for studying the metastable patterns of the Reaction-Diffusion equation, approximates the phase space of the solutions of the Reaction-Diffusion equation. In the thesis the same results are proved for the Cahn-Hilliard equation. More specifically, if a Cahn Hilliard solution at a fixed time is close enough to an element of the Invariant Manifold, then afterward it goes closer and remains close to the Invariant Manifold. For the proof of this result, the eigenvalues of the integrated Cahn-Hilliard operator are estimated. Additionally, a system of ODEs that describes the evolution in terms of the Invariant Manifold is derived. Finally some results regarding the stochastic Cahn-Hilliard equation are presented. All the results are obtained using the integrated form of the Cahn-Hilliard equation. The reason is that it has the useful property of self-adjointness of the tensor of integrated Cahn-Hilliard which is used in the proof of the results regarding dynamics.

Περίληψη

Σε αυτή την εργασία παρουσιάζονται τα αποτελέσματα των P.W. Bates και J. Xun για τα μεταευσταθή πρότυπα στην εξίσωση Cahn-Hilliard. Το κύριο εργαλείο που χρησιμοποιείται για τη μελέτη της δυναμικής των προτύπων είναι η αναλλοίωτη πολλαπλότητα. Η αναλλοίωτη πολλαπλότητα, η οποία δημιουργήθηκε και προτάθηκε από τους J. Carr και R.L. Pego για τη μελέτη των μεταευσταθών προτύπων της εξίσωσης αντίδρασης διάχυσης, προσεγγίζει το χώρο φάσεων της εξίσωσης αντίδρασης διάχυσης. Στην παρούσα εργασία τα ίδια αποτελέσματα αποδεικνύονται για την εξίσωση Cahn-Hilliard. Πιό συγκεκριμένα, εάν μια λύση της εξίσωσης Cahn-Hilliard σε μιαδεδομένη χρονική στιγμή είναι αρκετά κοντά σε ένα σημείο της αναλλοίωτης πολλαπλότητας, τοτε από εκείνη τη χρονική στιγμή και μετά παραμένει κοντά στην αναλλοίωτη πολλαπλότητα. Για την απόδειξη αυτού του αποτελέσματος, εκτιμώνται οι ιδιοτιμές του τελεστή της εξίσωσης Cahn-Hilliard στην ολοκληρωτική του μορφή. Επιπλέον παράγεται ένα σύστημα συνήθων διαφορικών εξισώσεων που περιγράφει τη δυναμική της εξίσωσης πάνω στην πολλαπλότητα. Τέλος παρουσιάζονται ορισμένα αποτελέσματα σχετικά με τη στοχαστική εξίσωση Cahn-Hilliard. Ολα τα αποτελέσματα έχουν παραχθεί με χρήση της ολοκληρωτικής μορφής της εξίσωσης Cahn-Hilliard. Ο λόγος είναι η αξιοποίηση της ιδιότητας της αυτοσυζυγίας του τελεστή της ολοκληρωτικής μορφής της εξίσωσης Cahn-Hilliard, που χρησιμοποιείται στις αποδείξεις.

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Chapter 1

Introduction

In this thesis the metastable patterns for the one-dimensional Cahn-Hilliard differential equation are studied. More specifically the results of P. W. Bates and J. Xun in [BX1, BX2] are reviewed. The goal is studying and understanding every aspect of their work. I have attempted to review these results in a comprehensive and I hope interesting way.

In the rest of the introduction we have the opportunity of taking a first look at this subject and at what we are going to do. Firstly the natural problem that is described by the Cahn-Hilliard equation is introduced. Secondly the equation is defined. Thirdly an overview of the structure and chapters of the thesis is given.

1.1 The Natural Problem

The natural problem which is described by the Cahn-Hilliard equation is the evolution of metastable patterns. For a detailed introduction to them see [GD]. The most famous metastable patterns phenomenon that the equation describes is the behaviour of a melted binary homogeneous alloy with given concentrations of two components. When it is quenched rapidly to a temperature between the melting temperatures of the two components, the homogeneous alloy divides into two different concentration phases. The evolution after the quench has two stages: the Separation, which is relatively fast, and the Coarsening, which is extremely slow.

For a more detailed description of the alloy and the phases described, see Figure 1.1. The components of the alloy are A and B with concentrations C_A and $C_B = 1 - C_A$ respectively. The melting temperatures of A, B are T_A, T_B. T_0 is the temperature that is quenched. T_0 is valued between T_A and T_B. The two phases are the solid and liquid phase. If we assume that λ is the concentration of the liquid phase, then we have the following equations.

$$\begin{aligned} C_B &= C_B^L(T_0)\lambda + C_B^S(T_0)(1-\lambda) \\ C_A &= 1 - C_B \end{aligned}$$

. Now what we have to do is to investigate the dynamics of the two phases. Equivalently we can investigate just the dynamics of one of the two phases or of the layer points between them. Nevertheless, before we start thinking about it, we should be familiar with the equation.

1.2 The Equation

One of the most celebrated models that describes the patterns mentioned in the previous section is the Cahn-Hilliard equation. It is suggested by Cahn and Hilliard (see [C, CH]). For a general reference on differential equations see [KPDE, KFA]. The Cahn-Hilliard equation is defined as:

$$u_t = (-\epsilon^2 u_{xx} + W'(u))_{xx}, \quad x \in (0,1), \quad t > 0, \tag{1.1}$$

where $0 < \epsilon << 1$ is the interaction length and $W(u) = 1/4(u^2-1)^2$ the potential.
The unknown function u is the rescaled concentration of one of the components of the binary alloy and takes values between -1 and 1, the potential with the boundary conditions:

$$u_x = u_{xxx} = 0, \quad x \in \{0,1\}, \quad t > 0. \tag{1.2}$$

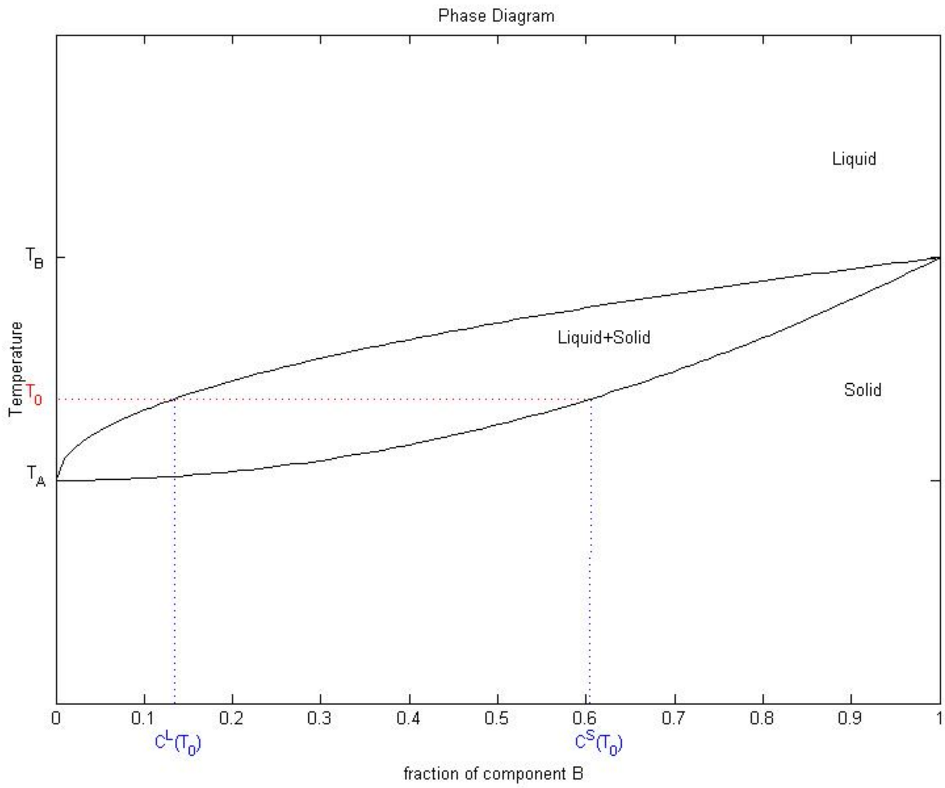

Figure 1.1: A typical phase diagram of two components A and B

What is the reason for (1.2) boundary conditions? While u is the rescaled concentration of one of the components and the fraction of it is fixed, there must exist a constant $-1 < M < 1$ such that $\int_0^1 u dx = M$ for all $t > 0$, or equivalently

$$\begin{aligned}
(\int_0^1 u dx)_t &= 0 \\
\int_0^1 u_t dx &= 0 \\
\int_0^1 (-\epsilon^2 u_{xx} + W'(u))_{xx} dx &= 0 \\
\int_0^1 (-\epsilon^2 u_{xxx} + W''(u) u_x)_x dx &= 0 \\
\left[-\epsilon^2 u_{xxx} + W''(u) u_x\right]_0^1 &= 0
\end{aligned}$$

which is satisfied because of (1.2) boundary conditions.

Free Energy Functional

Before we close this section, we need to mention the free energy functional:

$$J(u) = \int_0^1 ((\epsilon^2/2) u_x^2 + W(u)) dx$$

It describes the free energy of the metastable patterns. Figure 1.2 shows that the solution likes to be valued near 1 and -1 in order to has less energy. The Cahn-Hilliard equation is derived from the free energy functional. Finally a lot of results regarding the solutions of the Cahn-Hilliard equation have been generated by it:

- Every solution u of the Cahn-hilliard equation approaches an equilibrium state as $t \to \infty$ (see [EZ]).
- The limit of a Cahn-Hilliard solution u as $t \to \infty$ is a local minimizer of J in the world of fixed mass functions (see [H]).
- Let u be a Cahn-Hilliard solution. The $J(u)$, as t function, is monotonically decreasing (see [AA, CGS]).

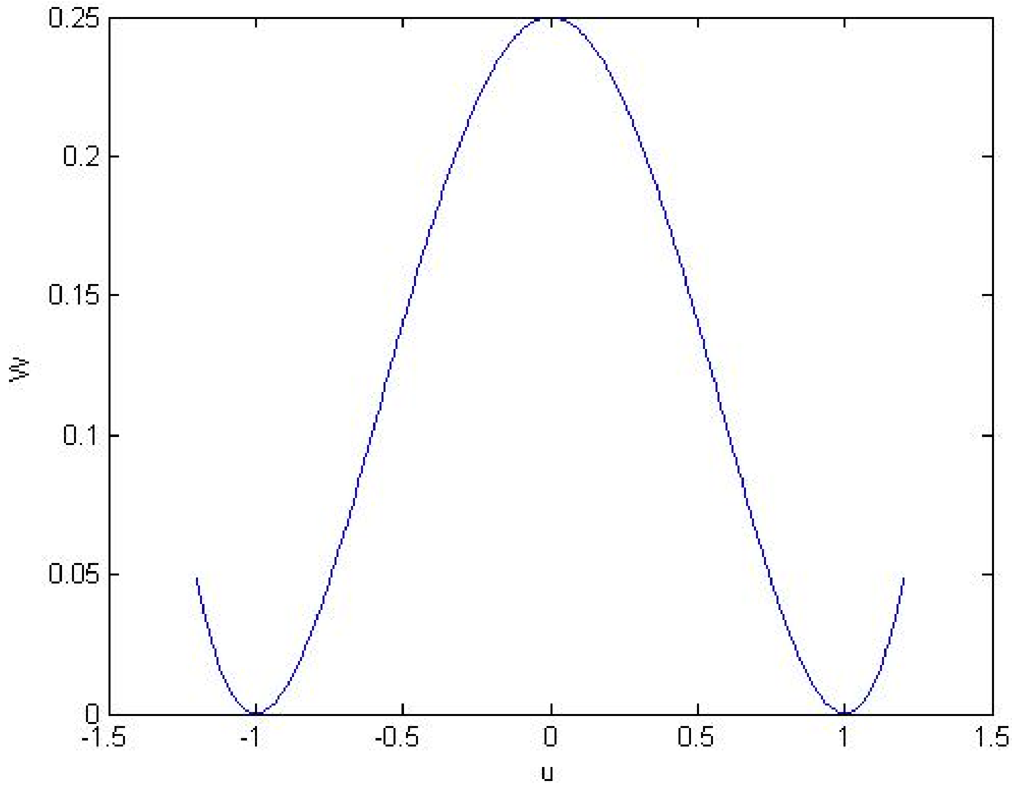

Figure 1.2: The graph of $W(u)$

1.3 Structure and Chapters of the Thesis

Before we start reading the main text let us have an overview of it.

In Chapter 2 some preliminaries are given, in order to provide us with some tools needed for the investigation of the equation. The basic tool of this thesis, which is the Invariant Manifold, is defined in this chapter. Nevertheless, firstly we define the first approximation manifold as a first step to the final construction of the Invariant Manifold. Invariant Manifold is like phase space of the solutions of the problem. Finally, the integrated Cahn-Hilliard equation is defined. (In the rest of the thesis we work with the integrated form because...) In this chapter we do not have important results about our problem but we build the base for the next two chapters.

In Chapter 3 Theorem A and Theorem B are proved. Theorem A gives estimations for the eigenvalues of the integrated Cahn-Hilliard operator. The estimations of Theorem A are used for the proof of Theorem B, which provides a good description of the dynamics of the solutions of the equation. Theorem B is based on the idea of Invariant Manifold. We will see that the Invariant Manifold is not an invariant manifold, with the strict meaning of the term but it is almost invariant.

And we finish with Chapter 4. In Chapter 3 the dynamics of the equation have already been investigated. Nevertheless, we have no analytical approximations of the solutions. In Chapter 4 we derive an analytical approximation of the ODEs system of the problem. This system sheds more light on the behaviour of the solutions. The thesis is closed with a reference to the dynamics of the stochastic Cahn-Hilliard equation.

Chapter 2

Preliminaries

2.1 First Approximation Manifold

The basic idea used during this thesis is Invariant Manifold. Invariant Manifold comes from works related with the Reaction-Diffusion (or Allen Cahn) equation. It is used by G.Fusco, J.Hale [FH] and J.Carr, R.L.Pego [CP] for results regarding Reaction Diffusion equation. P.W. Bates, J. Xun uses Invariant Manifold in their work [BX1, BX2] regarding Cahn-Hilliard equation. Invariant Manifold in both, Reaction-Diffusion and Cahn-Hilliard equation, is an approximation of the phase space of solutions.

In order to reach the key idea of Invariant Manifold, we have to answer the question: what is Reaction-Diffusion equation;

Definition 2.1.1 *The Reaction-Diffusion Differential Equation is*

$$u_t = \epsilon^2 u_{xx} - W'(u), \quad x \in (0,1), \quad t > 0,$$

where ϵ *as in (1.1) .*
With the boundary condition

$$u_x = 0, \quad x \in \{0,1\}, \quad t > 0.$$

Figure 2.1 helps us understanding the Reaction-Diffusion Equation.

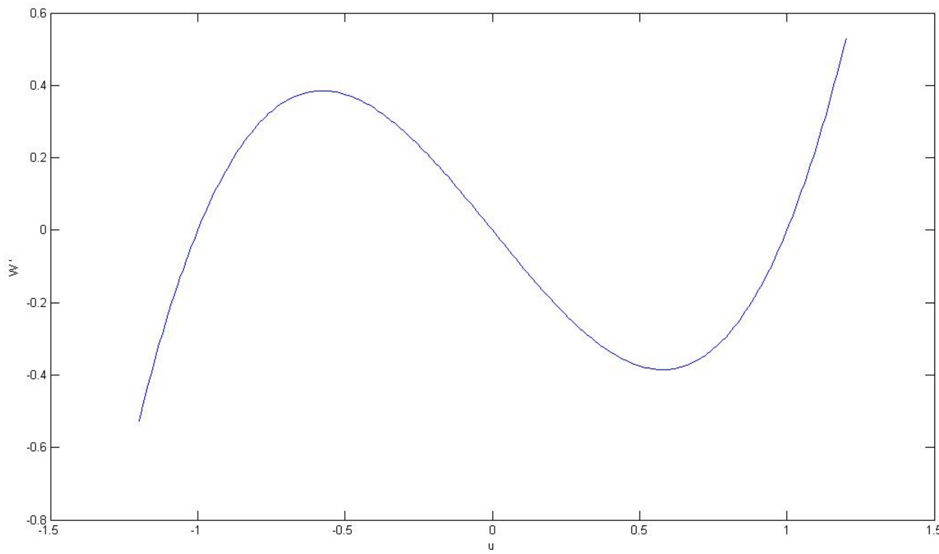

Figure 2.1: The graph of $W'(u)$

The elements of the Invariant Manifold, which are approximately components of the phase space of the solutions of the Cahn-Hilliard equation, are constructed in terms of solutions of the Reaction-Diffusion stationary problem. Obviously Reaction-Diffusion stationary problem is a specific case of the Cahn-Hilliard stationary problem. This is the main reason for the compatibility of the Invariant Manifold in both equations.

Definition 2.1.2 *The Reaction-Diffusion stationary problem is*

$$\begin{aligned} \epsilon^2 \phi_{xx} - W'(\phi) &= 0 \quad x \in (-l/2 - \epsilon, l/2 + \epsilon), \\ \phi &= 0 \quad x \in \{-l/2, l/2\}, \end{aligned}$$

where $l > 0$*.*

This ODE is autonomous which means that we don't need to be concerned about the domain of the solution ϕ. Nevertheless we define the domain as above for technical and practical convenience.

And now we just need evidence of the existence of the solutions of the stationary problem. It is given by the next theorem.

