THE PERIODIC TODA LATTICE

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Αφιερώνεται
στους γονείς μου,
Θωμά και Βάια
και στις μικρές
μου αδερφές,
Εύα και Γεωργία.

Contents

Preface

In the following pages, I tried to write a short introduction to the theory of nonlinear lattices (and especially the periodic Toda lattice) as part of my master's thesis. The problem was inrtoduced to me in the Spring of 2010 in a seminar held at the department of Applied Mathematics by Professor Spyridon Kamvissis. I was interested in that and so I asked him to give me more information about it. And so he did. He advised me that I should start with Toda's book [8].

I actually started studying the book in September and when I finished it, I went on with another one this time more "mathematically rigorous". I'm talking about the monograph of Teschl [7] (mainly I read the second part of that book concerning the Toda lattice). Here the exposition follows [8] and specifically:

- Chapter 1 explains how the origin of the problem lies in the so-called Fermi-Pasta-Ulam lattice and describes how Toda chose a specific potential that made the lattice periodic.
- The second chapter is concerned with some analytic tools that enable us solve the initial value problem with periodic conditions.
- In Chapter 3 the notions of spectrum and auxiliary spectrum are studied in depth in order to acquire the first results. The chapter ends with a short paragraph on Riemann surfaces the usefulness of which will be clear in the following chapter.
- The fourth Chapter reveals the role of Riemann surfaces and subsequently of theta functions. This chapter is the heart of the theory. Some tricky calculations are necessary to provide useful formulas in terms of theta functions.
- The final chapter (namely Chapter 5) combines all the previously acquired results in order to study the time evolution and finally state the solution of the problem being discussed.

  Literature

  As I've already stated, I used mainly two books in order to write this article.

Toda's [8] and Teschl's [7]. These concern the study of nonlinear lattices and include the theory used in the present article.

Furthermore, I used the classic book of Springer [5] to enter the world of Riemann surfaces. There's also the book of Farkas and Kra [4] which in my opinion is great. When I was finishing the typing of this article I came up with another book in the same direction. I'm talking about Donaldson's Riemann surfaces [3]. The reader who's not familiar with this topic is advised to take a look at one of these.

In order to understand how Toda found the lattice with exponential interaction I had to study a few things about elliptic functions and integrals. I used the book of Byrd and Friedman [2]. Also, I used Whitham's book [9] to gain some knowledge in nonlinear waves (lattice-solitons) but the reader doesn't have to know a thing about these in order to proceed.

Finally, the reader has to be familiar with some basic facts from complex analysis like Cauchy's Theorem, the Residue Theorem, etc.. I recommend the classics [1] and [6].

Acknowledgments

I want to express my deep gratitude to my advisor, Professor Spyridon Kamvissis for being so kind to answer my questions. He devoted considerable time and effort trying to explain to me all the questions that I was persistently asking him. I have profited from the discussions we had and I feel very lucky for the collaboration we had.

Also, I would like to thank the reading committee of my dissertation, Prof. Ioannis Platis and Prof. Nikolaos Efremidis for reading the manuscript and providing me useful advice and comments.

The support of the departments of Mathematics and Applied Mathematics was great during these two years of my graduate studies. All the staff was very friendly.

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Finally, I want to thank my family for supporting me all these years now. My father, my mother and my little two sisters. I am obliged to them. To them I owe the most.

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Chapter 1

An introduction to the problem

1.1 A few things about the problem

In 1955 *Enrico Fermi*, *John Pasta* and *Stanislaw Ulam* carried out a seemingly innocent computer experiment at *Los Alamos*. They considered a simple model for a nonlinear one-dimensional crystal describing the motion of a chain of particles with nearest neighbor interaction (in other words *non-linear lattice*).

They wanted to study the problem from the perspective of *energy*. They wanted to know how energy is shared between modes. They expected that because of the *nonlinearity* of interaction, *energy flow* between modes would take place finally establishing *energy equipartition*. So, they wanted to verify their expectation numerically. However, contrary to their assumption only a little energy partition occurred and the state of the system was found to almost return to the initial state.