Theorem 2.1.3 *There exists $\rho_0 > 0$, such that if $l > \epsilon/\rho_0$, there exist*

- *a unique solution $\phi(x, l, +)$ satisfying Definition 2.1.2 and $\phi(x, l, +) > 0$ for $|x| < l/2$, $W''(\phi(0, l, +)) > 0$,*
- *a unique solution $\phi(x, l, -)$ satisfying Definition 2.1.2 and $\phi(x, l, -) < 0$ for $|x| < l/2$, $W''(\phi(0, l, -)) > 0$.*

Proof

$$\epsilon^2\phi_{xx} - W'(\phi) = 0$$

or

$$(W(\phi) - \frac{1}{2}\epsilon^2\phi_x^2)_x = 0$$

or

$$W(\phi) - \frac{1}{2}\epsilon^2\phi_x^2 = \alpha,$$

where α is a constant, or

$$\epsilon^2\phi_x^2 = 2(W(\phi) - \alpha)$$

or

$$\epsilon\phi_x = 2^{1/2}(W(\phi) - \alpha)^{1/2}$$

or

$$\frac{1}{\epsilon\phi_x} = \frac{1}{2^{1/2}(W(\phi) - \alpha)^{1/2}}.$$

Integrating from $x = -l/2$ to $x = l/2$ with respect to ϕ we obtain

$$\int_{x=0}^{x=l/2} \frac{1}{\epsilon\phi_x} d\phi = \int_{x=0}^{x=l/2} \frac{1}{2^{1/2}(W(\phi) - \alpha)^{1/2}} d\phi$$

or

$$\frac{l}{2\epsilon\phi_x} = \int_{\phi(0)}^{0} \frac{1}{2^{1/2}(W(\phi) - \alpha)^{1/2}} d\phi$$

where $\alpha = W(\phi(0))$
The solution exists if and only if there exists a value for $\phi(0)$ where $W''(\phi(0)) > 0$ such that the last equation is satisfied.
According to the results of J. Carr, M. Gurtin and M. Slemrod [CGS] the last integral tent to infinity as $\phi(0)$ approaches -1 from above, and it is a monotone function of $\phi(0)$ for $\phi(0)$ close to -1. Hence for ρ_0 sufficiently small, there exists a unique value for $\phi(0)$ such that the last equation is satisfied.
$\blacksquare$

In Figure 2.2 we can see an example of a solution provided by Theorem 2.1.3.

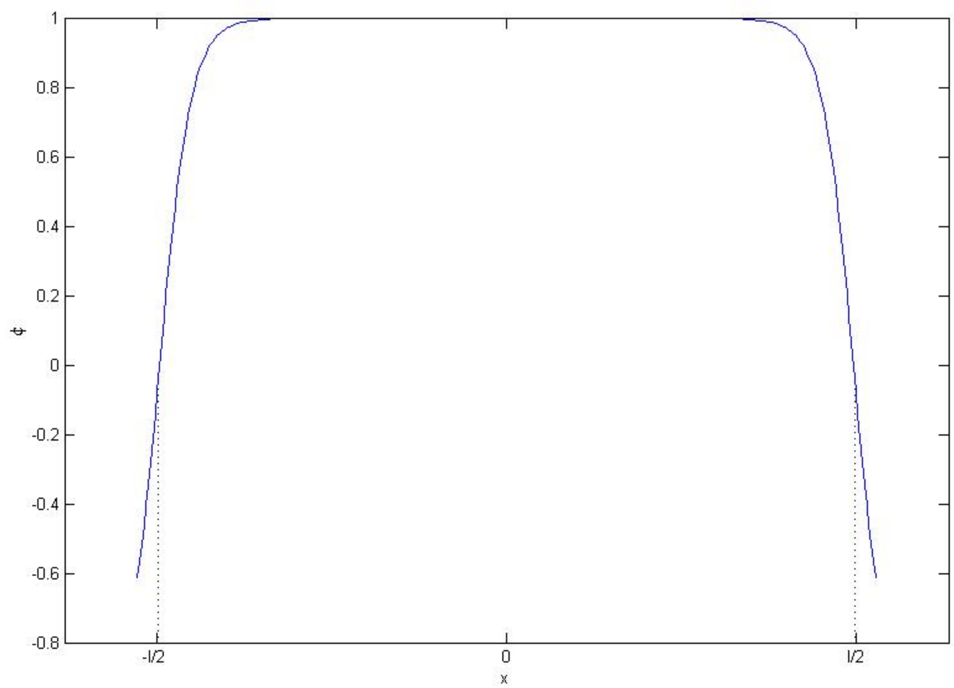

Figure 2.2: Simulation of a solution of 2.1.2. According to Theorem 2.1.3 this is the unique positive solution $\phi(x, l, +)$ with parameters $\epsilon = 0.01$ and $l = 0.316$.

The elements of the Invariant Manifold are constructed by piecing together solutions of the reaction diffusion stationary problem provided by Theorem 2.1.3. The connecting points between different solutions are called transition layers of the manifold. The term comes from the natural problem. As we mentioned in the Introduction, the problem we are studying has two phases. The points that are called transition layers are the points at which the phase changes. The whole number of these points, as we will see later, is the dimension of the manifold. The way that the solutions of the stationary reaction diffusion problem are pieced together is as smooth as the solutions of the problem that are pieced together. For an overview of this process we can take a look at Figures 2.3, 2.4, 2.5 and 2.6 or above, where it is strictly defined.

Let us start from the transition layer. Strictly the transition layers of a N+1 dimensional manifold are valued in $\Omega_{\rho_0} = \{h \in R^{N+1} \,|\, 0 < h_1 < \cdots < h_{N+1},\ \epsilon/\rho_0 < (h_j - h_{j-1})\}$, where ρ_0 as in Theorem 2.1.3, $h_0 := -h_1$ and $h_{N+2} := 2 - h_{N+1}$

Before we define the Invariant Manifold we define something more general, the first approximation Manifold. The first approximation manifold $\mathcal{M}_1$ is defined by the following rule:

Let $\chi : R \to [0,1]$ be C^∞ (in order to have a manifold as smooth as the solutions of the stationary reaction diffusion problem ϕ that are pieced together) with $\chi(x) = 0$ for $\chi \leq -1$ and $\chi(x) = 1$ for $\chi \geq 1$.
Let $m_j = (h_{j-1} + h_j)/2$ for $j = 1, 2, ..., N+2$.
For given $h \in \Omega_{\rho_0}$, we define

$$\begin{aligned} u^h(x) &= (1 - \chi(\frac{x - h_j}{\epsilon}))\phi(x - m_j, h_j - h_{j-1}, (-1)^j) \\ &\quad + \chi(\frac{x - h_j}{\epsilon})\phi(x - m_{j+1}, h_{j+1} - h_j, (-1)^{j+1}), \end{aligned}$$

where $x \in [m_j, m_{j+1}]$, $j = 1, 2, ..., N+1$.

$$\mathcal{M}_1 := \{u^h \,|\, h \in \Omega_{\rho_0}\}.$$

In Figure 2.7 we can see a simulation of a component of the first approximation manifold.

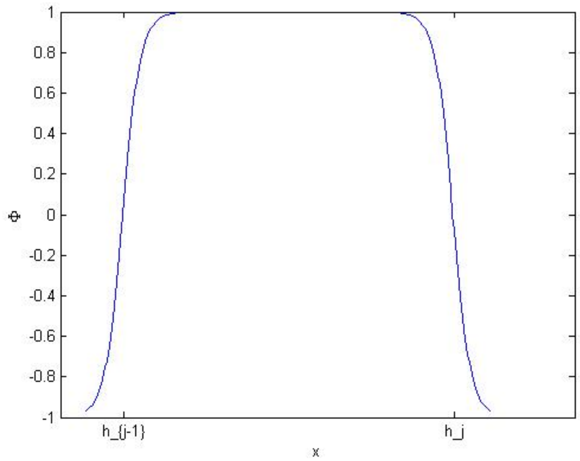

Figure 2.3: Construction of the first approximation manifold, step 1: $\phi(x - m_j, h_j - h_{j-1}, (-1)^j)$.

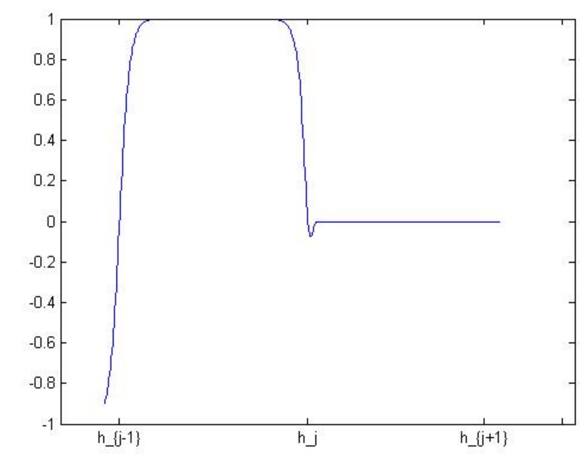

Figure 2.4: Construction of the first approximation manifold, step 2: $[1 - \chi(\frac{x-h_j}{\epsilon})]\phi(x - m_j, h_j - h_{j-1}, (-1)^j)$. It is smooth.

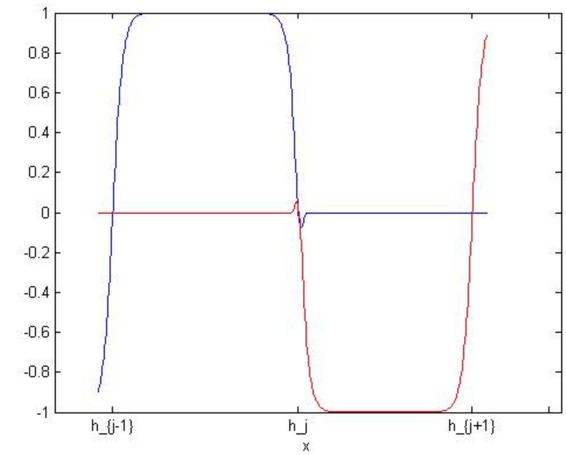

Figure 2.5: Construction of the first approximation manifold, step 3: $[1 - \chi(\frac{x-h_j}{\epsilon})]\phi(x - m_j, h_j - h_{j-1}, (-1)^j)$ like in Figure 2.4 with blue and $\chi(\frac{x-h_j}{\epsilon})\phi(x - m_{j+1}, h_{j+1} - h_j, (-1)^{j+1})$ with red. They are smooth.

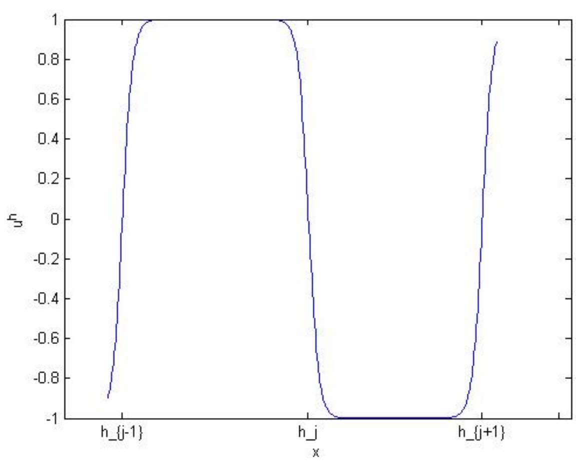

Figure 2.6: Construction of the first approximation manifold, step 4: The sum of the two functions of Figure 2.5, $[1 - \chi(\frac{x-h_j}{\epsilon})]\phi(x - m_j, h_j - h_{j-1}, (-1)^j) + \chi(\frac{x-h_j}{\epsilon})\phi(x - m_{j+1}, h_{j+1} - h_j, (-1)^{j+1})$. This is the way that solutions of reaction diffusion problem are pieced together for constructing the first approximate manifold. This way of piecing conserves smoothness.

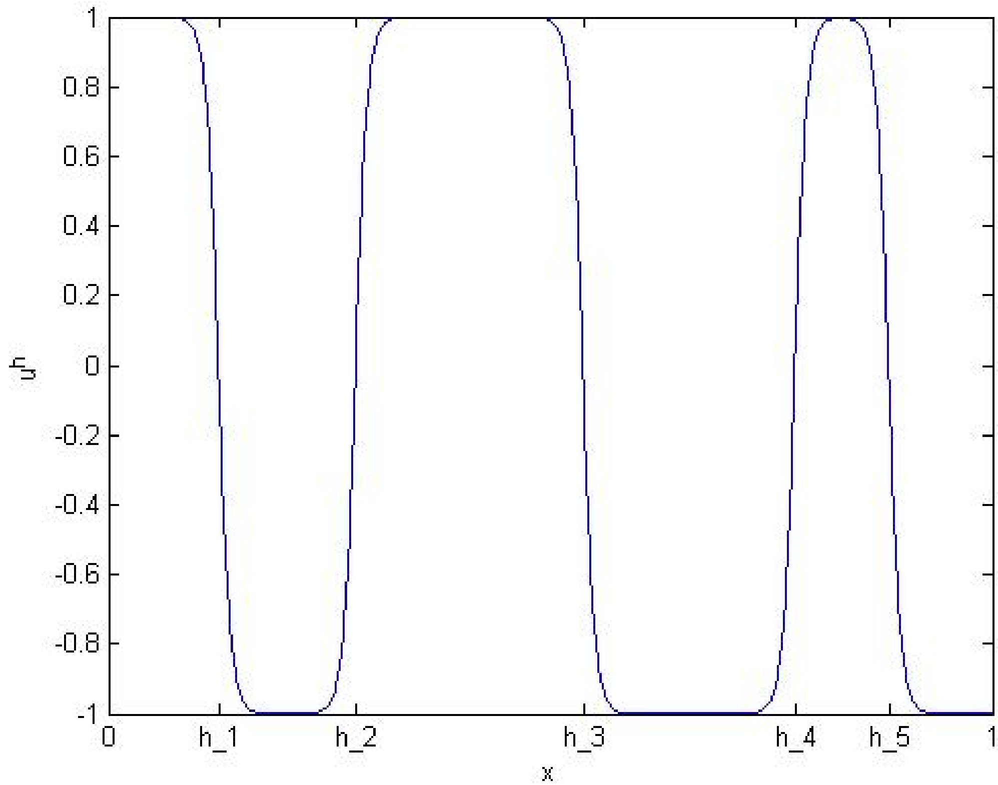

Figure 2.7: Simulation of a component of the first approximation manifold with $\epsilon = 0.01$, N=4 and $h = (0.1250, 0.2800, 0.5370, 0.7770, 0.8830)$.

The first approximation manifold is a good step. Nevertheless, the boundary conditions (1.2) have not been used for the construction of it. Unfortunately before we involve the boundary conditions, we have to face some propositions and some notation. After that section 2.3 is dedicated to the Invariant Manifold, which involves the boundary conditions.

2.2 Notation and Useful Propositions

Basic notation

- For any function $\phi(x)$, $\tilde{\phi}(x) := \int_0^x \phi(y)dy$
- $\mathcal{L}^c(\tilde{\phi}) := -\epsilon^2\tilde{\phi}_{xxxx} + (W'(\tilde{\phi}_x))_x$ the integrated Cahn-Hilliard differential operator
- $\mathcal{L}^b(\phi) := \epsilon^2\phi_{xx} - W'(\phi)$, the bistable reaction diffusion differential operator
- $L^c(\tilde{\phi}) := -\epsilon^2\tilde{\phi}_{xxxx} + (W''(u^h)\tilde{\phi}_x)_x$, this is $\mathcal{L}^c$ linearized at $\tilde{u^h}$
- $u^h_j := \partial u^h/\partial h_j$, $j = 1, 2, ..., N+1$
- $\alpha^j = W(u^h(m_j))$, $\beta^j = 1 - |u^h(m_j)|$,
  $l_j := h_j - h_{j-1}$, $r_j := \epsilon/l_j$
- $\alpha(r) := \max \alpha^j(r)$, $\beta(r) := \max \beta^j(r)$,
  $l := \min l_j$, $r = \max r_j = \epsilon/l$
- $I_j := [m_j, m_{j+1}]$
- $\chi^j(x) := \chi(\frac{x-h_j}{\epsilon})$
- $\phi^j(x) := \phi(x - m_j, l_j, (-1)^j)$

The thesis is written by using the notation above. Below some important prepositions.

Propositions

Proposition 2.2.1 *Let $\Phi(x)$ be the unique solution of*

$$\epsilon^2\Phi_{xx} - W'(\Phi) = 0$$
$$\Phi(0) = 0$$

with

$$\Phi(x) \to \pm 1 \text{ as } x \to \pm\infty$$

and K_1, δ_1, δ_2 *be positive with* $K_1 > 1$ *and such that* $\min \mid \Phi(x) \pm 1 \mid < \delta_1/2$ *for every* $\pm x > \epsilon K_1$. *Then there is a* $\rho_1 = \rho_1(K_1, \delta_1, \delta_2) > 0$ *such that if* $\rho \le \rho_1$ *and* $h \in \Omega_\rho$ *we have*

$$\mid u^h(x) - \Phi((x - h_j)(-1)^{j-1}) \mid < \delta_2 \;\; x \in [h_j - \epsilon K_1, h_j + \epsilon K_1]$$

and

$$\mid u^h(x) - (-1)^j \mid < \delta_2 \;\; x \in [h_{j-1} + \epsilon K_1, h_j - \epsilon K_1] \cap [0, 1]$$

for $j = 1, 2, ..., N + 1$.

Proof See Proposition 2.2 of [BX1]. ∎

Proposition 2.2.2 *For* $x \in [-l/2, l/2]$

$$2\phi_l(x, l, \pm 1) = -sgn(x)\phi_x(x, l, \pm 1) + 2w(x, l, \pm 1)$$

where, for $x \neq 0$

$$w(x, l, \pm 1) = \epsilon^{-1} l^{-2} \alpha'_\pm(r) \phi(|x|, l, \pm 1) \int_{l/2}^{|x|} \phi_x(s, l, \pm 1)^{-2} ds$$

and

$$w(0, l, \pm 1) = \frac{-\epsilon^{-1} l^{-2} \alpha'_\pm(r)}{(0, l, \pm 1)}.$$

Proof See Proposition 2.8 of [BX1]. ∎

Proposition 2.2.3 *The interval* $[h_{j-1} - \epsilon, h_{j+1} + \epsilon]$ *contains the support of* u_j^h *and*

$$u_j^h(x) = \begin{cases} \chi^{j-1} w^j, & x \in I_{j-1} \\ (1-\chi^j)(-\phi_x^j + w^j) + \chi^j(-\phi_x^{j+1} - w^{j+1}) & \\ +\chi_x^j(\phi^j - \phi^{j+1}), & x \in I_j \\ -(1-\chi^{j+1})w^{j+1}, & x \in I_{j+1} \end{cases}$$

where w_j *as in Proposition 2.2.2 when we replace* $\phi(x, l, \pm)$ *by* $\phi^j(x, l, \pm)$.

Proof See Proposition 2.11 of [BX1]. ∎

Proposition 2.2.4 *There exists* r_0 *such that for* $0 < r < r_0$,
$|w(x, l, \pm 1)| \leq C\epsilon^{-1}\beta_{\pm}(r)$ *for* $x \in [-l/2 - \epsilon, l/2 + \epsilon]$
$|w(x, l, \pm 1)| \leq C\epsilon^{-1}\alpha_{\pm}(r)$ *for* $x \in [-l/2 - \epsilon, l/2 - \epsilon]$ *and* $x \in [-l/2 + \epsilon, l/2 + \epsilon]$.

Proof See Proposition 2.9 of [BX1]. ∎

Proposition 2.2.5 *For* $x \in [-l/2 - \epsilon, l/2 + \epsilon]$

$$\begin{aligned} |w_x(x, l, \pm 1)| &\leq C\epsilon^{-2} r^{-1} \beta_{\pm}(r) \\ |w_{xx}(x, l, \pm 1)| &\leq C\epsilon^{-3} \beta_{\pm}(r). \end{aligned}$$

Proof See Proposition 2.10 of [BX1]. ∎

Proposition 2.2.6 *There is a* $\rho_0 > 0$ *such that if* $0 < \rho < \rho_0$ *and* $u^h \in \mathcal{M}$ *with* $h \in \Omega_\rho$, *then we have*

$$\begin{aligned} |\ \mathcal{L}^c \tilde{u}^h(x)\ | &\leq C\alpha(r)\epsilon^{-1}, \ \ x \in [0, 1] \\ \|\ \mathcal{L}^c \tilde{u}^h\ \| &\leq C\alpha(r)\epsilon^{-1} \end{aligned}$$

for some constant C *independent of* r *and* u^h

Proof See Proposition 2.2.6. ∎

Proposition 2.2.7 *There is a $\rho_0 > 0$ such that if $0 < \rho < \rho_0$ and $u^h \in \mathcal{M}$ with $h \in \Omega_\rho$, then we have*

$$
\begin{aligned}
\left| \frac{\partial}{\partial h_j} \mathcal{L}^b u^h(x) \right| &\leq C\beta(r)\epsilon^{-2}, \ \ x \in [0,1] \\
\left| \frac{\partial}{\partial h_j} \frac{\partial}{\partial x} \mathcal{L}^b u^h \right| &\leq C\beta(r)\epsilon^{-4}, \ \ x \in [0,1]
\end{aligned}
$$

for some constant C independent of r and u^h

Proof See Proposition 2.2.7 ∎

Proposition 2.2.8 *There exists $r_0 > 0$ such that, for $0 < r < r_0$,*

$$
|w(x,l,\pm 1)| \leq \begin{cases} C\epsilon^{-1}\beta(r), & x \in [-\frac{1}{2}l - \epsilon, \frac{1}{2}l + \epsilon], \\ C\epsilon^{-1}\alpha(r), & x \in [-\frac{1}{2}l - \epsilon, -\frac{1}{2}l + \epsilon] \cup [\frac{1}{2}l - \epsilon, \frac{1}{2}l + \epsilon], \end{cases}
$$

Where w as in Proposition 2.2.2.

Proof See Lemma 7.9 of [CP]. ∎

Proposition 2.2.9 *There is a $\rho_0 > 0$ such that if $0 < \rho < \rho_0$ and $u^h \in \mathcal{M}$ with $h \in \Omega_\rho$, then we have*

$$
|\mathcal{L}^b(u^\xi)| \leq C|\alpha^{i+1} - \alpha^i|
$$

for $x \in I_{i+1}$

Proof See Theorem 3.5 of [CP]. ∎

2.3 Invariant Manifold

Definition and explanation of the Invariant Manifold

First approximation manifold is a good step. Nevertheless we need to consider the boundary conditions. According to (1.2) we have $\int_0^1 u^h(x) = M$

where $-1 < M < 1$. Consequently we define the Invariant Manifold (or the second approximate manifold according to the [BX1, BX2]).

$$\mathcal{M} := \{u^h \in \mathcal{M}_1 \,|\, \int_0^1 u^h(x) = M\},$$

As opposed to the first approximate manifold, it decreases from $N+1$ to N-dimensional and can be parametrized it by $(h_1, h_2, ..., h_N)$. In other words we have the following expression

$$h_{N+1} := h_{N+1}(h_1, h_2, ..., h_N)$$

This is quite obvious to understand. If we order the first N layers the last layer should go to a specific place in order to be satisfied the conservation of mass. This claim is proved using the Implicit function theorem. More strictly and detailed we have the following proposition.

Proposition 2.3.1 *Let* $M(h) = \int_0^1 u^h(x)dx$ *for* $h \in \Omega_\rho$*, then* $M(h)$ *is a smooth function of* h *and* $\partial M/\partial h_j = 2(-1)^{j+1} + O(\epsilon^{-1}\beta(r))$.

Proof By Proposition 2.2.3, $[h_{j-1} - \epsilon, h_{j+1} + \epsilon]$ contains the support and in I_j we have

$$\begin{aligned}
u_j^h &= (1-\chi^j)(-\phi_x^j + w^j) + \chi^j(-\phi_x^{j+1} - w^{j+1}) + \chi_x^j(\phi^j - \phi^{j+1}) \\
&= -[(1-\chi^j)\phi_x^j + \chi^j\phi_x^{j+1} + \chi_x^j(\phi^{j+1} - \phi^j)] + (1-\chi^j)w^j - \chi^j w^{j+1} \\
&= -u_x^h + (1-\chi^j)w^j - \chi^j w^{j+1}
\end{aligned}$$

By Propositions 2.2.4 and 2.2.3, there exists a $r_0 > 0$ such that if $0 < r < r_0$, then

$$\begin{aligned}
|\int_{I_{j+1}\cup I_{j-1}} u_j^h(x)dx| &\leq C\epsilon^{-1}\beta(r) \\
|\int_{I_j} [(1-\chi^j)w^j - \chi^j w^{j+1}]dx| &\leq C\epsilon^{-1}\beta(r)
\end{aligned}$$

for some constant C.
Thus

$$\begin{aligned}
\frac{\partial}{\partial h_j} M(h) &= \int_0^1 u_j^h dx \\
&= \int_{I_j} u_j^h dx + \int_{I_{j-1} \cup I j+1} u_j^h dx \\
&= \int_{I_j} [-u_x^h + (1-\chi^j) w^j - \chi^j w^{j+1}] dx + \int_{I_{j-1} \cup I j+1} u_j^h dx \\
&= \int_{I_j} -u_x^h dx + \int_{I_j} [(1-\chi^j) w^j - \chi^j w^{j+1}] dx + \int_{I_{j-1} \cup I j+1} u_j^h dx \\
&= \int_{I_j} -u_x^h dx + O(\epsilon^{-1} \beta(r)) \\
&= u^h(m_j) - u^h(m_{j+1}) + O(\epsilon^{-1} \beta(r)) \\
&= (-1)^j (\beta(r_j) - 1) - (-1)^{j+1} (\beta(r_{j+1}) - 1) + O(\epsilon^{-1} \beta(r)) \\
&= 2(-1)^{j+1} + O(\beta(r)) + O(\epsilon^{-1} \beta(r)) \\
&= 2(-1)^{j+1} + O(\epsilon^{-1} \beta(r))
\end{aligned}$$

$\blacksquare$

By Proposition 2.3.1 and the implicit function Theorem, if $u^h \in \mathcal{M}$ we can think of h_{N+1} as a function of $h_1, h_2, ..., h_N$. Furthermore,

$$\frac{\partial}{\partial h_j} M(h) + \frac{\partial}{\partial h_{N+1}} M(h) \frac{\partial h_{N+1}}{\partial h_j} = 0,$$

or

$$\frac{\partial h_{N+1}}{\partial h_j} = \frac{-\frac{\partial}{\partial h_j} M(h)}{\frac{\partial}{\partial h_{N+1}} M(h)},$$

or

$$\frac{\partial h_{N+1}}{\partial h_j} = \frac{-2(-1)^{j+1} + O(\epsilon^{-1} \beta(r))}{2(-1)^{N+2} + O(\epsilon^{-1} \beta(r))}$$

or

$$\frac{\partial h_{N+1}}{\partial h_j} = (-1)^{j-N} + O(\epsilon^{-1} \beta(r)) \tag{2.1}$$

While for the Invariant Manifold $\mathcal{M}$, we are interested in, h_{N+1} is dependent on $h_1, h_2, ..., h_N$ we denote u^h by u^ξ, where $\xi = (\xi_1, \xi_2, ..., \xi_N) := (h_1, h_2, ..., h_N)$.

Consequently we denote

$$u_j^\xi := \frac{\partial u^\xi}{\partial \xi_j} = \frac{\partial u^h}{\partial h_j} + \frac{\partial u^h}{\partial h_{N+1}} \frac{\partial h_{N+1}}{\partial h_j}$$

while u_j^h still means $\partial u^h / \partial h_j$.

Integrated Cahn-Hilliard Equation

We are going to slightly change the way we study this problem. What we change is the Cahn-Hilliard equation. We convert our equation to an integrated form and consequently we mutually convert the elements of the Invariant Manifold. Integrated Cahn Hilliard equation, which is equivalent to Cahn-Hilliard is defined as follows:

$$\begin{aligned} \tilde{u}_t = -\epsilon^2 \tilde{u}_{xxxx} + (W'(\tilde{u}_x))_x, \quad x \in (0,1), \quad t > 0, \\ \tilde{u}(0,t) = 0, \quad \tilde{u}(1,t) = M, \quad t > 0, \\ \tilde{u}_{xx} = 0, \quad x \in \{0,1\}, \quad t > 0, \end{aligned} \tag{2.2}$$

where $\tilde{u}(x,t) = \int_0^x u(y,t)dy$.

What we have to do is to study the dynamics of (2.2) in a neighbourhood of $\mathcal{M}$. For this we adopt the following coordinate system

$$\tilde{u} \longrightarrow (\xi, \tilde{v}) \textit{ meaning } \tilde{u} = \tilde{u}^\xi + \tilde{v}$$

where $\langle \tilde{v}, E_j^\xi \rangle = 0$ for $j = 1, 2, ..., N$

and $\tilde{v} = \tilde{v}_{xx} = 0$ at $x = 0, 1$, where E_j^ξ is defined by:

$$\begin{aligned} E_j^\xi(x) &:= \bar{w}_j(x) - Q_j(x) \\ \bar{w}_j(x) &:= \tilde{u}_j^h(x) + \tilde{u}_{j+1}^h(x),\ j = 1, 2, ..., N, \end{aligned}$$

and

$$Q_j(x) := (-\frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{1}{3}x)\bar{w}_{jxx}(0) + \frac{1}{6}(x^3 - x)\bar{w}_{jxx}(1) + x\bar{w}_j(1)$$

substituting $\tilde{u} = \tilde{u}^\xi + \tilde{v}$ to the ICH, we get

$$\sum_{j=1}^N \tilde{u}_j^\xi \dot{\xi}_j + \tilde{v}_t = -\epsilon^2(\tilde{u}_{xxxx}^\xi + \tilde{v}_{xxxx}) + (W'(\tilde{u}_x^\xi + \tilde{v}_x))_x \tag{2.3}$$

Differentiating $\langle \tilde{v}_i, E_i^\xi \rangle = 0$ with respect to t, we get

$$\begin{aligned} \langle \tilde{v}, E_i^\xi \rangle_t &= 0 \\ \langle \tilde{v}_t, E_i^\xi \rangle + \langle \tilde{v}, (E_i^\xi)_t \rangle &= 0 \\ \langle \tilde{v}_t, E_i^\xi \rangle + \langle \tilde{v}, \sum_{j=1}^N E_{ij}^\xi \dot{\xi}_j \rangle &= 0 \end{aligned}$$

or, $\langle \tilde{v}_t, E_i^\xi \rangle = -\langle \tilde{v}, \sum_{j=1}^N E_{ij}^\xi \dot{\xi}_j \rangle$ for $i = 1, 2, ..., N$.

Denote $a_{ij} = \langle \tilde{u}_j^\xi, E_i^\xi \rangle - \langle \tilde{v}, E_{ij}^\xi \rangle$ for $i, j = 1, 2, ..., N$ where $E_{ij}^\xi = (\partial/\partial \xi_j) E_i^\xi$.

taking the inner product for $i = 1, 2, ..., N$ we have

$$\begin{aligned}
\langle \sum_{j=1}^{N} \tilde{u}_j^{\xi} \dot{\xi}_j + \tilde{v}_t, E_i^{\xi} \rangle &= \langle -\epsilon^2 (\tilde{u}_{xxxx}^{\xi} + \tilde{v}_{xxxx}) + (W'(\tilde{u}_x^{\xi} + \tilde{v}_x))_x, E_i^{\xi} \rangle \\
\langle \sum_{j=1}^{N} \tilde{u}_j^{\xi} \dot{\xi}_j, E_i^{\xi} \rangle + \langle \tilde{v}_t, E_i^{\xi} \rangle &= \langle -\epsilon^2 (\tilde{u}_{xxxx}^{\xi} + \tilde{v}_{xxxx}) + (W'(\tilde{u}_x^{\xi} + \tilde{v}_x))_x, E_i^{\xi} \rangle \\
\langle \sum_{j=1}^{N} \tilde{u}_j^{\xi} \dot{\xi}_j, E_i^{\xi} \rangle - \langle \tilde{v}, \sum_{j=1}^{N} E_{ij}^{\xi} \dot{\xi}_j \rangle &= \langle -\epsilon^2 (\tilde{u}_{xxxx}^{\xi} + \tilde{v}_{xxxx}) + (W'(\tilde{u}_x^{\xi} + \tilde{v}_x))_x, E_i^{\xi} \rangle \\
\sum_{j=1}^{N} \langle \tilde{u}_j^{\xi}, E_i^{\xi} \rangle \dot{\xi}_j - \sum_{j=1}^{N} \langle \tilde{v}, E_{ij}^{\xi} \rangle \dot{\xi}_j &= \langle -\epsilon^2 (\tilde{u}_{xxxx}^{\xi} + \tilde{v}_{xxxx}) + (W'(\tilde{u}_x^{\xi} + \tilde{v}_x))_x, E_i^{\xi} \rangle
\end{aligned}$$

So

$$\sum_{j=1}^{N} \alpha_{ij} \dot{\xi}_j = \langle -\epsilon^2 (\tilde{u}_{xxxx}^{\xi} + \tilde{v}_{xxxx}) + (W'(\tilde{u}_x^{\xi} + \tilde{v}_x))_x, E_i^{\xi} \rangle \tag{2.4}$$

This gives the motion of coordinate point on $\mathcal{M}$

Denoting

$$f_2 = \int_0^1 (1-\tau) W'''(\tilde{u}_x^{\xi} + \tau \tilde{v}_x) d\tau$$

we can get the following equation

$$\sum_{j=1}^{N} \tilde{u}_j^{\xi} \dot{\xi}_j + \tilde{v}_t = \mathcal{L}^c \tilde{u}^{\xi} + L^c \tilde{v} + (f_2 \tilde{v}_x^2)_x. \tag{2.5}$$

Chapter 3

Dynamics

As mentioned at the start of the previous chapter, the basic tool used in the study is Invariant Manifold. It is also known by [CP] that it approximates the phase space of the solutions of Reaction-Diffusion (or Allen Cahn) equation. So we need to prove that Invariant Manifold also approximates the phase space of the solutions of the Cahn-Hilliard equation. This result is given by the following Theorem.

Theorem 3.0.2 *(Theorem B of [BX1]) Let $\tilde{u}^{\xi(t)} + \tilde{v}$ be an orbit of the integrated Cahn-Hilliard equation starting outside the slow channel. If the $H^2 - norm$ of $\tilde{v}(0)$ is $o(\epsilon^7)$, then the $H^2 - norm$ of $\tilde{v}$ will decrease exponentially until $\tilde{u}(t)$ enters the slow channel. Then it will remain in the channel while $h \in \Omega_\rho$ and will follow the approximate manifold with speed $O(e^{-c/r})$. Here c is less than $W''(\pm 1)$ but close to it and $r = \epsilon/l$, therefore $\tilde{u}$ will stay in the slow channel for an exponentially long time.*

Where the slow channel is defined by the following Definition

Definition 3.0.3 *Let $A_\epsilon(\tilde{v}) = \int_0^1 [\epsilon^2 \tilde{v}_{xx} + W''(u^h)\tilde{v}_x^2]dx$. The slow channel is defined to be the set*

$$\Gamma = \{\tilde{u} : \tilde{u} = \tilde{u}^\xi + \tilde{v},\ A_\epsilon(\tilde{v}) < b\epsilon^{-5}\alpha^2(r)\},$$

where b is a positive number.