So as mentioned above, the problem being considered is *classical mechanics* of one-dimensional lattices (i.e. chains) of particles with nearest neighbor interaction. First of all, we restrict ourselves to a *uniform system* (also referred to as a system without *impurities*). This means that each particle has *mass* m. Also, we denote by y_n the *displacement* of the n^{th} particle and by $\phi(y_{n+1} - y_n)$ the *interaction potential* between neighboring particles. The *mechanical analogue* is this: we can think of the above system as a chain of infinitely many particles joined together with "nonlinear" *springs* (see Figure 1.1).

Then, if

$$f(r) \equiv -\phi'(r) = -\frac{d\phi(r)}{dr}$$

is the *force* of the spring when it is stretched by the amount r and

$$r_n = y_{n+1} - y_n$$

is the *mutual displacement*, *Newton's law* tells us that the **equations of motion** are given by:

$$m\frac{d^2y_n}{dt^2} = \phi'(y_{n+1} - y_n) - \phi'(y_n - y_{n-1}), \quad n \in \mathbb{Z} \tag{1.1}$$

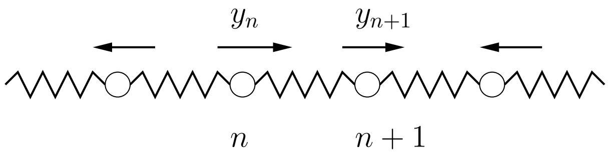

Figure 1.1: A model for one-dimensional lattice.

1.2 The harmonic interaction

As an intuitive example let's first of all consider the **linear case**. When $f(r)$ is proportional to r, that is when *Hooke's law* is obeyed, the spring is said to be *linear* and the *potential* (in this case we say that we have **harmonic interaction**) can be written as

$$\phi(r) = \frac{\kappa}{2}r^2.$$

Then, the equations of motion take the form:

$$m\frac{d^2y_n}{dt^2} = \kappa(y_{n-1} - 2y_n + y_{n+1}). \tag{1.2}$$

If $y_n^{(1)} \equiv y_n^{(1)}(t)$ and $y_n^{(2)} \equiv y_n^{(2)}(t)$ are solutions of (1.2), then the *linear superposition*

$$y_n = y_n^{(1)} + y_n^{(2)}$$

is also a solution of the *linear equation* (1.2). In particular, when the particles $n = 0$ and $n = N + 1$ (where $N \in \mathbb{Z}$) are fixed, then

$$\begin{cases} y_n^{(l)}(t) = C_n \sin\left(\frac{\pi l}{N+1}n\right)\cos(\omega_l t + \delta_l) \\ \omega_l = 2\sqrt{\frac{\kappa}{m}}\sin\left(\frac{\pi l}{2(N+1)}\right), \qquad l = 1, 2, \ldots, N \end{cases} \tag{1.3}$$

is the l^{th} **normal mode**, and the *general motion* is given by a linear superposition of such modes. The amplitude C_n of each mode is a *constant* determined by the *initial conditions* and no energy transfer occurs between the modes. The linear lattice is therefore *nonergodic* and cannot be an object of statistical mechanics unless some modification is made.

1.3 Nonlinear interaction

Fermi did some work on similar problems when he was young and after computers were developed he came back to this as one of the problems computers might solve. He thought that if one added a nonlinear term to the force between particles in a one-dimensional lattice, energy would flow from mode to mode eventually leading to a statistical equilibrium state where the energy is shared equally among linear modes (equipartition of energy).

They tested potentials one with a *cubic* term (α is the nonlinearity constant)

$$\phi(r) = \frac{\kappa}{2} r^2 + \frac{\kappa\alpha}{3} r^3, \tag{1.4}$$

another with a *quartic* term (α' is the nonlinearity constant)

$$\phi(r) = \frac{\kappa}{2} r^2 + \frac{\kappa\alpha'}{4} r^4,$$

and a third one with *broken linear force*

$$f(r) = \begin{cases} -\kappa r, & if \quad |r| \leq r_0, \\ -(\kappa - \kappa')r_0 - \kappa' r, & if \quad |r| > r_0 \end{cases}$$

where κ, κ' and r_0 are positive constants such that $\kappa \neq \kappa'$.