In order to prove this important theorem we need one more theorem

Theorem 3.0.4 *(Theorem A of [BX1]) There exists a positive number C independent of ϵ and ρ such that if $h \in \Omega_\rho$ with $\rho < \rho_0$, then L^c has exactly N exponentially small eigenvalues $\psi_1, \psi_2, ..., \psi_N$ and all the other eigenvalues $\psi_{N+1}, \psi_{N+2}, ...$ are valued with values less than $-C$.*
For technical convenience, the eigenvalues $\{\psi_i\}_{i=1}^{\infty}$ are defined to be $\psi_1 \geq \psi_2 \geq \cdots$.

3.1 Proof of Theorem A

The proof of Theorem A has two parts. In the first part it is proved that all the eigenvalues except N are valued with values less than $-C$. In the second part it is proved that there are N eigenvalues that are exponentially small.

First part: All the eigenvalues except N are valued with values less than $-C$.

Lemma 3.1.1 *There exists a $\rho_0 > 0$ such that for $0 < \rho < \rho_0$, and if $u^h \in \mathcal{M}$ with $h \in \Omega_\rho$, then $\psi_2', \psi_3', ..., \psi_{N+1}'$ are linearly independent, where $\{\psi_i\}_{i=1}^{N+1}$ are orthonormal eigenfunctions of*

$$\begin{aligned} \epsilon^2 \psi'' - W''(u^h)\psi = \mu(\epsilon)\psi, &\qquad 0 < x < 1 \\ \psi' = 0, &\qquad x = 0, 1 \end{aligned}$$

Proof If we suppose that $\psi_2', \psi_3', ..., \psi_{N+1}'$ are not linearly independent, then we can write them as

$$\sum_{i=2}^{N+1} C_i \psi_i' = 0.$$

where C_i are constants which are not all zero.

Integrating from 0 to x, we get,

$$\sum_{i=2}^{N+1} C_i \psi_i = C,$$

where C is constant.

In order to finish this proof we can follow two ways.

(1) In [BX1] it is proved with this argument:
If we take the inner product with ψ_1 of $\sum_{i=2}^{N+1} C_i \psi_i = C$, we get

$$0 = \int_0^1 \psi_1 dx,$$

so $C = 0$ which is a contradiction because $\psi_2, \psi_3, ..., \psi_{N+1}$ are linearly independent.

(2) Another argument for the same result:
is based on the fact that C is an eigenfunction of our problem, with eigenvalue $\mu = -W''(u^h)$.
The functions of the set $\{\psi_2, \psi_3, ..., \psi_{N+1}, C\}$ are linearly dependent which is a contradiction because it is a subset of $\{\psi_1, \psi_2, ...\}$. ∎

Lemma 3.1.2 *Let $\lambda_i^{\xi}(\epsilon)$, $i > N$ be an eigenvalue of the following problem*

$$\begin{aligned} L^c H &= \lambda(\epsilon) H, \qquad 0 < x < 1 \\ H'' &= H = 0, \qquad x = 0, 1. \end{aligned}$$

Then for $h \in \Omega_\rho$ with ρ sufficiently small

$$\lambda_i^{\xi}(\epsilon) \leq -C < 0,$$

where C is constant independent of ϵ and ξ.

Proof Applying the variational characterization of the eigenvalues for the problem

$$-\lambda_{N+1} = \max_{\mathcal{F}} \min_{\mathcal{H}} \frac{\int_0^1 [\epsilon^2 H_{xx}^2 + W''(u^\xi) H_x^2] dx}{\int_0^1 H^2 dx},$$

where

$$\mathcal{H} = \{H : H(0) = H(1) = 0,\ \langle H, \phi_i \rangle = 0,\ i = 1, 2, ..., N\}$$

and

$$\mathcal{F} = \{\{\phi_i\}_{i=1}^N : \{\phi_i\}_{i=1}^N\, a\, set\, of\, N\, linearly\, independent\, functions\}.$$

The inequality we want to show can be written as

$$-\lambda_i^\xi(\epsilon) \geq C > 0.$$

By the variational characterization, it can be written as

$$\max_{\mathcal{F}} \min_{\mathcal{H}} \frac{\int_0^1 [\epsilon^2 H_{xx}^2 + W''(u^\xi) H_x^2] dx}{\int_0^1 H^2 dx} \geq C > 0.$$

In order to prove what we want, it suffices to find a component $\{\phi_i\}_{i=1}^N$ in $\mathcal{F}$ such that

$$\min_{\mathcal{H}} \frac{\int_0^1 [\epsilon^2 H_{xx}^2 + W''(u^\xi) H_x^2] dx}{\int_0^1 H^2 dx} \geq C > 0.$$

The component with which we attempt to minimize it is defined using Lemma 3.1.1: $\phi_1 = \psi_2'$, $\phi_2 = \psi_3'$, ..., $\phi_N = \psi_{N+1}'$, where ψ_i as in Lemma 3.1.1.

Additionally, instead of minimizing $\frac{\int_0^1 [\epsilon^2 H_{xx}^2 + W''(u^\xi) H_x^2] dx}{\int_0^1 H^2 dx}$ under $\mathcal{H}$ we are going to minimize it under the following set

$$\mathcal{H}' = \{H : H = \int_0^x h(x) dx,\ \|h\| = 1,\ \int_0^1 h(x) dx = 0,$$

$$\langle H, \psi_1 \rangle \geq 0,\ \langle H, \psi_i \rangle = 0,\ i = 2, ..., N + 1\}.$$

So before calculate $\min_{\mathcal{H}'} \frac{\int_0^1 [\epsilon^2 H_{xx}^2 + W''(u^\xi) H_x^2] dx}{\int_0^1 H^2 dx}$ we prove that

$$\min_{\mathcal{H}'} \frac{\int_0^1 [\epsilon^2 H_{xx}^2 + W''(u^\xi) H_x^2] dx}{\int_0^1 H^2 dx} = \min_{\mathcal{H}} \frac{\int_0^1 [\epsilon^2 H_{xx}^2 + W''(u^\xi) H_x^2] dx}{\int_0^1 H^2 dx} \tag{3.1}$$

$\{\frac{\int_0^1 [\epsilon^2 H_{xx}^2 + W''(u^\xi) H_x^2] dx}{\int_0^1 H^2 dx} : H \in \mathcal{H}\} \geq \mathcal{H}'$

First we prove that for every $H \in \mathcal{H}$ there exists a $H' \in \mathcal{H}'$ such that $\frac{\int_0^1 [\epsilon^2 H_{xx}^2 + W''(u^\xi) H_x^2] dx}{\int_0^1 H^2 dx} = \frac{\int_0^1 [\epsilon^2 H_{xx}'^2 + W''(u^\xi) H_x'^2] dx}{\int_0^1 H'^2 dx}$.
Let $H \in \mathcal{H}$, then because H is two times differentiable $h = H_x$ is continuous and consequently $H = \int_0^x h(t) dt$.
$0 = H(1) = \int_0^1 h(x) dx$

$$\begin{aligned} \langle H', \psi_{i+1} \rangle + \langle H, \psi'_{i+1} \rangle &= [H\psi_i]_0^1 \\ \langle h, \psi_{i+1} \rangle + \langle H, \phi_i \rangle &= 0 - 0 \\ \langle h, \psi_{i+1} \rangle + 0 &= 0 \end{aligned}$$

there are two more conditions, $\|h\| = 1$ and $\langle h, \psi_1 \rangle \geq 0$. These conditions determine a real valued constant C' such that

$H' = C'H$ Obviously this fact does not influence the number given by the type $\frac{\int_0^1 [\epsilon^2 H_{xx}^2 + W''(u^\xi) H_x^2] dx}{\int_0^1 H^2 dx}$
We have proved that

$$\min_{\mathcal{H}'} \frac{\int_0^1 [\epsilon^2 H_{xx}^2 + W''(u^\xi) H_x^2] dx}{\int_0^1 H^2 dx} \leq \min_{\mathcal{H}} \frac{\int_0^1 [\epsilon^2 H_{xx}^2 + W''(u^\xi) H_x^2] dx}{\int_0^1 H^2 dx}.$$

In the same way someone can prove and the opposite inequality. It is not written because it is not necessary for the continuing of the proof.

Calculation of $\min_{\mathcal{H}'} \frac{\int_0^1 [\epsilon^2 H_{xx}^2 + W''(u^\xi) H_x^2] dx}{\int_0^1 H^2 dx}$ is accomplished in the steps (I)-(V):

(I)

$$\begin{aligned}\langle Ah, h\rangle &:= \int_0^1 [\epsilon^2 h_x^2 + W''(u^\xi)h^2]dx \\ &= \sum_{k=1}^{\infty} a_k^2 \bar{\mu}_k,\end{aligned}$$

where μ_k are the eigenvalues defined on the problem of Lemma 3.1.1 and $\bar{\mu}_k := -\mu_k$. It is known that μ_1, $\mu_2, ...,$ μ_{N+1} are exponentially small and that the rest of them are negative and bounded away from zero uniformly in ϵ and $\xi \in \Omega_{\rho_0}$. This result is known by the [ABF].

Thus we can write

$$\begin{aligned}\langle Ah, h\rangle &\geq a_1^2\bar{\mu}_1 + \cdots + a_{N+1}^2\bar{\mu}_{N+1} + \bar{\mu}_{N+2}\sum_{k=N+2}^{\infty} a_k^2 \\ &= \bar{\mu}_{N+2} \parallel h \parallel^2 - \sum_{k=1}^{N+1}(\bar{\mu}_{N+2} - \bar{\mu}_k)a_k^2.\end{aligned}$$

By Definition of $\mathcal{H}'$ we have $\parallel h \parallel = 1$, so

$$\begin{aligned}\langle Ah, h\rangle &\geq \bar{\mu}_{N+2} - (\bar{\mu}_{N+2} - \bar{\mu}_1)a_1^2 \\ &= \bar{\mu}_{N+2}(1 - a_1^2) + \bar{\mu}_1 a_1^2.\end{aligned}$$

Let

$$\delta(\epsilon) := 1 - a_1 \leq 1 - a_1^2,$$

then

$$\begin{aligned}\langle Ah, h\rangle &\geq \bar{\mu}_{N+2}\delta(\epsilon) + \bar{\mu}_1 a_1^2 \\ &\geq \bar{\mu}_{N+2}\delta(\epsilon) - O(e^{-c/\epsilon}),\end{aligned}$$

thus

$$\langle Ah, h\rangle \geq C\delta(\epsilon) - O(e^{-c/\epsilon}), \tag{3.2}$$

where C independent of ϵ and ξ. This happens because, according to [ABF], $\bar{\mu}_{N+2}$ is positive and bounded away from zero uniformly in ϵ and $\xi \in \Omega_\rho$.

(II)Let $\hat{\phi} = \int_0^1 \psi(x)dx$. Then

$$\begin{aligned}
\| h - (\psi_1 - \hat{\psi}_1) \|^2 &= \langle h, h\rangle - 2\langle h, \psi_1 - \hat{\psi}_1\rangle + \| \psi_1 - \hat{\psi}_1 \|^2 \\
&= \langle h, h\rangle - 2\langle h, \psi_1\rangle + 2\langle h, \hat{\psi}_1\rangle + \langle \psi_1 - \hat{\psi}_1, \psi_1 - \hat{\psi}_1\rangle \\
&= \langle h, h\rangle - 2\langle h, \psi_1\rangle + 2\langle h, \hat{\psi}_1\rangle + \langle \psi_1, \psi_1\rangle - 2\langle \hat{\psi}_1, \psi_1\rangle + \langle \hat{\psi}_1, \hat{\psi}_1\rangle \\
&= 1 - 2\langle h, \psi_1\rangle + 2\hat{\psi}_1 \int_0^1 h dx + 1 - 2\hat{\psi}_1 \int_0^1 \psi_1 dx + \hat{\psi}_1^2 \\
&= 1 - 2\langle h, \psi_1\rangle + 2\hat{\psi}_1 0 + 1 - 2\hat{\psi}_1^2 + \hat{\psi}_1^2 \\
&= 2[1 - \langle h, \psi_1\rangle] - \hat{\psi}_1^2,
\end{aligned}$$

thus

$$\| h - (\psi_1 - \hat{\psi}_1) \|^2 + \hat{\psi}_1^2 = 2[1 - \langle h, \psi_1\rangle] = 2\delta(\epsilon).$$

(III) Estimation of $\| H \|^2$

$$\begin{aligned}
|H(x)| &= |\int_0^x h(t)dt| \\
&\leq |\int_0^x (\psi_1 - \hat{\psi}_1)dt| + |\int_0^x h - (\psi_1 - \hat{\psi}_1)dt| \\
&\leq \hat{\psi}_1 + \| h - (\psi_1 - \hat{\psi}_1) \| \\
&\leq \sqrt{2}[\hat{\psi}_1^2 + \| h - (\psi_1 - \hat{\psi}_1) \|^2]^{1/2} \\
&= 2\delta(\epsilon)^{1/2},
\end{aligned}$$

thus

$$\| H \|^2 \leq 4\delta(\epsilon). \tag{3.3}$$

(IV) $\delta(\epsilon) \geq K\epsilon$ for some positive constant K:

$$\delta(\epsilon) = 1 - a_1 = \frac{1 - a_1^2}{1 + a_1} \geq \frac{(1 - a_1^2)}{2}$$

Let $Q(k) := \langle k, \psi_1 \rangle^2$ where $k \in L^2$. Then

$$\max_{k \in \mathcal{H}'} Q(k) \leq \max_{\int_0^1 kdx = 0, \|k\| = 1} Q(k) = Q\left(\frac{\psi_1 - \hat{\psi}_1}{\| \psi_1 - \hat{\psi}_1 \|}\right) = 1 - \hat{\psi}_1^2.$$

Therefore

$$a_1^2 = Q(h) \leq 1 - \hat{\psi}_1^2$$

or

$$\delta(\epsilon) \geq \frac{1}{2} \hat{\psi}_1^2.$$

thus

$$\delta(\epsilon) \geq K\epsilon.$$

(V) Combining (3.1), (3.2) and (3.3)

$$\begin{aligned} -\lambda_{N+1} &= \min_{H \in \mathcal{H}'} \frac{\int_0^1 [\epsilon^2 H_{xx}^2 + W''(u^\xi) H_x^2] dx}{\int_0^1 H^2 dx} \\ &= \min_{H \in \mathcal{H}'} \frac{\langle Ah, h \rangle}{\| H \|^2} \\ &\geq \frac{[C\delta(\epsilon) - O(e^{-c/\epsilon})]}{4\delta(\epsilon)} \\ &\geq C > 0 \end{aligned}$$

$\blacksquare$

We have already proved that all the eigenvalues except N are valued with values less than $-C$.

Second part: There are N eigenvalues that are exponentially small.

In this part we prove that the first N eigenvalues are exponentially small. In order to reach our goal, we approximate the first N eigenfunctions of L^c. The functions that approximate the first N eigenfunctions are E_j^{ξ}, $1 \leq j \leq N$ defined as follows:

$$\begin{aligned} E_j^{\xi}(x) &:= \bar{w}_j(x) - Q_j(x) \\ \bar{w}_j(x) &:= \tilde{u}_j^h(x) + \tilde{u}_{j+1}^h(x),\ j = 1, 2, ..., N, \end{aligned}$$

and

$$Q_j(x) := (-\frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{1}{3}x)\bar{w}_{jxx}(0) + \frac{1}{6}(x^3 - x)\bar{w}_{jxx}(1) + x\bar{w}_j(1)$$

with

$$E_j = E_{jxx} = 0 \ for\ x = 0, 1.$$

Before we prove this claim (that E_j approximate the eigenvalues) in Lemma 3.1.3 we need some calculations.
First of all we need to calculate the following term

$$\tilde{u}_j^h = \int_0^x \frac{\partial u^h}{\partial h_i}(y)dy$$

By Proposition 2.2.3 we have the following formula

$$u_j^h(x) = \begin{cases} \chi^{j-1}w^j, & x \in I_{j-1} \\ (1-\chi^j)(-\phi_x^j + w^j) + \chi^j(-\phi_x^{j+1} - w^{j+1}) & \\ +\chi_x^j(\phi^j - \phi^{j+1}), & x \in I_j \\ -(1-\chi^{j+1})w^{j+1}, & x \in I_{j+1} \\ 0, & otherwise \end{cases} \tag{3.4}$$

Therefore it is easy to obtain

$$u_j^h(x) = -u_x^h(x) + (1-\xi^j)w^j - \xi^j w^{j+1},\ x \in I_j$$

Using Propositions 2.2.2, 2.2.4 and 2.2.5, we also obtain

$$\bar{w}_j = \begin{cases} 0, & x \le m_{j-1} \\ e, & x \in I_{j-1} \\ u^h(m_j) - u^h(x) + e, & x \in I_j \cup I_{j+1} \\ u^h(m_j) - u^h(m_{j+2}) + e, & x \ge m_{j+2} \\ 0, & otherwise \end{cases} \tag{3.5}$$

Where e is an error estimated as

In I_{j-1}

$$e = \int_{m_{j-1}}^{x} \xi^{j-1}(y) w^j(y) dy,$$

so by Proposition 2.2.4

$$|e| \le C\epsilon^{-1}\beta(r).$$

In I_j

$$e = \int_{m_{j-1}}^{m_j} \xi^{j-1}(y) w^j(y) dy \int_{m_j}^{x} (1-\xi^j(y)) w^j(y) dy,$$

so by Proposition 2.2.4

$$|e| \le C\epsilon^{-1}\beta(r).$$

In I_{j+1}

$$e = C\epsilon^{-1}\beta(r) - \int_{m_{j+1}}^{x} \xi^{j+1}(y) w^{j+2}(y) dy,$$

so by Proposition 2.2.4

$$|e| \le C\epsilon^{-1}\beta(r).$$

For $x \ge m_{j+2}$

$$e = C\epsilon^{-1}\beta(r) - \int_{m_{j+2}}^{x} (1-\xi^{j+2}(y)) w^{j+2}(y) dy,$$

so by Proposition 2.2.4

$$|e| \leq C\epsilon^{-1}\beta(r).$$

Therefore

$$|e| \leq C\epsilon^{-1}\beta(r) \ \ everywhere\ in\ [0,1] \tag{3.6}$$

So after the approaches of e in I_{j-1}, I_j, I_{j+1} and I_{j+2} we know that $\bar{w}_j$ looks like Figure 3.1

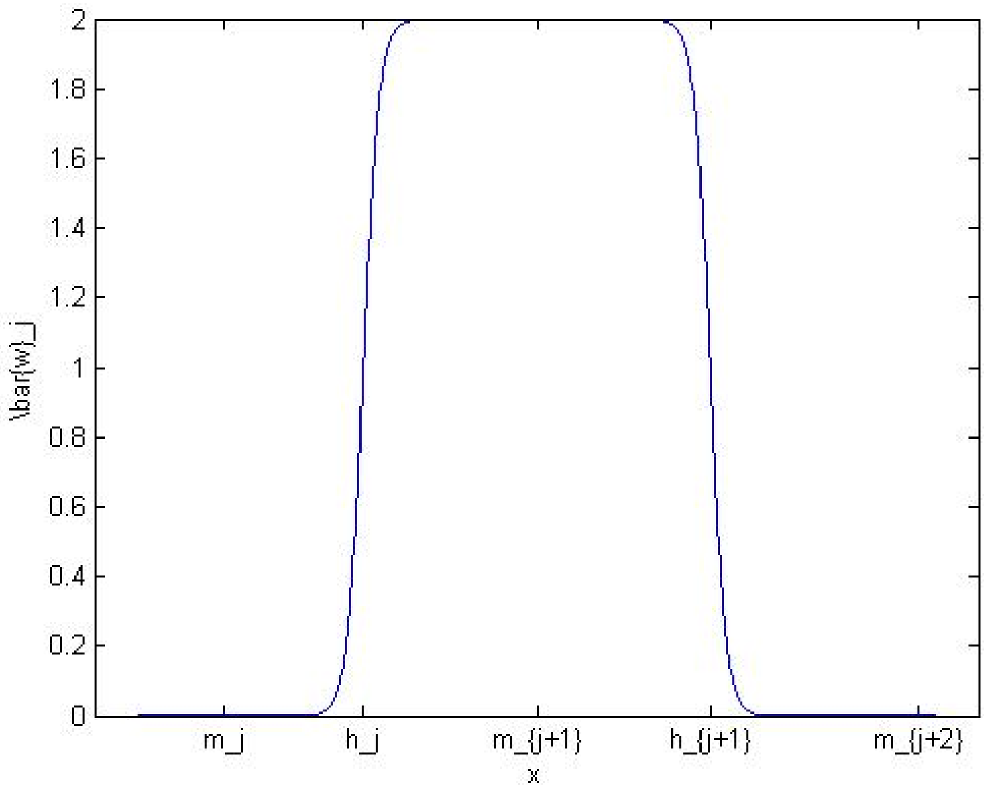

Figure 3.1: Simulation of the $\bar{w}_j$

continuing we can take

$$|\bar{w}_j(1)| \leq C\epsilon^{-1}\beta(r)$$

$$|\bar{w}_{jxx}(0)|,\ |\bar{w}_{jxx}(1)| \leq C\epsilon^{-3}\beta(r) \tag{3.7}$$

Therefore

$$E_j^h(x) = \begin{cases} e, & x \leq m_j \\ u^h(m_j) - u^h(x) + e, & x \in I_j \cup I_{j+1} \\ u^h(m_j) - u^h(m_{j+2}) + e, & x \geq m_{j+2} \end{cases} \tag{3.8}$$

where e satisfies (3.6)

Now we are ready to take some rigorous results regarding the approximation of the eigenfunctions by E_j

Lemma 3.1.3 *Let $H_1, H_2, ..., H_N$ be the orthonormal eigenfunctions of L^c corresponding to the first N eigenvalues $\lambda_1, \lambda_2, ..., \lambda_N$ respectively. Let R_i, $1 \leq i \leq N$ satisfies the equation below*

$$E_i = \sum_{j=1}^{N} C_{ij} H_j + R_i,\ \langle R_i, H_j \rangle = 0.$$

Then if $h \in \Omega_\rho$ with ρ sufficiently small, we have

(i)

$$\begin{aligned} \| R \| &\leq C\epsilon^{-5}\beta(r), \\ \| R' \| &\leq C\epsilon^{-6}\beta(r), \\ \| R'' \| &\leq C\epsilon^{-7}\beta(r). \end{aligned}$$

(ii) Both (C_{ij}) and $\epsilon(C_{ij})^{-1}$ are bounded uniformly with respect to ϵ when ϵ is enough small.

Proof proof of (i)

(1) By 3.8 in $I_i \cup I_{i+1}$, we have

$$L^c E_i = L^c e - L^c u^h$$

and

$$L^c u^h = -(\mathcal{L}^b(u^h))_{xx}.$$

As in the proof of Proposition 2.2.6

$$|L^c u^h| \leq C\epsilon^{-2}\alpha(r)$$

And also

$$|L^c e| \leq C\epsilon^{-5}\beta(r)$$

Finally we have

$$|L_i^E| \leq C\epsilon^{-5}\beta(r) \tag{3.9}$$

(2) Applying L^c to the $E_i = \sum_{j=1}^{N} C_{ij}H_j + R_i$ we obtain

$$\begin{aligned} L^c E_i &= \sum_{j=1}^{N} C_{ij} L^c H_j + L^c R_j \\ &= \sum_{j=1}^{N} C_{ij} \lambda_j H_j + L^c R_j \end{aligned}$$

Taking the inner product with R_i, we get

$$\begin{aligned} |\langle L^c R_i, R_i\rangle| &= |\langle L^c E_i, R_i\rangle - \langle \sum_{j=1}^{N} C_{ij}\lambda_j H_j, R_i\rangle| \\ &= |\langle L^c E_i, R_i\rangle| \end{aligned}$$

so

$$|\langle L^c R_i, R_i\rangle| \leq C\epsilon^{-5}\beta(r)\|R_i\|. \tag{3.10}$$

by Lemma 3.1.2

$$\lambda_i \geq C > 0, \; i > N$$

so

$$|\langle L^c R_i, R_i\rangle| \geq C\|R_i\|^2$$

Therefore from the last two inequalities we take

$$\| R_i \| \leq C\epsilon^{-5}\beta(r). \tag{3.11}$$

(3)

$$\begin{aligned}\langle -L^c R_i, R_i\rangle &= \int_0^1 [\epsilon^2 (R_i'')^2 + W''(u^h)(R_i')^2]dx \\ &\geq \epsilon^2 \parallel R_i'' \parallel^2 - C \parallel R_i' \parallel^2\end{aligned}$$

By interpolation

$$\parallel R_i' \parallel^2 \leq \eta \parallel R_i'' \parallel^2 + \eta^{-1} \parallel R_i \parallel^2$$

where $\eta = \frac{\epsilon^2}{2}\frac{1}{C}$.
Substituting,

$$\langle -L^c R_i, R_i\rangle \geq \frac{\epsilon^2}{2} \parallel R_i'' \parallel^2 - \frac{2}{\epsilon^2} C \parallel R_i \parallel^2$$

On the other hand, by (3.10) (3.11) we have

$$\begin{aligned}|\langle -L^c R_i, R_i\rangle| &\leq C\epsilon^{-5}\beta(r)\|R_i\| \\ &\leq C\epsilon^{-10}\beta(r)^2 \\ &= C\epsilon^{-10}\alpha(r),\end{aligned}$$

thus

$$\begin{aligned}\frac{\epsilon^2}{2}\|R_i''\|^2 &\leq \langle L^c R_i, R_i\rangle + \frac{2}{\epsilon^2} C\|R_i\|^2 \\ &\leq C\epsilon^{-10}\alpha(r) + \frac{2}{\epsilon^2} C(C\epsilon^{-5}\beta(r))^2 \\ &\leq C\epsilon^{-10}\alpha(r) + \frac{2}{\epsilon^2} C\epsilon^{-10}\alpha(r) \\ &= C\epsilon^{-12}\alpha(r)\end{aligned}$$

Consequently

$$\|R_i''\|^2 \leq C\epsilon^{-14}\alpha(r)$$

and

$$\|R_i'\|^2 \leq C\epsilon^{-12}\alpha(r)$$

proof of (ii)
Let $A = (a_{ij})$ a matrix such as $A = CC^T$ where $C = C_{ij}$ and C^T the

transpose of C. We have three different cases: for $|i-k|>1$, for $i=k$ and for $i=k+1$.
For $|i-k|>1$

$$|\langle E_i, E_k\rangle| \leq C\epsilon^{-6}\alpha(r)$$

For $k=i$

$$\begin{aligned}
\langle E_i, E_k\rangle &= \langle E_i, E_i\rangle \\
&= \int_{m_i}^{m_{i+2}} E_i^2 dx + O(\epsilon^{-3}\beta(r)) \\
&= \int_{m_i}^{m_{i+2}} [u^h(x) - u^h(m_i)]^2 dx + O(\epsilon^{-3}\beta(r))
\end{aligned}$$

by Proposition 2.2.1

$$\begin{aligned}
\langle E_i, E_i\rangle &\geq \frac{1}{2}[m_{i+2} - m_i] + O(\epsilon^{-3}\beta(r)) \\
&\geq \min_{1\leq j\leq N}(h_{j+1} - h_j) + O(\epsilon^{-3}\beta(r))
\end{aligned}$$

thus

$$\langle E_i, E_i\rangle \geq \frac{\epsilon}{\rho}$$

For $k=i+1$

$$\begin{aligned}
\langle E_i, E_k\rangle &= \langle E_i, E_{i+1}\rangle \\
&= \int_{m_{i+1}}^{m_{i+2}} E_i E_{i+1} dx + O(\epsilon^{-3}\beta(r)) \\
&= \int_{m_{i+1}}^{m_{i+2}} [u^h(x) - u^h(m_i)][u^h(x) - u^h(m_{i+1})] dx + O(\epsilon^{-3}\beta(r))
\end{aligned}$$

While x is valued in I_{i+1} by Proposition 2.2.1 we have the following statements:
I f $x > h_{i+1}$ then the first factor of the integral $[u^h(x) - u^h(m_i)]$ is small.
Otherwise if $x < h_{i+1}$ then the second factor $[u^h(x) - u^h(m_{i+1})]$ is small.

In fact we can show that

$$\langle E_i, E_{i+1} \rangle = O(\epsilon)$$

as $r \to \infty$. Consequently we have for ρ sufficiently small and $h \in \Omega_\rho$, the matrix A is diagonally dominant. Since

$$|C_{ij}| \leq \| H_j \| \| E_i \|$$

is bounded and

$$C^{-1} = C^T A^{-1}$$

we can see that the elements of C and ϵC^{-1} are uniformly bounded as $\epsilon \to 0$
$\blacksquare$

The proof of Theorem A is completed by the next lemma

Lemma 3.1.4 *Let $\lambda_1, \lambda_2, ..., \lambda_N$ be the the first N eigenvalues of EVP. Then*

$$\lambda_i = O(\epsilon^{-6}\beta(r)), \; i = 1, 2, ..., N$$

Proof proof for i=1:
By the variational characterization of eigenvalue λ_1

$$\lambda_1 = -\int_0^1 [\epsilon^2 H_1''^2 + W''(u^h) H_1'^2]$$

where $\| H_1 \| = 1$.
By Lemma 3.1.3 for some constants a_i, b_i, we have

$$H_1 = \sum_{i=1}^{N} [a_i E_i + b_i R_i],$$

where ϵa_i, ϵb_i are uniformly bounded as $\epsilon \to 0$.

Let

$$Z = \sum_{i=1}^{N} a_i E_i,$$

then we have

$$\begin{aligned}
\int_0^1 [\epsilon^2 H_1''^2 + W''(u^h) H_1'^2] dx & \\
&= \epsilon^2 \int_0^1 (H_1 - Z)''^2 dx + \int_0^1 W''(u^h)(H_1 - Z)'^2 dx \\
&\quad + \int_0^1 [\epsilon^2 Z''^2 + W''(u^h) Z'^2] dx \\
&\quad +2\{\epsilon^2 \int_0^1 (H_1 - Z)'' Z'' dx + \int_0^1 W''(u^h)(H_1 - Z)' Z' dx\} \\
&=: I + II + III,
\end{aligned}$$

where

$$\begin{aligned}
I &:= \epsilon^2 \int_0^1 (H_1 - Z)''^2 dx + \int_0^1 W''(u^h)(H_1 - Z)'^2 dx \\
II &:= + \int_0^1 [\epsilon^2 Z''^2 + W''(u^h) Z'^2] dx \\
III &:= +2\{\epsilon^2 \int_0^1 (H_1 - Z)'' Z'' dx + \int_0^1 W''(u^h)(H_1 - Z)' Z' dx\}
\end{aligned}$$

By (3.9),

$$L^c Z = O(\epsilon^{-6} \beta(r))$$

Consequently

$$L^c (H_1 - Z) = \lambda_1 H_1 + O(\epsilon^{-6} \beta(r))$$

$$-\epsilon^2 \int_0^1 (H_1 - Z)'''' Z dx + \int_0^1 (W''(u^h)(H_1 - Z)')' Z dx$$

$$= \lambda_1 \int_0^1 H_1 Z dx + O(\epsilon^{-6} \beta(r)),$$

Integrating by parts we take

$$\begin{aligned}
\lambda_1 \int_0^1 H_1 Z dx + O(\epsilon^{-6}\beta(r)) &= -\epsilon^2 \int_0^1 (H_1 - Z)'' Z'' dx \\
&\quad - \int_0^1 W''(u^h)(H_1 - Z)' Z' dx \\
&= -\frac{1}{2} III.
\end{aligned}$$

By 3.9 the second term is estimated as

$$\begin{aligned}
II &= \int_0^1 L^c Z Z dx \\
&= O(\epsilon^{-7}\beta(r))
\end{aligned}$$

and by (3.10), (3.11)we take

$$I = O(\epsilon^{-5}\beta(r)).$$

Therefore because

$$\lambda_1 = -I - II - III$$

we have

$$\lambda_1 = 2\lambda_1 \int_0^1 H_1 Z dx + O(\epsilon^{-7}\beta(r))$$

so

$$\lambda_1 (1 - 2 \int_0^1 H_1 Z dx) = O(\epsilon^{-7}\beta(r))$$

and

$$\begin{aligned}\int_0^1 H_1 Z dx &= \int_0^1 H_1(H_1 - \sum_{i=1}^N b_i R_i) dx \\ &= 1 - \sum_{i=1}^N b_i \int_0^1 H_1 R_i dx \\ &= 1.\end{aligned}$$

Thus

$$\lambda_1 = O(\epsilon^{-7}/\beta(r)).$$

∎

3.2 Proof of Theorem B

Now we are ready to prove Theorem B.

Definition 3.2.1 *Let* $\tilde{v} \in C^2[0,1]$, $\tilde{v} = 0$ *at* $x = 1, 2$,

- $A_\epsilon(\tilde{v}) := \int_0^1 [\epsilon^2 \tilde{v}_{xx}^2 + W''(u^h)\tilde{v}_x^2] dx$,
- $B_\epsilon(\tilde{v}) := \int_0^1 [\epsilon^2 \tilde{v}_{xx}^2 + \tilde{v}_x^2] dx$.

Under this definition we have the following lemma.

Lemma 3.2.2 *Let* $\tilde{v} \in C^2[0,1]$, $\tilde{v} = 0$ *at* $x = 0, 1$,

$$\begin{aligned}\|\tilde{v}\|_\infty^2 &\leq B_\epsilon(\tilde{v}) \\ \epsilon\|\tilde{v}\|_\infty^2 &\leq (1+\epsilon) B_\epsilon(\tilde{v}).\end{aligned}$$

Proof It is not so hard to see that for every x

$$\begin{aligned} |\tilde{v}(x)| &= \left| \int_0^x \tilde{v}(t)dt \right| \\ &\leq \left| \int_0^x \right| \tilde{v}(t) \mid dt \\ &\leq \|\tilde{v}_x\| \\ &\leq B_\epsilon(\tilde{v})^{1/2} \end{aligned}$$

thus

$$\begin{aligned} |\tilde{v}(x)|^2 &\leq B_\epsilon(\tilde{v}) \\ \max_x\{|\tilde{v}(x)|^2\} &\leq B_\epsilon(\tilde{v}) \\ \{\max_x |\tilde{v}(x)|\}^2 &\leq B_\epsilon(\tilde{v}) \\ \|\tilde{v}\|_\infty^2 &\leq B_\epsilon(\tilde{v}) \end{aligned}$$

We have already finished with the first inequality.