For these potentials the results turned out to be qualitatively similar. They treated lattices with $N = 32$ and $N = 64$ particles so that both ends ($n = 0$ and $n = N + 1$) were fixed. The lattice was initially at rest and given the displacement

$$y_n(0) = B \sin \frac{\pi n}{N+1}.$$

This means that they *excited the lowest mode.* They observed that after a certain time almost all the energy went back to the initial mode. The displacement of each particle went back to the initial state too. This is the so-called **FPU recurrence phenomenon**. Computer experiments sometimes yield unexpected findings and the *FPU recurrence phenomenon* is one of them. This was reconfirmed by many researchers. It can be said that if the energy is not too large, recurrence phenomena will occur.

These results combined with subsequent ones from *Ford et al.* showed that nonlinear lattices have rather stable motion (for this reason *Ford* introduced the term *nonlinear normal modes*). This remarkable property led to the finding of an explicitly solvable one-dimensional lattice in the particular case of *exponential* interaction.

1.4 A useful interpretation

The *lattice with exponential interaction* was found after looking for a system with explicit exactly periodic solutions. The concept of *dual systems* pointed in that way.

Systems **A** and **B** are said to be **dual to one another** if **B** is obtained from **A** by replacing particles by springs and springs by particles following certain rules (e.g. for a harmonic lattice we can replace heavier/lighter particles by weaker/stronger springs in such way that the normal mode frequencies (1.3) are the same in both systems).

We can *generalize* the idea of dual systems by the following consideration. The *Hamiltonian* which gives rise to the equation of motion (1.1) is:

$$\mathcal{H} = \frac{1}{2m}\sum_{n\in\mathbb{Z}} p_n^2 + \sum_{n\in\mathbb{Z}} \phi(r_n),$$

where the *momentum* p_n is related to the *kinetic energy*:

$$\mathcal{K} = \frac{1}{2}\sum_{n\in\mathbb{Z}} m\dot{y}_n^2.$$

Differentiating $\mathcal{K}$ with respect to the *velocity* $\dot{y}_n$, we obtain

$$p_n = \frac{\partial\mathcal{K}}{\partial\dot{y}_n} = m\dot{y}_n.$$

We shall now use the mutual displacement r_n as the *generalized coordinate*. For brevity's sake, we consider here an infinite lattice where only N particles can move and the rest are pinched down. For example, assuming that the left end particle $n = 0$ is fixed we have

$$\begin{cases} y_0 = 0, \quad y_1 = r_0, \quad y_2 = r_0 + r_1, \quad \dots \\ \dot{y}_0 = 0, \quad \dot{y}_1 = \dot{r}_0, \quad \dot{y}_2 = \dot{r}_0 + \dot{r}_1, \quad \dots \end{cases}$$

So, for a lattice with N movable particles

$$\mathcal{K} = \frac{1}{2}\sum_{n=0}^{N-1} m(\dot{r}_0 + \dot{r}_1 + \cdots + \dot{r}_n)^2.$$

The momentum s_n *conjugate* to r_n is defined by

$$\begin{aligned} s_n \equiv \frac{\partial\mathcal{K}}{\partial\dot{r}_n} &= m\Big((\dot{r}_0 + \dot{r}_1 + \cdots + \dot{r}_n) + (\dot{r}_0 + \dot{r}_1 + \cdots + \dot{r}_{n+1}) + \cdots \\ &\quad + (\dot{r}_0 + \dot{r}_1 + \cdots + \dot{r}_{N-1})\Big) \\ &= m\sum_{k=n}^{N-1}\sum_{j=0}^{k} \dot{r}_j. \end{aligned}$$

Therefore we have

$$s_{n-1} - s_n = m\dot{y}_n, \quad n = 1, \dots, N-1 \tag{1.5}$$

$$s_N = 0$$

and the Hamiltonian becomes:

$$\mathcal{H} = \frac{1}{2m}\sum_{n=0}^{N-1}(s_n - s_{n+1})^2 + \sum_{n=0}^{N-1}\phi(r_n).$$

Then, the **canonical equations of motion** are:

$$\dot{r}_n = \frac{\partial \mathcal{H}}{\partial s_n} = -\frac{s_{n-1} - 2s_n + s_{n+1}}{m}, \tag{1.6}$$

$$\dot{s}_n = -\frac{\partial \mathcal{H}}{\partial r_n} = -\phi'(r_n) \tag{1.7}$$

If we eliminate s_n from these equations, we obtain:

$$m\ddot{r}_n = \phi'(r_{n-1}) - 2\phi'(r_n) + \phi'(r_{n+1}) \tag{1.8}$$

which is, however, the difference of (1.1) with the same equation in which n is replaced by $n+1$ and therefore is *not* a new equation.

If (1.7) admits an *inverse*, we may write

$$r_n = -\frac{1}{m}\chi(\dot{s}_n). \tag{1.9}$$

(χ being a function of $\dot{s}_n$). Then, we can eliminate r_n from (1.6) to obtain:

$$\frac{d}{dt}\chi(\dot{s}_n) = s_{n-1} - 2s_n + s_{n+1}. \tag{1.10}$$

This is an equation **dual** to (1.8). If we think of s_n as the "displacement" then the right-hand side of (1.10) can be interpreted as the force of linear springs and in the left-hand side, $\chi(\dot{s}_n)$ can be interpreted as the momentum associated to the "speed" $\dot{s}_n$. Then (1.10) turn out to be the *mechanical equations of motion*.

The force f_n of the spring is related to $\dot{s}_n$ by

$$f_n = -\phi'(r_n) = \dot{s}_n \tag{1.11}$$

and the equation of motion (1.8) is rewritten as:

$$\frac{d^2}{dt^2}\chi(f_n) = f_{n-1} - 2f_n + f_{n+1}.$$

Further, we introduce the integral of s_n by

$$S_n = \int_0^t s_n(u)du$$

Choosing the integration constant appropriately, we have from (1.6), (1.5):

$$y_n = \frac{1}{m}(S_{n-1} - S_n),$$

$$r_n = -\frac{1}{m}(S_{n-1} - 2S_n + S_{n+1}),$$

and the equations of motion take the form:

$$\chi(\ddot{S}_n) = S_{n-1} - 2S_n + S_{n+1}.$$

1.5 An integrable lattice

Since *Fermi et al.* indicated that there are nonlinear lattices which admit periodic behavior at least when the energy is not too high, it is reasonable to look for a nonlinear lattice which admits periodic waves.

We seek such a lattice here, and we shall show that a **lattice with exponential interaction** has the desired properties. This means finding a potential function $\phi(r)$ such that the equations of motion (1.1) can be integrated. Equation (1.1) or equivalently (1.8) is ubiquitous but it proved hard to find such a potential $\phi(r)$. On the contrary, (1.10) can be considered as a *recurrence formula* expressing s_{n+1} in terms of s_n, s_{n-1} and the derivative of some function at s_n, which is related to the inverse function of the potential $\phi(r)$. Under these conditions, many functions $\phi(r)$ were tried.

In the case of the harmonic lattice, typical periodic waves are sinusoidal. Therefore, it was quite natural to think of *elliptic functions* (for elliptic integrals and Jacobian functions see [2]) as possible candidates because they are in a sense extensions of trigonometric functions. However, the first obvious choice of a solution proportional to *Jacobi's sn* or *cn* functions did not work.

On the other hand, an *addition formula* for sn^2 had been noticed:

$$sn^2(u+v) - sn^2(u-v) = 2\frac{d}{dv}\Big(\frac{sn(u)\cdot cn(u)\cdot dn(u)\cdot sn^2(v)}{1-k^2sn^2(u)\cdot sn^2(v)}\Big)$$

which led to the lattice being searched for.

Indeed, using

$$dn^2(u) = 1 - k^2 sn^2(u)$$

we define a function $\varepsilon(u)$ by

$$\varepsilon(u) = \int_0^u dn^2(v)dv$$

hence

$$\varepsilon'(u) = dn^2(u)$$

$$\varepsilon''(u) = -2k^2 sn(u) \cdot cn(u) \cdot dn(u).$$

Thus, we obtain

$$\varepsilon(u-v) - 2\varepsilon(u) + \varepsilon(u+v) = \frac{\varepsilon''(u)}{\frac{1}{sn^2(v)} - 1 + \varepsilon'(u)}. \tag{1.12}$$

Though $\varepsilon(u)$ is *not* a periodic function, the function defined by

$$Z(u) = \varepsilon(u) - \frac{E}{K}u$$

is a periodic function with period $2K$, where K and E are respectively the *complete elliptic integrals* of the *first* and *second kind* (see [2]).