For the second inequality we assume that $\tilde{v}_x^2(x_1) = \|\tilde{v}_x\|_\infty^2$. By the Roll theorem and the boundary conditions $\tilde{v} = 0$ at $x = 0, 1$ we know that there exists $x_0 \neq x_1$ such that $\tilde{v}_x(x_0) = 0$ and consequently we have

$$\tilde{v}_x^2(x_0) \leq B_\epsilon(\tilde{v}).$$

We can assume that $x_1 \geq x_0$, without lost of generality because otherwise we can reflect the function around $x = 1/2$.

$$\begin{aligned} 0 &\leq (\epsilon\tilde{v}_{xx} - \tilde{v}_x)^2 \\ 0 &\leq \epsilon^2\tilde{v}_{xx}^2 + \tilde{v}_x^2 - 2\epsilon\tilde{v}_{xx}\tilde{v}_x \\ 2\epsilon\tilde{v}_{xx}\tilde{v}_x &\leq \epsilon^2\tilde{v}_{xx}^2 + \tilde{v}_x^2. \end{aligned}$$

Integrating from x_0 to x_1 we get

$$
\begin{aligned}
\int_{x_0}^{x_1} 2\epsilon\tilde{v}_{xx}(x)\tilde{v}_x(x)dx &\leq \int_{x_0}^{x_1} \epsilon^2\tilde{v}_{xx}^2(x) + \tilde{v}_x^2(x)dx \\
\int_{x_0}^{x_1} 2\epsilon\tilde{v}_{xx}(x)\tilde{v}_x(x)dx &\leq \int_0^1 \epsilon^2\tilde{v}_{xx}^2(x) + \tilde{v}_x^2(x)dx \\
\left[\epsilon\tilde{v}_x^2\right]_{x_0}^{x_1} &\leq B_\epsilon(\tilde{v}) \\
\epsilon\tilde{v}_x^2(x_1) - \epsilon\tilde{v}_x^2(x_0) &\leq B_\epsilon(\tilde{v}) \\
\epsilon\tilde{v}_x^2(x_1) &\leq B_\epsilon(\tilde{v}).
\end{aligned}
$$

Consequently

$$\epsilon\|\tilde{v}_x(x_1)\|_\infty^2 \leq B_\epsilon(\tilde{v})$$

∎

Lemma 3.2.3 *There is a $\rho_0 > 0$ such that if $0 < \rho < \rho_0$ and $h \in \Omega$, then for any $\tilde{v} \in C^2$ with $\tilde{v} = 0$ at $x = 0, 1$ and $\langle \tilde{v}, E_j \rangle = 0,\ 1 \leq j \leq N$, there is a constant C such that*

$$CA_\epsilon(\tilde{v}) \geq \epsilon^2 B_\epsilon(\tilde{v})$$

Proof By Lemmas 3.1.2 and 3.1.4 we have

$$
\begin{aligned}
A_\epsilon(\tilde{v}) &= -\sum_{j=1}^{\infty} \lambda_j \langle H_j, \tilde{v}\rangle^2 \\
&\geq \sum_{j=1}^{N} -\lambda_j \langle H_j, \tilde{v}\rangle^2 - \lambda_{N+1} \sum_{j=N+1}^{\infty} \langle H_j, \tilde{v}\rangle^2 \\
&\geq (\lambda_{N+1} - \lambda_1)\sum_{j=1}^{N} \langle H_j, \tilde{v}\rangle^2 - \lambda_{N+1}\sum_{j=1}^{\infty} \langle H_j, \tilde{v}\rangle^2 \\
&= (\lambda_{N+1} - \lambda_1)\sum_{j=1}^{N} \langle H_j, \tilde{v}\rangle^2 - \lambda_{N+1}\|\tilde{v}\|^2 \\
&\geq -\frac{\lambda_{N+1}}{2}\|\tilde{v}\|^2
\end{aligned}
$$

where H_j, $j = 1, 2, ..., N$ as in Lemma 3.1.3
$\blacksquare$

Suppose that...

Lemma 3.2.4 *Let $h \in \Omega_\rho$ where ρ small enough, $\tilde{u}(t,x) = \tilde{u}^\xi + \tilde{v}$ is such that $\tilde{v}$ satisfies (2.5) $\xi(t)$ satisfies (2.4) . Additionally $\tilde{u}$ is so close to $\mathcal{M}$ such that $B_\epsilon(\tilde{v}) \leq \epsilon^7 O(1)$, then*

$$|\dot{\xi}_i| \leq C[\epsilon^{-5}\beta(r)B_\epsilon(\tilde{v})^{1/2} + \epsilon^{-2}B_\epsilon(\tilde{v}) + \epsilon^{-2}\alpha(r)].$$

Proof by (2.5) and (2.4)

$$\langle -\epsilon^2(\tilde{u}^\xi_{xxxx} + \tilde{v}_{xxxx}) + (W'(\tilde{u}^\xi_x + \tilde{v}_x))_x, E^\xi_i\rangle$$

$$= \langle \mathcal{L}^c \tilde{u}^\xi + L^c \tilde{v} + (f_2 \tilde{v}_x^2)_x, E_i\rangle$$

where

$$f_2 = \int_0^1 (1-\tau)W'''(\tilde{u}^\xi_x + \tau\tilde{v}_x)d\tau. \tag{3.12}$$

By Proposition 2.2.6

$$|\langle \mathcal{L}^c \tilde{u}^\xi, E_i\rangle| \leq C\epsilon^{-1}\alpha(r)$$

additionally

$$\begin{aligned} \langle L^c \tilde{v}, E_i\rangle &= \langle \tilde{v}, L^c E_i\rangle \\ &= \langle \tilde{v}, \frac{\partial}{\partial h_i}\frac{\partial}{\partial x}\mathcal{L}^b u^\xi\rangle + \langle \tilde{v}, \frac{\partial}{\partial h_{i+1}}\frac{\partial}{\partial x}\mathcal{L}^b u^\xi\rangle - \langle \tilde{v}, L^c Q_i\rangle \end{aligned}$$

consequently

and by Proposition 2.2.7

$$|\frac{\partial}{\partial h_i}\frac{\partial}{\partial x}\mathcal{L}^b u^h| \leq C\epsilon^{-4}\beta(r)$$

So

$$\langle L^c\tilde{v}, E_i\rangle \leq C\epsilon^{-4}\beta(r)\|\tilde{v}\| \tag{3.13}$$

thus

$$\langle L^c\tilde{v}, E_i\rangle \leq C\epsilon^{-4}\beta(r)\|\tilde{v}\|_\infty \tag{3.14}$$

By Lemma 3.2.2

$$\langle L^c\tilde{v}, E_i\rangle \leq C\epsilon^{-4}\beta(r)B_\epsilon(\tilde{v})^{1/2}$$

if we assume that

$$\epsilon^{-1}B_\epsilon(\tilde{v}) \ \ \textit{is bounded} \tag{3.15}$$

and by (3.12), (3.4) and integration by parts , we have

$$|\langle (f_2\tilde{v}_x^2)_x, E_i\rangle| \leq C\epsilon^{-1}B_\epsilon(\tilde{v})$$

.

Next we need to estimate the coefficient matrix (a_{ij}) defined as in (2.4)

$$a_{ij} := \langle \tilde{u}_j^\xi, E_i\rangle - \langle \tilde{v}, E_{ij}\rangle.$$

Let

$$b_{ij} := \langle \tilde{u}_j^\xi, E_i\rangle$$

we can rewrite $\tilde{u}_i^\xi$ as

$$\tilde{u}_i^\xi := \bar{w}_i - \bar{w}_{i+1} + \cdots + (-1)^{N-i}\bar{w}_N + e_i u_{N+1}^h,$$

where $\bar{w}_i$ is defined as in (3.3)

$$\bar{w}_i := \tilde{u}_i^\xi(x) + \tilde{u}_{i+1}^\xi(x)$$

and

$$e_i = O(\epsilon^{-1}\beta(r)).$$

Therefore by the proof of Lemma 3.1.3 and because $E_j = \bar{w}_j(x) - Q_j(x)$, we get

$$b_{ij} = O(\epsilon^{-1}\beta(r)) + \begin{cases} 0, & j > i+1 \\ O(\epsilon), & j = i+1 \\ (-1)^{i-j} \parallel \|w_i\| O(\epsilon), & j \leq i \end{cases}$$

also

$$| \frac{\partial}{\partial \xi_i} \bar{w} | \leq C\epsilon^{-1}$$

and by the Poincare inequality

$$\tilde{v} \leq C B_\epsilon(\tilde{v})^{1/2}.$$

So

$$|\langle \tilde{v}, E_{ij} \rangle| \leq C\epsilon^{-1} B_\epsilon(\tilde{v})^{1/2}.$$

▮

Proof of Theorem B

First we need some estimations on $A_\epsilon(\tilde{v})$. We start with this term $\frac{1}{2}\frac{\partial}{\partial t}\langle -L^c\tilde{v}, \tilde{v}\rangle$.

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\langle -L^c\tilde{v}, \tilde{v}\rangle &= -\frac{1}{2}\langle (L^c\tilde{v})_t, \tilde{v}\rangle - \frac{1}{2}\langle L^c\tilde{v}, \tilde{v}_t\rangle \\
&= -\frac{1}{2}\langle (-\epsilon^2\tilde{v}_{xxxx} + (W''(u^\xi)\tilde{v}_x)_x)_t, \tilde{v}\rangle - \frac{1}{2}\langle L^c\tilde{v}, \tilde{v}_t\rangle \\
&= -\frac{1}{2}\langle -\epsilon^2\tilde{v}_{txxxx} + (W''(u^\xi)\tilde{v}_{tx})_x + (\frac{\partial}{\partial t}W''(u^\xi)\tilde{v}_x)_x, \tilde{v}\rangle - \frac{1}{2}\langle L^c\tilde{v}, \tilde{v}_t\rangle \\
&= -\frac{1}{2}\langle L^c\tilde{v}_t + (\frac{\partial}{\partial t}W''(u^\xi)\tilde{v}_x)_x, \tilde{v}\rangle - \frac{1}{2}\langle L^c\tilde{v}, \tilde{v}_t\rangle \\
&= -\frac{1}{2}\langle L^c\tilde{v}_t, \tilde{v}\rangle - \frac{1}{2}\langle (\frac{\partial}{\partial t}W''(u^\xi)\tilde{v}_x)_x, \tilde{v}\rangle - \frac{1}{2}\langle L^c\tilde{v}, \tilde{v}_t\rangle \\
&= -\langle L^c\tilde{v}, \tilde{v}_t\rangle + \langle -\frac{1}{2}(\frac{\partial}{\partial t}W''(u^\xi)\tilde{v}_x)_x, \tilde{v}\rangle
\end{aligned}$$

where we have used selfadjoint structure (see [BF]):

$$\langle L^c \tilde{v}_t, \tilde{v}\rangle = \langle \tilde{v}_t, L^c \tilde{v}\rangle.$$

Now

$$\begin{aligned}
\tilde{v}_t &= \mathcal{L}^c \tilde{u} - \sum_{j=1}^{N} \tilde{u}_j^{\xi} \dot{\xi}_j \\
&= \mathcal{L}^c (\tilde{u}^{\xi} + \tilde{v}) - \sum_{j=1}^{N} \tilde{u}_j^{\xi} \dot{\xi}_j \\
&= \mathcal{L}^c \tilde{u}^{\xi} + L^c \tilde{v} + (f_2 \tilde{v}_x^2)_x - \sum_{j=1}^{N} \tilde{u}_j^{\xi} \dot{\xi}_j,
\end{aligned}$$

where f_2 as in (3.12).
Therefore

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\langle -L^c \tilde{v}, \tilde{v}\rangle &= -\langle L^c \tilde{v}, L^c \tilde{v}\rangle \\
&\quad -\langle L^c \tilde{v}, \mathcal{L}^c \tilde{u}^{\xi}\rangle \\
&\quad -\langle L^c \tilde{v}, (f_2 \tilde{v}_x^2)_x\rangle \\
&\quad +\sum_{j=1}^{N} \langle L^c \tilde{v}, \tilde{u}_j^{\xi} \dot{\xi}_j\rangle \\
&\quad +\langle -\frac{1}{2}(W''(u^{\xi})_t \tilde{v}_x)_x, \tilde{v}\rangle \\
&=: (1) + (2) + (3) + (4) + (5).
\end{aligned}$$

where

$$\begin{aligned}
(1) &:= -\langle L^c\tilde{v}, L^c\tilde{v}\rangle \\
(2) &:= -\langle L^c\tilde{v}, \mathcal{L}^c\tilde{u}^\xi\rangle \\
(3) &:= -\langle L^c\tilde{v}, (f_2\tilde{v}_x^2)_x\rangle \\
(4) &:= +\sum_{j=1}^{N}\langle L^c\tilde{v}, \tilde{u}_j^\xi\dot{\xi}_j\rangle \\
(5) &:= +\langle -\frac{1}{2}(W''(u^\xi)_t\tilde{v}_x)_x, \tilde{v}\rangle
\end{aligned}$$

Then we estimate these terms

$$|(2)| \leq \|L^c\tilde{v}\|\|\mathcal{L}^c\tilde{u}^\xi\| \leq \frac{1}{4}\|L^c\tilde{v}\| + \|\mathcal{L}^c\tilde{u}^\xi\| \tag{3.16}$$

In order to estimate (3), we expand $(f_2\tilde{v}_x^2)_x$

$$\begin{aligned}
(f_2\tilde{v}_x^2)_x &= 2\tilde{u}_x\tilde{u}_{xx}\int_0^1(1-\tau)W'''(\tilde{u}_x^\xi+\tau\tilde{v}_x)d\tau \\
&\quad +\tilde{v}_x^2\int_0^1(1-\tau)W''''(\tilde{u}_x^\xi+\tau\tilde{v}_x)(\tilde{u}_{xx}^\xi+\tau\tilde{v}_{xx})d\tau
\end{aligned}$$

So under the assumption (3.15)

$$\begin{aligned}
|(3)| &\leq C[\|L^c\tilde{v}\|\|\tilde{v}_x\|_\infty\|\tilde{v}_{xx}\| + \|L^c\tilde{v}\|\|\tilde{v}_x\|_\infty^2(\frac{1}{\epsilon}+\|\tilde{v}_{xx}\|)] \\
&\leq \frac{1}{8}\|L^c\tilde{v}\|^2 + C(\epsilon^{-4}B_\epsilon(\tilde{v})^2)
\end{aligned}$$

$$\begin{aligned}
|(4)| &\leq \|L^c\tilde{v}\|\|\tilde{u}_j^\xi\||\dot{\xi}_j| \\
&\leq C\epsilon^{-1/2}\|L^c\tilde{v}\||\dot{\xi}_j| \\
&\leq \frac{1}{8}\|L^c\tilde{v}\| + C\epsilon^{-1}|\dot{\xi}_j|^2
\end{aligned}$$

$$\begin{aligned}|(5)| &= \frac{1}{2} \mid \langle [W''(u^\xi)]_t \tilde{v}_x, \tilde{v}_x \rangle \mid \\ &\leq C \sum_{j=1}^{N} |\dot{\xi}_j| \|u_j^\xi\|_1 \|\tilde{v}_x\|_\infty^2 \\ &\leq C\epsilon^{-1} \sum_{j=1}^{N} |\dot{\xi}_j| B_\epsilon(\tilde{v}),\end{aligned}$$

because by Proposition 2.2.5 we know that $\| u_j^\xi \|_1$ is bounded.

Using the approximations $|(2)|$, $|(3)|$, $|(4)|$ and $|(5)|$, we get

$$\frac{1}{2}\frac{\partial}{\partial t}\langle -L^c\tilde{v}, \tilde{v}\rangle + \frac{1}{2} \parallel L^c\tilde{v} \parallel^2 \leq \parallel \mathcal{L}^c(\tilde{u}^\xi) \parallel^2 + C\epsilon^{-1}|\dot{\xi}_j|^2 + C\epsilon^{-4}B_\epsilon^2(\tilde{v}) \quad (3.17)$$

so using Proposition 2.2.6

$$\frac{1}{2}\frac{\partial}{\partial t}\langle -L^c\tilde{v}, \tilde{v}\rangle + \frac{1}{2} \parallel L^c\tilde{v} \parallel^2 \leq C[\epsilon^{-2}\alpha^2(r) + \epsilon^{-1}|\dot{\xi}_j|^2 + \epsilon^{-4}B_\epsilon^2(\tilde{v})]. \quad (3.18)$$

Using Lemma 3.2.4 we can get

$$|\dot{\xi}_j|^2 \leq C[\epsilon^{-10}\beta^2(r)B_\epsilon(\tilde{v}) + \epsilon^{-4}B_\epsilon^2(\tilde{v}) + \epsilon^{-4}\alpha^2(r)]$$

Combining this inequality with (3.18), we get

$$\frac{1}{2}\frac{\partial}{\partial t}\langle -L^c\tilde{v}, \tilde{v}\rangle + \frac{1}{2} \parallel L^c\tilde{v} \parallel^2 \leq C[\epsilon^{-5}\alpha^2(r) + \epsilon^{-5}B_\epsilon^2(\tilde{v}) + \epsilon^{-11}\beta(r)B_\epsilon(\tilde{v})].$$

By Lemma 3.2.3, we have

$$\epsilon^2 B_\epsilon(\tilde{v}) \leq C \parallel L^c\tilde{v} \parallel^2 .$$

Thus

$$\frac{1}{2}\frac{\partial}{\partial t}\langle -L^c\tilde{v},\tilde{v}\rangle+\frac{1}{2}\parallel L^c\tilde{v}\parallel^2\leq C[\epsilon^{-5}\alpha^2(r)+(\epsilon^{-5}B_\epsilon(\tilde{v})+\epsilon^{-11}\beta(r))\epsilon^{-2}\parallel L^c\tilde{v}\parallel^2].$$

Using the assumption $\epsilon^{-7}B_\epsilon(\tilde{v})=o(1)$ of Lemma 3.2.4, we get

$$\frac{1}{2}\frac{\partial}{\partial t}\langle -L^c\tilde{v},\tilde{v}\rangle+\frac{1}{3}\parallel L^c\tilde{v}\parallel^2\leq C\epsilon^{-3}\alpha^2(r)\leq C\epsilon^{-5}\alpha^2(r).$$

Using Lemma 3.1.2, we can get

$$\frac{\partial}{\partial t}A_\epsilon(\tilde{v})+aA_\epsilon(\tilde{v})\leq C\epsilon^{-5}\alpha^2(r),$$

where

$$a=\frac{1}{3}|\lambda_{N+1}|>0.$$

Integration gives

$$A_\epsilon(\tilde{v}(t))\leq A_\epsilon(\tilde{v}(0))e^{-at}+C\epsilon^{-5}\alpha^2(r)(1-e^{-at}),$$

so

$$A_\epsilon(\tilde{v}(t))\leq \max\left\{A_\epsilon(\tilde{v}(0)),C\epsilon^{-5}\alpha^2(r)\right\}, \tag{3.19}$$

Where C, a are positive constants, independent of ϵ and $\tilde{v}$.
By Lemma 3.2.2 and 3.19, we write the inequality of Lemma 3.2.4 as

$$\begin{aligned}|\dot{\xi}_i| &\leq C[\epsilon^{-6}\beta(r)A_\epsilon^{1/2}(\tilde{v})+\epsilon^{-4}A_\epsilon(\tilde{v})+\epsilon^{-2}\alpha(r)]\\ &\leq C[\epsilon^{-6}\beta(r)(A_\epsilon(\tilde{v}(0))^{1/2}e^{-(at)/2}+\epsilon^{-5/2}\alpha(r))+\epsilon^{-4}A_\epsilon(\tilde{v}(0))e^{-at}+\epsilon^{-2}\alpha(r)],\end{aligned}$$

thus

$$|\dot{\xi}_i|\leq C[\epsilon^{-6}\beta(r)(A_\epsilon(\tilde{v}(0))^{1/2}e^{-(at)/2}+\epsilon^{-4}A_\epsilon(\tilde{v}(0))e^{-at}+\epsilon^{-2}\alpha(r)].$$

The slow channel is defined by the following inequality

$$A_\epsilon(\tilde{v}(0)) \leq C\epsilon^{-5}\alpha^2(r).$$

Therefore from the last two inequalities we get

$$|\dot{\xi}_j| \leq C\epsilon^{-2}a(r) \tag{3.20}$$

The proof of Theorem B follows from (3.19) and (3.20).
The slow channel as we know is defined by Definition 3.0.3 to be

$$\Gamma = \{\tilde{u} : \tilde{u} = \tilde{u}^\xi + \tilde{v},\, A_\epsilon(\tilde{v}) < b\epsilon^{-5}\alpha^2(r)\}.$$

Additionally if we define

$$\Gamma' = \{\tilde{u} : \tilde{u} = \tilde{u}^\xi + \tilde{v},\, \epsilon^{-7+\delta_1}B_\epsilon(\tilde{v}) < C\},$$

with some fixed δ_1, then we have the conclusion of Figure 3.2. If a solution starts in Γ', then by (3.19) it will decay exponentially towards Γ until u^ξ leaves the boundary of the Invariant Manifold. Additionally, if it enters the Γ then it will stay there and it can leave it only through the ends of the channel.

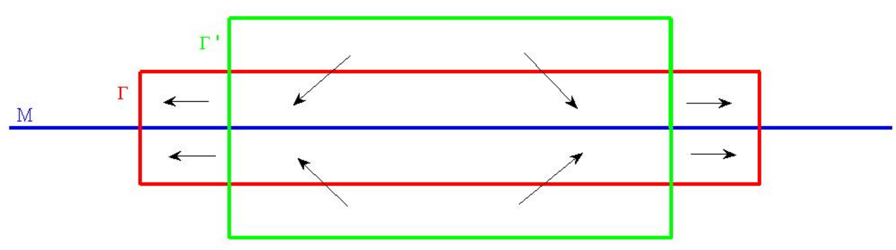

Figure 3.2: The Invariant Manifold is in blue and Γ and Γ' are in red and green respectively. The arrows shows the motion of the solutions

Chapter 4

System of ODEs

For any function $\phi(x)$, $\tilde{\phi}(x) := \int_0^1 \phi(y)dy$

4.1 The System of ODEs

We have already proven that the Invariant Manifold attracts the solutions which are close to its neighbourhood called "slow channel" (Theorem B). Nevertheless, we have not investigated the evolution of the solution close to the Invariant Manifold. Because v is negligibly small, what we have to find is the behaviour of $\xi = (\xi_1, \xi_2, ..., \xi_N)$. The answer to this question is given rigorously by the following ODE system.

$$
\begin{aligned}
\frac{d\xi_1}{dt} &= \frac{1}{4(\xi_2 - \xi_1)}(\alpha^3 - \alpha^1) + O(\epsilon\alpha) \\
\frac{d\xi_2}{dt} &= \frac{1}{4(\xi_2 - \xi_1)}(\alpha^3 - \alpha^1) + \frac{1}{4(\xi_3 - \xi_2)}(\alpha^4 - \alpha^2) + O(\epsilon\alpha) \\
\frac{d\xi_3}{dt} &= \frac{1}{4(\xi_3 - \xi_2)}(\alpha^4 - \alpha^2) + \frac{1}{4(\xi_4 - \xi_3)}(\alpha^5 - \alpha^3) + O(\epsilon\alpha) \\
&\vdots \\
\frac{d\xi_N}{dt} &= \frac{1}{4(\xi_N - \xi_{N-1})}(\alpha^{N+1} - \alpha^{N-1}) + \frac{1}{4(h_{N+1} - \xi_N)}(\alpha^{N+2} - \alpha^N) + O(\epsilon\alpha)
\end{aligned}
$$

In order to take an easier view of this system, we can imagine h_{N+1} as a variable adding one more equation

$$\frac{dh_{N+1}}{dt} = \frac{1}{4(h_{N+1} - \xi_N)}(\alpha^{N+2} - \alpha^N) + O(\epsilon\alpha).$$

The derivation of this system is given in the next section. As we can see, between two neighbour quantities of the same phase (ξ_{j-1}, ξ_j) and (ξ_{j+1}, ξ_{j+2}) the bigger of them two out to pump quantity from the smaller. This happens because of the sign of number $\alpha^{j+2} - \alpha^j$. The tempo of this process is proportional to $(\alpha^{j+2} - \alpha^j)/(\xi_{j+1} - \xi_j)$.

Regarding the layer points, because they are between two different phases, their movement depends on the pumping of both phases. That's why in the system every $\dot{\xi}_i$ is equal to the sum of two terms. An example is given in Figure 4.1

4.2 Derivation of the ODE System

In this section we derive the system mentioned in the previous section. We work with the integrated equation and with inner product, so:

$$\sum_{j=1}^{N} \alpha_{ij}\dot{\xi}_j = \langle -\epsilon^2(\tilde{u}^{\xi}_{xxxx} + \tilde{v}_{xxxx}) + (W'(\tilde{u}^{\xi}_x + \tilde{v}_x))_x, E^{\xi}_i \rangle$$

The key, in order to produce the system, is estimation of $\langle -\epsilon^2(\tilde{u}^{\xi}_{xxxx} + \tilde{v}_{xxxx}) + (W'(\tilde{u}^{\xi}_x + \tilde{v}_x))_x, E^{\xi}_i \rangle$ and of the matrix α_{ij}. Afterwards we invert the matrix α_{ij} to α^{-1}_{ij} and we multiply with it both , right and left hand, sides of the equation.

Approximation of $\langle -\epsilon^2(\tilde{u}^{\xi}_{xxxx} + \tilde{v}_{xxxx}) + (W'(\tilde{u}^{\xi}_x + \tilde{v}_x))_x, E^{\xi}_i \rangle$

$\langle -\epsilon^2(\tilde{u}^{\xi}_{xxxx} + \tilde{v}_{xxxx}) + (W'(\tilde{u}^{\xi}_x + \tilde{v}_x))_x, E^{\xi}_i \rangle$
$\langle \mathcal{L}^c\tilde{u}^{\xi} + L^c\tilde{v} + (f_2\tilde{v}^2_x)_x, E_i \rangle =: I_1 + I_2 + I_3$

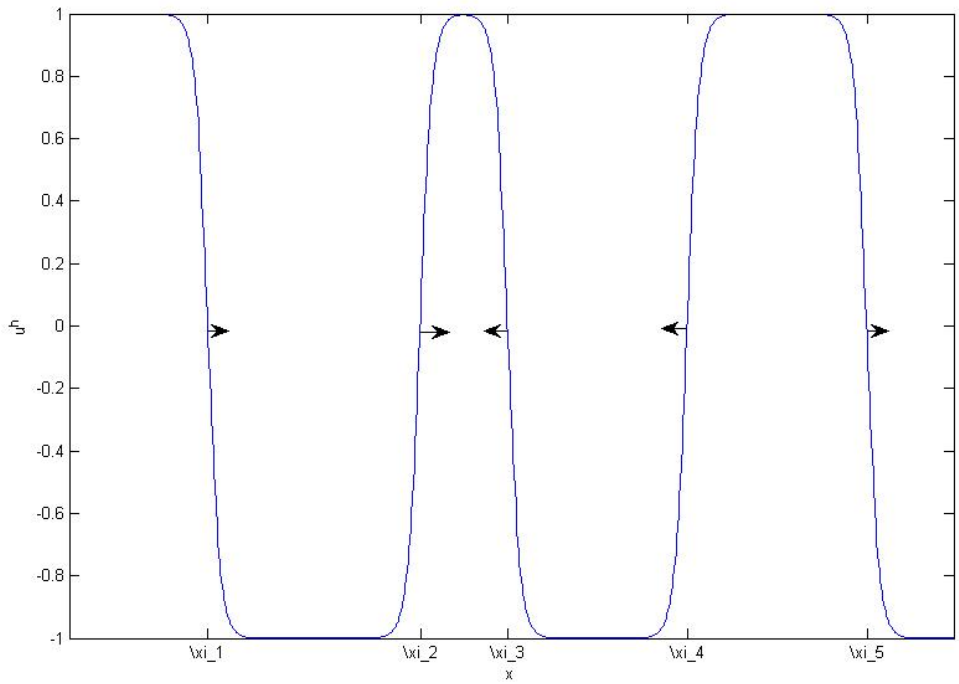

Figure 4.1: In this Figure are the dynamics described by the system of ODEs. The arrows are the velocities of ξ_is with measure $|\dot{\xi}_i|$.

where

$$
\begin{aligned}
I_1 &= \langle \mathcal{L}^c, E_i \rangle \\
I_2 &= \langle L^c \tilde{v} +, E_i \rangle \\
I_3 &= \langle (f_2 \tilde{v}_x^2)_x, E_i \rangle
\end{aligned}
$$

Estimation of I_1:

$$
\begin{aligned}
I_1 &= \langle \mathcal{L}^c u^\xi, E_i \rangle \\
&= \langle -(\mathcal{L}^b u^\xi)_x, E_i \rangle \\
&= \langle \mathcal{L}^b u^\xi, E_{ix} \rangle \\
&= \langle \mathcal{L}^b u^\xi, u_i^\xi + u_{i+1}^\xi - Q_{ix} \rangle
\end{aligned}
$$

by Proposition 2.2.3 we can take some results about $u_i^\xi + u_{i+1}^\xi$

$$
u_i^\xi + u_{i+1}^\xi = \begin{cases}
\chi^{i-1} w^i, & x \in I_{i-1} \\
(1-\chi^i)(-\phi_x^i + w^i) + \chi^i(-\phi_x^{i+1} - w^{i+1}) & \\
+\chi_x^i(\phi^i - \phi^{i+1}) + \chi^i w^{i+1}, & x \in I_i \\
-(1-\chi^{i+1}) w^{i+1} + (1-\chi^{i+1})(-\phi_x^{i+1} + w^{i+1}) & \\
+\chi^{i+1}(-\phi_x^{i+2} - w^{i+2}) + \chi_x^{i+1}(\phi^{i+1} - \phi^{i+2}), & x \in I_{i+1} \\
-(1-\chi^{i+2}) w^{i+2}, & x \in I_{i+2}
\end{cases}
$$

or,

$$
u_i^\xi + u_{i+1}^\xi = \begin{cases}
\chi^{i-1} w^i, & x \in I_{i-1} \\
(1-\chi^i)(-\phi_x^i + w^i) - \chi^i \phi_x^{i+1} + \chi_x^i(\phi^i - \phi^{i+1}), & x \in I_i \\
-(1-\chi^{i+1}) \phi_x^{i+1} + \chi^{i+1}(-\phi_x^{i+2} - w^{i+2}) & \\
+\chi_x^{i+1}(\phi^{i+1} - \phi^{i+2}), & x \in I_{i+1} \\
-(1-\chi^{i+2}) w^{i+2}, & x \in I_{i+2}
\end{cases}
$$

or,

$$
u_i^\xi + u_{i+1}^\xi = \begin{cases}
\chi^{i-1} w^i, & x \in I_{i-1} \\
(1-\chi^i) w^i - (1-\chi^i)\phi_x^i - \chi^i \phi_x^{i+1} + \chi_x^i(\phi^i - \phi^{i+1}), & x \in I_i \\
-\chi^{i+1} w^{i+2} - (1-\chi^{i+1})\phi_x^{i+1} - \chi^{i+1}\phi_x^{i+2} + \chi_x^{i+1}(\phi^{i+1} - \phi^{i+2}), & x \in I_{i+1} \\
-(1-\chi^{i+2}) w^{i+2}, & x \in I_{i+2}
\end{cases}
$$

or,

$$u_i^\xi + u_{i+1}^\xi = \begin{cases} \chi^{i-1}w^i, & x \in I_{i-1} \\ (1-\chi^i)w^i + u_x^\xi, & x \in I_i \\ -\chi^{i+1}w^{i+2} + u_x^\xi, & x \in I_{i+1} \\ -(1-\chi^{i+2})w^{i+2}, & x \in I_{i+2} \end{cases}$$

Therefore we have

$$I_1 = \int_{I_i \cup I_{i+1}} \mathcal{L}^b(u^\xi)u_x^\xi dx + \langle \mathcal{L}^b u^\xi, e\rangle$$

where

$$e = \begin{cases} -Q_{ix} + \chi^{i-1}w^i, & x \in I_{i-1} \\ -Q_{ix} + (1-\chi^i)w^i, & x \in I_i \\ -Q_{ix} - \chi^{i+1}w^{i+2}, & x \in I_{i+1} \\ -Q_{ix} - (1-\chi^{i+2})w^{i+2}, & x \in I_{i+2} \\ -Q_{ix}, & elsewhere \end{cases}$$

$$\begin{aligned} \int_{I_i \cup I_{i+1}} \mathcal{L}^b(u^\xi)u_x^\xi dx &= \int_{m_i}^{m_{i+2}} (\epsilon^2 u_{xx}^\xi - W'(u^\xi))u_x^\xi dx \\ &= [\frac{\epsilon^2}{2}{u_x^\xi}^2 - W(u^\xi)]_{m_i}^{m_{i+2}} \\ &= \frac{\epsilon^2}{2}u_x^\xi(m_{i+2})^2 - W(u^\xi(m_{i+2})) - \frac{\epsilon^2}{2}u_x^\xi(m_i)^2 + W(u^\xi(m_i)) \\ &= 0 - \alpha^{i+2} - 0 + \alpha^i \\ &= \alpha^i - \alpha^{i+2} \end{aligned}$$

by Proposition 2.2.9 $|\mathcal{L}^b(u^\xi)| \leq C|\alpha^{i+1} - \alpha^i|$ for $x \in I_{i+1}$

by (3.7) and Proposition 2.2.8, $|e| \leq C\epsilon^{-3}\beta(r)$.
Thus

$$I_1 = \alpha^i - \alpha^{i+2} + O(\epsilon^{-3}\beta(r)\alpha(r))$$

The new Estimation of I_2: By (3.14)

$$|\langle L^c\tilde{v}, E_i\rangle| \leq C\epsilon^{-4}\beta(r)B_\epsilon(\tilde{v})^{1/2}$$

By Lemma 3.2.3

$$|\langle L^c\tilde{v}, E_i\rangle| \leq C\epsilon^{-5}\beta(r)A_\epsilon(\tilde{v})^{1/2}$$

By Definition 3.0.3 of the slow channel Γ

$$|\langle L^c\tilde{v}, E_i\rangle| \leq Cb^{1/2}\epsilon^{-15/2}\alpha(r)\beta(r)$$

Thus

$$I_2 = O(\epsilon^{-15/2}\alpha(r)\beta(r))$$

Estimation of I_3:

By (3.15)

$$\langle (f_2\tilde{v}_x^2)_x, E_i\rangle \leq C\epsilon^{-1}B_\epsilon(\tilde{v})$$

By Proposition 3.2.3

$$\langle (f_2\tilde{v}_x^2)_x, E_i\rangle \leq C\epsilon^{-3}A_\epsilon(\tilde{v})$$

By Definition 3.0.3 of the slow channel Γ

$$\begin{aligned} |\langle (f_2\tilde{v}_x^2)_x, E_i\rangle| &\leq C\epsilon^{-3}b\epsilon^{-5}\alpha^2(r) \\ |\langle (f_2\tilde{v}_x^2)_x, E_i\rangle| &\leq Cb\epsilon^{-8}\alpha^2(r) \end{aligned}$$

Thus

$$I_3 = O(\epsilon^{-8}\alpha^2(r))$$

Now it remains to estimate the matrix α_{ij}.

$$\alpha_{ij} = \langle \tilde{u}_j^\xi, E_i^\xi\rangle - \langle \tilde{v}, \frac{\partial}{\partial\xi_j}E_i^\xi\rangle$$

Consequently, it suffices to estimate $\langle \tilde{u}_j^\xi, E_i^\xi\rangle$ and $\langle \tilde{v}, \frac{\partial}{\partial\xi_j}E_i^\xi\rangle$.