Rewriting (1.12), we have

$$Z(u-v) - 2Z(u) + Z(u+v) = \frac{d}{du}\log\left(1 + \frac{1}{\frac{1}{sn^2(u)} - 1 + \frac{E}{K}} Z'(u)\right)$$

which is to be compared with (1.10). Thus we see that (1.10) is satisfied when we put

$$\begin{cases} u = 2\left(\nu t \pm \frac{n}{\lambda}\right)K \\ v = 2\frac{K}{\lambda} \end{cases}$$

where λ (the *wavelength*) and ν (the *frequency*) are constants, and identify the functions s_n and χ with

$$s_n(t) = \frac{2K\nu}{b/m} Z(u)$$

and

$$\chi(\dot{s}) = \frac{m}{b}\log\left(1 + \frac{\frac{b/m}{(2K\nu)^2}}{\frac{1}{sn^2(v)} - 1 + \frac{E}{K}}\dot{s}\right) - m\sigma, \tag{1.13}$$

where b and σ are constants. $\chi(\dot{s})$ is the inverse function of $\dot{s} = -\phi'(r)$ and must *not* contain ν and v which means that the factor of $\dot{s}$ in (1.13) is a constant *independent* of ν and v. Therefore, the relation

$$(2K\nu)^2 = \frac{a}{b}\Big(\frac{1}{sn^2(2K/\lambda)} - 1 + \frac{E}{K}\Big)^{-1}$$

must hold where a is a constant and in order that the right-hand side is positive, we must assume that $ab > 0$.

By (1.9) and (1.13), we have

$$r = -\frac{1}{b}\log\Big(1 + \frac{\dot{s}}{a}\Big) + \sigma$$

with $r = r_n$. Taking the inverse, by (1.7) we have

$$\dot{s} = a(e^{-b(r-\sigma)} - 1) = -\phi'(r).$$

Therefore, for the potential we obtain a function with *three parameters*: a, b and σ which can be written as:

$$\phi(r) = \frac{a}{b}e^{-b(r-\sigma)} + ar + const. \tag{1.14}$$

or

$$\phi(r) = Ae^{-br} + ar.$$

Take a look at Figure 1.2.

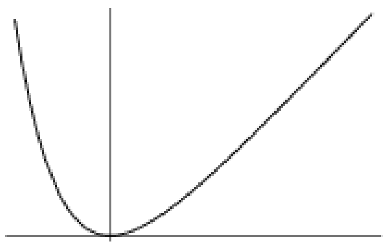

Figure 1.2: The *Toda potential* $\phi(r) = e^{-r} + r - 1$.

1.6 The Toda lattice

If we take the position of the *minimum* of $\phi(r)$ at the origin $r = 0$, the potential (1.14) takes the form:

$$\phi(r) = \frac{a}{b}e^{-br} + ar, \quad (ab > 0). \tag{1.15}$$

In the following, we use this expression for the interaction potential. The *lattice with exponential interaction* is now called the **Toda lattice**.

Observe that if we *expand* (1.15) assuming "small" r, we have

$$\phi(r) = const. + \frac{ab}{2}r^2 - \frac{ab^2}{6}r^3 + \cdots .$$

Thus, for sufficiently "small" amplitude motion, the lattice looks like a linear lattice with *spring constant*

$$\kappa = ab.$$

For a somewhat "larger" motion the nonlinear parameter of (1.4) is given by

$$\alpha = -\frac{b}{2}.$$

Considering the potential (1.15), from (1.1) and (1.8) we have the following **equations of motion**:

$$m\frac{d^2 y_n}{dt^2} = a(e^{-b(y_n - y_{n-1})} - e^{-b(y_{n+1} - y_n)})$$

and respectively:

$$m\frac{d^2 r_n}{dt^2} = a(-e^{-br_{n-1}} + 2e^{-br_n} - e^{-br_{n+1}}).$$

As for the equivalent **dual expression**, (1.10) yields:

$$\frac{d}{dt}\log(a + \dot{s}_n) = \frac{b}{m}(s_{n-1} - 2s_n + s_{n+1})$$

or

$$\frac{\ddot{s}_n}{a + \dot{s}_n} = \frac{b}{m}(s_{n-1} - 2s_n + s_{n+1}).$$

Differentiating the last equation, we have

$$\frac{d^2}{dt^2}\log\Big(1 + \frac{f_n}{a}\Big) = \frac{b}{m}(f_{n-1} - 2f_n + f_{n+1})$$

and integrating, we obtain

$$\log\Big(1 + \frac{\ddot{S}_n}{a}\Big) = \frac{b}{m}(S_{n-1} - 2S_n + S_{n+1}),$$

after choosing the integration constants appropriately. These are the *equations of motion for the lattice with exponential interaction*. The force of the spring is, by (1.11), given as

$$f_n = a(e^{-br_n} - 1) = \dot{s}_n.$$

Chapter 2

Some useful tools

2.1 Periodic systems

In this chapter we treat *periodic systems*. We are going to see that a *discrete* version of *Hill's Equation* comes in play. To solve the problem it is convenient to use the *spectrum* and the *auxiliary spectrum* for fixed boundary conditions of this equation. The *fundamental solutions* and the *discriminant* of the discrete Hill's Equation are introduced and actually play important roles. The discriminant is a *polynomial* of the spectrum. The *initial value problem* reduces to *inverse spectral theory*.

2.2 Matrix formalism

We discuss the initial value problem of a periodic lattice with exponential interaction between neighbors. We assume *no impurity* (we take $m = 1$) and a system composed of N particles. The equations of motions can then be written as:

$$\begin{cases} \dot{Q}_n = P_n \\ \dot{P}_n = e^{-(Q_n - Q_{n-1})} - e^{-(Q_{n+1} - Q_n)}, \end{cases} \tag{2.1}$$

where Q_n and P_n are to be interpreted as *displacement* and *momentum*, respectively. Further, we use the transformation:

$$\begin{cases} a_n = \frac{1}{2} e^{-(Q_{n+1} - Q_n)/2} \\ b_n = \frac{1}{2} P_n. \end{cases}$$

These are the so-called *Flaschka's variables*. Then the equations (2.1) give:

$$\begin{cases} \dot{a}_n = a_n(b_n - b_{n+1}) \\ \dot{b}_n = 2(a_{n-1}^2 - a_n^2). \end{cases} \tag{2.2}$$

Observe that these equations are not altered when we change the sign of a_n. Since we are interested in a periodic lattice consisting of N particles, we have

$$a_{n+N} = a_n, \quad b_{n+N} = b_n \tag{2.3}$$

We introduce the $N \times N$ matrices L and B by

$$L = \begin{bmatrix} b_1 & a_1 & & & & & & & a_N \\ a_1 & b_2 & & & & & & & \\ & & \ddots & & & & 0 & & \\ & & & b_{n-1} & a_{n-1} & & & & \\ & & & a_{n-1} & b_n & a_n & & & \\ & & & & a_n & b_{n+1} & & & \\ & & 0 & & & & \ddots & & \\ & & & & & & & b_{N-1} & a_{N-1} \\ a_N & & & & & & & a_{N-1} & b_N \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & -a_1 & & & & & & & a_N \\ a_1 & 0 & & & & & & & \\ & & \ddots & & & & 0 & & \\ & & & 0 & -a_{n-1} & & & & \\ & & & a_{n-1} & 0 & -a_n & & & \\ & & & & a_n & 0 & & & \\ & & 0 & & & & \ddots & & \\ & & & & & & & 0 & -a_{N-1} \\ -a_N & & & & & & & a_{N-1} & 0 \end{bmatrix}.$$

We write now (2.2) in matrix form. We have:

$$\frac{dL}{dt} = BL - LB \equiv [B, L]. \tag{2.4}$$

This is *Lax's formalism.* Lax's formalism is to write a *time evolution equation* in the form of (2.4). Here, it is essential that B is *antisymmetric* so that the matrix U defined by

$$\frac{dU}{dt} = BU, \quad U(0) = I$$

is *unitary.* In other words

$$\frac{dU^{-1}}{dt} = -U^{-1}B$$

$$UU^{-1} = U^{-1}U = I.$$

Therefore, we have

$$\frac{d}{dt}(U^{-1}LU) = 0$$

so that $U^{-1}LU$ is *time independent* and

$$L(t) = U(t)L(0)U^{-1}(t). \tag{2.5}$$

Thus, $L(t)$ and $L(0)$ are *unitarily equivalent.*

Let $\lambda(t)$ (a scalar function) and $\phi(t)$ (a $N \times 1$ - matrix valued function) denote the *eigenvalues* and *eigenfunctions* of $L(t)$ respectively; then at $t = 0$:

$$L(0)\phi(0) = \lambda(0)\phi(0)$$

and using (2.5), we have

$$L(t)U(t)\phi(0) = \lambda(0)U(t)\phi(0).$$

Comparing this with the equation

$$L(t)\phi(t) = \lambda(t)\phi(t), \tag{2.6}$$

at time t, we see that

$$\phi(t) = U(t)\phi(0)$$

or

$$\frac{d\phi}{dt} = B\phi \tag{2.7}$$

and

$$\lambda(t) = \lambda(0) = \lambda.$$

Therefore, the eigenvalues are *independent of time.* Furthermore, since all the elements of the matrix L are *real*, all the eigenvalues λ are also real. Thus, *motion in the lattice conserves its spectrum* λ (the isospectral deformation). By (2.6) the eigenvalues are determined by the *determinant equation*

$$\det(\lambda I - L) = 0.$$

The equation of motion (2.4) is *equivalent* to (2.6) and (2.7), namely to:

$$\begin{cases} L(t)\phi(t) = \lambda\phi(t) \\ \frac{d\phi}{dt} = B\phi. \end{cases}$$

To show this it is only necessary to differentiate (2.6) with respect to t, to obtain

$$\frac{dL}{dt}\phi + LB\phi = \lambda\frac{d\phi}{dt} \Rightarrow \frac{dL}{dt}\phi + LB\phi = \lambda B\phi = BL\phi$$

or

$$\Big(\frac{dL}{dt} - (BL - LB)\Big)\phi = 0$$

Thus, we have (2.4).

2.3 Discrete Hill's equation

Remember that for the *dynamical variables* we have chosen *Flaschka's variables* $a_n \equiv a_n(t)$, $b_n \equiv b_n(t)$. Remember also that the *periodic conditions* are expressed as $a_{n+N} = a_n$, $b_{n+N} = b_n$. It is convienient to consider an *infinite system* composed of such a_n and b_n. For this infinite system we discuss the equation:

$$(L\phi)_n \equiv a_{n-1}\phi(n-1) + b_n\phi(n) + a_n\phi(n+1) = \lambda\phi(n), \tag{2.8}$$

with a constant λ. Since the coefficients a_n and b_n are periodic the above is a *discrete* version of *Hill's Equation*

$$-\frac{d^2\phi}{dx^2} + (u-\lambda)\phi = 0, \quad u(x+\ell) = u(x),$$

and (2.8) is called the *discrete Hill's Equation.* This is a *difference relation of second rank* which can be solved for $\phi(n)$ when for example the values $\phi(0)$ and $\phi(1)$ at $n = 0$ and $n = 1$ respectively, are given. The function $\phi(n)$ obtained is *not* periodic in general and may tend to $+\infty$ as $n \to \pm\infty$. A solution is said to be *stable* if it is *bounded.* It will be shown below that **stable solutions correspond to bands of the spectrum** λ and **between these stable regions there are unstable regions** (gaps). Such a spectral structure depends on a_n, b_n and conversely the spectral structure *restricts* a_n, b_n to some extent. It will be shown that if a certain set of data regarding the initial conditions is given, we can determine the *future evolution of the lattice.* Since we want to determine a_n and b_n from the knowledge of the spectrum, this is an *inverse spectral problem.*