estimation of $\langle \tilde{u}_j^{\xi}, E_i^{\xi} \rangle$:

$$\langle \tilde{u}_j^{\xi}, E_i^{\xi} \rangle = \langle \bar{w}_j - \bar{w}_{j+1} + \cdots + (-1)^{N-j}\bar{w}_N + e_j \tilde{u}_{N+1}^h, E_i^{\xi} \rangle,$$

where by (3.6) and Figure 3.1

$$e_j = \partial h_{N+1}/\partial h_j - 1 = O(\epsilon^{-1}\beta(r))$$

Continuing we have

$$\begin{aligned}
\langle \tilde{u}_j^{\xi}, E_i^{\xi} \rangle &= \langle \bar{w}_j - \bar{w}_{j+1} + \cdots + (-1)^{N-j}\bar{w}_N + e_j \tilde{u}_{N+1}^h, E_i^{\xi} \rangle \\
&= \langle \bar{w}_j - \bar{w}_{j+1} + \cdots + (-1)^{N-j}\bar{w}_N + e_j \tilde{u}_{N+1}^h, \bar{w}_i - Q_i(x) \rangle \\
&= \langle \bar{w}_j - \bar{w}_{j+1} + \cdots + (-1)^{N-j}\bar{w}_N, \bar{w}_i \rangle \\
&\quad - \langle \bar{w}_j - \bar{w}_{j+1} + \cdots + (-1)^{N-j}\bar{w}_N + e_j \tilde{u}_{N+1}^h, Q_i(x) \rangle \\
&\quad + \langle e_j \tilde{u}_{N+1}^h, \bar{w}_i \rangle
\end{aligned}$$

By (3.7) we know that $|Q_i(x)| \leq C\epsilon^{-3}\beta(r)$

$$\langle \bar{w}_j - \bar{w}_{j+1} + \cdots + (-1)^{N-j}\bar{w}_N + e_j \tilde{u}_{N+1}^h, Q_i(x) \rangle$$

$$\leq \| \bar{w}_j - \bar{w}_{j+1} + \cdots + (-1)^{N-j}\bar{w}_N + e_j \tilde{u}_{N+1}^h \| \| Q_i(x) \|$$

$$\leq max|\bar{w}_j - \bar{w}_{j+1} + \cdots + (-1)^{N-j}\bar{w}_N + e_j \tilde{u}_{N+1}^h| \, max|\bar{w}_j - \bar{w}_{j+1} + \cdots + (-1)^{N-j}\bar{w}_N + e_j \tilde{u}_{N+1}^h|$$

$$\begin{aligned}
\langle e_j \tilde{u}_{N+1}^h, \bar{w}_i \rangle &\leq \| e_j \tilde{u}_{N+1}^h \| \| \bar{w}_i \| \\
&\leq max|e_j \tilde{u}_{N+1}^h| max|\bar{w}_i|
\end{aligned}$$

$$\langle \tilde{u}_j^{\xi}, E_i^{\xi} \rangle = \langle \bar{w}_j - \bar{w}_{j+1} + \cdots + (-1)^{N-j}\bar{w}_N, \bar{w}_i \rangle + O(\epsilon^{-4}\beta)$$

or

$$\langle \tilde{u}_j^{\xi}, E_i^{\xi} \rangle = \langle \bar{w}_j, \bar{w}_i \rangle - \langle \bar{w}_{j+1}, \bar{w}_i \rangle + \cdots + (-1)^{N-j} \langle \bar{w}_N, \bar{w}_i \rangle + O(\epsilon^{-4}\beta)$$

And now what we have to do is to estimate $\langle \bar{w}_j, \bar{w}_i \rangle$ for every $1 \le i, j \le N$.

for $i = j$ by (3.5), (3.6)

$$\int_0^1 \bar{w}_j^2 dx = 4l_{j+1} + C_j \epsilon + O(\beta)$$

Similarly for $i \neq j$

$$\int_0^1 \bar{w}_i \bar{w}_j dx = O(\epsilon).$$

finally

$$\langle \tilde{u}_j^\xi, E_i^\xi \rangle = \begin{cases} (-1)^{i+j} 4l_{j+1} + O(\epsilon), & i \ge j \\ O(\epsilon), & i < j. \end{cases}$$

Estimation of $\langle \tilde{v}, \frac{\partial}{\partial \xi_j} E_i^\xi \rangle$:

By Lemma 3.2.2 and Definition 3.0.3

$$|\tilde{u}| = O(\epsilon^{-7/2}\alpha)$$

or

$$\| \tilde{u} \| = O(\epsilon^{-7/2}\alpha)$$

Also $\| \frac{\partial}{\partial \xi_j} E_i^\xi \| = O(\epsilon^{-1/2})$ and

$$\langle \tilde{v}, \frac{\partial}{\partial \xi_j} E_i^\xi \rangle = O(\epsilon^{-4}\alpha).$$

And also

$$\alpha_{ij} = \begin{cases} (-1)^{i+j} 4l_{j+1} + O(\epsilon), & i \ge j \\ O(\epsilon), & i < j. \end{cases}$$

or

$$(a_{ij}) =$$

$$\begin{pmatrix} 4(\xi_2-\xi_1) & 0 & 0 & \cdots & 0 \\ -4(\xi_3-\xi_2) & 4(\xi_3-\xi_2) & 0 & \cdots & 0 \\ 4(\xi_4-\xi_3) & -4(\xi_4-\xi_3) & 4(\xi_4-\xi_3) & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ (-1)^{N-1}4(h_{N+1}-\xi_N) & (-1)^{N-2}4(h_{N+1}-\xi_N) & (-1)^{N-3}4(h_{N+1}-\xi_N) & \cdots & 4(h_{N+1}-\xi_N) \end{pmatrix}$$
$$+O(\epsilon)$$

or

$$(a_{ij})^{-1} = \begin{pmatrix} \frac{1}{4(\xi_2-\xi_1)} & 0 & 0 & \cdots & 0 \\ \frac{1}{4(\xi_2-\xi_1)} & \frac{1}{4(\xi_3-\xi_2)} & 0 & \cdots & 0 \\ 0 & \frac{1}{4(\xi_3-\xi_2)} & \frac{1}{4(\xi_4-\xi_3)} & \cdots & 0 \\ \cdots & 0 & \frac{1}{4(\xi_4-\xi_3)} & \cdots & \vdots \\ \cdots & \cdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{4(h_{N+1}-\xi_N)} \end{pmatrix} + O(\epsilon)$$

Multiplying both terms of the equation with (a_{ij}) we take the following system

$$\begin{aligned} \frac{d\xi_1}{dt} &= \frac{1}{4(\xi_2-\xi_1)}(\alpha^3-\alpha^1)+O(\epsilon\alpha) \\ \frac{d\xi_2}{dt} &= \frac{1}{4(\xi_2-\xi_1)}(\alpha^3-\alpha^1)+\frac{1}{4(\xi_3-\xi_2)}(\alpha^4-\alpha^2)+O(\epsilon\alpha) \\ \frac{d\xi_3}{dt} &= \frac{1}{4(\xi_3-\xi_2)}(\alpha^4-\alpha^2)+\frac{1}{4(\xi_4-\xi_3)}(\alpha^5-\alpha^3)+O(\epsilon\alpha) \\ &\vdots \\ \frac{d\xi_N}{dt} &= \frac{1}{4(\xi_N-\xi_{N-1})}(\alpha^{N+1}-\alpha^{N-1})+\frac{1}{4(h_{N+1}-\xi_N)}(\alpha^{N+2}-\alpha^N)+O(\epsilon\alpha) \end{aligned}$$

We can either see h_{N+1} as a value or as a function of ξ. If h_{N+1} is a value, the system has N equations but $N+1$ values.

by Proposition 2.3.1 and differentiating $\int_0^1 u^h(x)dx = M$ we find

$$\frac{\partial h_{N+1}}{\partial h_j} = (-1)^{N-j} + O(\epsilon^{-1}\beta)$$

and so differentiating h_{N+1} by time we find

$$\frac{dh_{N+1}}{dt} = \sum_{j=1}^{N}[(-1)^{N-j} + O(\epsilon^{-1}\beta)]\dot{\xi}_j.$$

Using the system it turns out to be

$$\frac{dh_{N+1}}{dt} = \frac{1}{4(h_{N+1}-\xi_N)}(\alpha^{N+2}-\alpha^N) + O(\epsilon\alpha)$$

and finally we have the system

$$\begin{aligned}
\frac{d\xi_1}{dt} &= \frac{1}{4(\xi_2-\xi_1)}(\alpha^3-\alpha^1) + O(\epsilon\alpha)\\
\frac{d\xi_2}{dt} &= \frac{1}{4(\xi_2-\xi_1)}(\alpha^3-\alpha^1) + \frac{1}{4(\xi_3-\xi_2)}(\alpha^4-\alpha^2) + O(\epsilon\alpha)\\
\frac{d\xi_3}{dt} &= \frac{1}{4(\xi_3-\xi_2)}(\alpha^4-\alpha^2) + \frac{1}{4(\xi_4-\xi_3)}(\alpha^5-\alpha^3) + O(\epsilon\alpha)\\
&\vdots\\
\frac{d\xi_N}{dt} &= \frac{1}{4(\xi_N-\xi_{N-1})}(\alpha^{N+1}-\alpha^{N-1}) + \frac{1}{4(h_{N+1}-\xi_N)}(\alpha^{N+2}-\alpha^N) + O(\epsilon\alpha)\\
\frac{dh_{N+1}}{dt} &= \frac{1}{4(h_{N+1}-\xi_N)}(\alpha^{N+2}-\alpha^N) + O(\epsilon\alpha).
\end{aligned}$$

We are thankful to P.W.Bates J.Xun for this elegant system.

Stochastic Cahn-Hilliard Equation

The Cahn-Hilliard equation is a very good model for describing the metastable patterns. Nevertheless, it is an idealization of the natural problem. This happens because of the noise of thermal fluctuations (see [BMW]). If we add an extra non-homogeneous stochastic term to the Cahn-Hilliard equation we obtain the following Stochastic-Cahn-Hilliard equation (or Cahn-Hilliard-Cook), which considers noise (for a basic reference on stochastic differential equations see [W]):

$$u_t = (-\epsilon^2 u_{xx} + W'(u))_{xx} + \partial_x \dot{B}, \ \ x \in (0,1), \ \ t > 0,$$

where $0 < \epsilon << 1$ is the interaction length, $W(u) = 1/4(u^2-1)^2$ the potential and B is a Q-Wiener process and accounts for the noise.
With the boundary conditions

$$u_x = u_{xxx} = 0, \ \ x \in \{0,1\}, \ \ t > 0.$$

The Q-Wiener process is defined by the following definition.

Definition 4.2.1 *Let $(H, <>)$ be a Hilbert space. The H-valued stochastic process B is called Q-Wiener process if and only if:*

- $B(0) = 0$ *(with probability 1), B has continuous paths.*
- *The differences $B(t_1) - B(s_1)$, $B(t_2) - B(s_2)$ are independent for every $0 \leq s_1 < t_1 \leq s_2 < t_2$.*
- *B is Gaussian, meaning $(< B(t_1), h_1 >, ..., < B(t_n), h_n >)$ is a vector valued Gaussian random variable for every $h_i \in H, \ t_i \geq 0$.*
- *$E(B(t)) = 0$, meaning $E < B(t), h >$ is a vector valued Gaussian random variable for every $h \in H, \ t \geq 0$.*
- *B has covariance operator Q, meaning Q is a positive semidefinite symmetric operator, such that $E < B(t), h >< B(s), g >= min\{t, s\} < Qh, g >$ for every $h, g \in H, \ t, s \geq 0$.*

We can also convert Stochastic Cahn-Hilliard equation to an integrated form

$$\tilde{u}_t = -\epsilon^2 \tilde{u}_{xxxx} + (W'(\tilde{u}_x))_x + \dot{B}, \ \ x \in (0,1), \ \ t > 0,$$

$$\tilde{u}(0,t) = 0, \quad \tilde{u}(1,t) = M, \quad t > 0,$$

$$\tilde{u}_{xx} = 0, \quad x \in \{0,1\}, \quad t > 0,$$

where $\tilde{u}(x,t) = \int_0^x u(y,t)dy$.

The next step in order to obtain a solution is to integrate by time. As opposed to deterministic integrals, the integral used is the Itô integral:

Definition 4.2.2 *The Itô integral of a stochastic process $v(t)$ is defined by the sum*

$$\int_0^t vdB = \lim_{n\to\infty} \sum_{k=0}^{n-1} v(t_k)(B(t_{k+1}) - B(t_k)),$$

where $0 = t_0 < t_2 < ... < t_n = t$ *and* $t_{k+1} - t_k \to 0$ *as* $n \to \infty$.

Now the theoretical background is complete. Is it possible to derive a system of Stochastic ODEs for the Stochastic-Cahn-Hilliard equation? The answer is given by D.Antonopoulou, D.Blomker, G.Karali, in [ABK].

Appendix A

Numerical Simulations

All the Figures of the thesis are made using the MathWorks Matlab technical computing language. The m-code for simulating the Invariant Manifold, like in Figures 2.7 4.1, is given in this appendix .

```
%parameter epsilon of the Cahn-Hilliard equation
epsilon=0.01;
%step for the simulations
h=0.001;

%input for the transition layers
% u(1) is the value of u^h at x=0
% du is the derivative of u^h at x=0
% duu(i) is the derivative of u^h at x=h_i
u(1)=0.9999999;
du=0;
u(2)=u(1)+du*h;
duu(1)=-70.651678;
duu(2)=70.6516786207036;
duu(3)=-70.65167862070;
duu(4)=70.651;
duu(5)=-70.65167862070;
%duu(6)=70.5516786207035;
%duu(7)=-70.6516786207036;
%duu(8)=70.5516786207036;
```

```
for i=1:1:1/h+1
  x(i)=h*(i-1);
end

%simulation of the Reaction-Diffusion equation using method of finite
%elements. The step used is h.
k=1;
for i=2:1:1/h

    if u(i-1)*u(i)<0
        k=k+1;
        hh(k-1)=x(i);
        hh(k)=10;
        u(i)=duu(k-1)*h;
        u(i-1)=0;
    end

    u(i+1)=-u(i-1)+2*u(i)+(h/epsilon)^2*(u(i)^3-u(i));
end

for i=1:1:1/h+1
    ou(i)=u(i);
end


%the manifold is the sum of two terms

%the first term
l=1;
i=0;
while i<1/h+1
    i=i+1;
    uu(i)=u(i);
    if abs(x(i)-hh(l))<epsilon
        while abs(x(i)-hh(l))<epsilon
             u(i)=-u(i-2)+2*u(i-1)+(h/epsilon)^2*(u(i-1)^3-u(i-1));

             if mod(l,2)==0
```

```
                uu(i)=(0.5-(0.5)*sin(0.5*3.1416*(x(i)-hh(l))/epsilon))*u(i);
            end

            if mod(l,2)==1
                uu(i)=(1-(0.5-(0.5)*sin(0.5*3.1416*(x(i)-hh(l))/epsilon)))*u(i);
            end

            if i<1/h+1
                if i<1/h+1
                    if (abs(x(i+1)-hh(l))>epsilon || (x(i+1)-hh(l))==epsilon)
                        l=l+1;
                    end
                end
            end
            i=i+1;

        end
        i=i-1;
    end
end


for i=1:1:1/h+1
    u(i)=ou(i);
end


%the second term
 l=k-1;
 i=1/h+2;
while i>1
    i=i-1;
 uuu(i)=0;
    if abs(x(i)-hh(l))<epsilon
        while abs(x(i)-hh(l))<epsilon
            u(i)=-u(i+2)+2*u(i+1)+(h/epsilon)^2*(u(i+1)^3-u(i+1));

            if mod(l,2)==0
                uuu(i)=(0.5+(0.5)*sin(0.5*3.1416*(x(i)-hh(l))/epsilon))*u(i);
```

```
            end

            if mod(l,2)==1
                uuu(i)=(1-(0.5+(0.5)*sin(0.5*3.1416*(x(i)-hh(l))/epsilon)))*u(i);
            end

            if i>1
                if (abs(x(i-1)-hh(l))>epsilon || (x(i-1)-hh(l))==epsilon)
                    l=l-1;
                    if l==0
                        l=k;
                    end
                end
            end
            i=i-1;

        end
        i=i+1;
    end
end


%sum of the first and the second terms
for i=1/h+1:-1:1
    u(i)=uuu(i)+uu(i);
end


plot(x,u)
%plot(x,u,'r')
xlabel('x')
ylabel('u^h')
set(gca,'XTick',hh)
set(gca,'XTickLabel',{'h_1','h_2','h_3','h_4','h_5'})

%the l_{N+2}
ll=2*(1-hh(k-1))
```

The m-code for calculating the $u_x^h(h_{N+1})$

In the code given above duu(N+1), which approximates $u_x^h(h_{N+1})$, takes a unique value. The reason is that duu(N+1) is dependent on duu(1), duu(2), ...,duu(N) because $l_{N+1} = 2(1 - h_{N+1})$. The method used in the following code, for calculating duu(N+1), is bisection method.

```
%parameter epsilon of the Cahn-Hilliard equation
epsilon=0.01;
%step for the simulations
h=0.001;

l=l1;

%Bisection method in the interval [a,b]
a=70;
b=70.6516786208;
e=10;

while abs(e)>0.001

    m=(a+b)/2;
    i=2;
    u(1)=0;
    u(2)=m*h+u(1);
    while u(i)-u(i-1)>0
        u(i+1)=-u(i-1)+2*u(i)+(h/epsilon)^2*(u(i)^3-u(i));
        i=i+1;
    end

    e=2*(i-2)*h-l;
    if e<0
        a=m;
    else
        b=m;
    end

end
```

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