2.4 Some formulas

To begin with, we describe certain properties of the *discrete Hill's Equation.* Since this is a difference equation of the second rank, for every fixed λ we have *two linearly independent* solutions (called **fundamental solutions**). Any other arbitrary solution (for the same λ of course) can be written as a *linear combination* of these *fundamental solutions.* Let us denote the fundamental solutions of (2.8) by

$$\phi_1(n) \equiv \phi_1(n,\lambda), \quad \phi_2(n) \equiv \phi_2(n,\lambda).$$

Any arbitrary solution $\phi(n) \equiv \phi(n,\lambda)$ can be written as

$$\phi(n) = c_1\phi_1(n) + c_2\phi_2(n). \tag{2.9}$$

From now on, we consider the *fundamental solutions* defined by the *boundary conditions*

$$\begin{cases} \phi_1(0) = 1, & \phi_1(1) = 0 \\ \phi_2(0) = 0, & \phi_2(1) = 1 \end{cases} \tag{2.10}$$

Writing down (2.8) for $\phi_1(n)$, we have

$$\begin{cases} a_0 + a_1\phi_1(2) = 0 \\ b_2\phi_1(2) + a_2\phi_1(3) = \lambda\phi_1(2) \\ a_2\phi_1(2) + b_3\phi_1(3) + a_3\phi_1(4) = \lambda\phi_1(3) \\ \qquad\vdots \end{cases}$$

and solving successively, we obtain

$$\begin{cases} \phi_1(2) = -\frac{a_0}{a_1} \\ \phi_1(3) = -\frac{a_0}{a_1 a_2}(\lambda - b_2) \\ \phi_1(4) = -\frac{a_0}{a_1 a_2 a_3}\Big(\lambda^2 - (b_1 + b_2)\lambda + b_2 b_3 - a_2^2\Big) \\ \qquad\vdots \end{cases}$$

In general, by *induction* we have :

$$\phi_1(n) = -a_0\Big(\prod_{j=1}^{n-1} a_j\Big)^{-1}\Bigg(\lambda^{n-2} - \Big(\sum_{j=2}^{n-1} b_j\Big)\lambda^{n-3} + \cdots\Bigg), \quad n \geq 2. \tag{2.11}$$

Similarly, we have:

$$\phi_2(n) = \Big(\prod_{j=1}^{n-1} a_j\Big)^{-1}\Bigg(\lambda^{n-1} - \Big(\sum_{j=1}^{n-1} b_j\Big)\lambda^{n-2} + \cdots\Bigg), \quad n \geq 2. \tag{2.12}$$

2.5 The spectrum

Replacing n in (2.8) by $n + N$ and remembering (2.3) we see that $\phi_1(n + N)$ and $\phi_2(n+N)$ also satisfy (2.8). So, they can be expressed as linear combinations of $\phi_1(n)$ and $\phi_2(n)$. To be more precise: we consider the *shift operator* acting on sequences, i.e. $\mathcal{S} : \ell(\mathbb{Z}, \mathbb{R}) \to \ell(\mathbb{Z}, \mathbb{R})$ (where $\ell(\mathbb{Z}, \mathbb{R})$ is the space of real valued sequences from $\mathbb{Z}$) such that $\ell(\mathbb{Z}, \mathbb{R}) \ni \phi \longmapsto \mathcal{S}\phi \in \ell(\mathbb{Z}, \mathbb{R})$ and the n-term of the *sequence* $\mathcal{S}\phi$ is given by: $\Big(\mathcal{S}\phi\Big)(n) = \phi(n + 1)$ (or else is the $(n + 1)$-term of the sequence ϕ). In the same spirit, $\mathcal{S}^N$ denotes the composition of N-copies of the operator $\mathcal{S}$. In other words, $\left(\mathcal{S}^N\phi\right)(n) = \phi(n+N)$. Replacing now n in (2.8) by $n+N$ and remembering (2.3) yields

$$a_{n-1}\left(\mathcal{S}^N\phi\right)(n-1) + b_n\left(\mathcal{S}^N\phi\right)(n) + a_n\left(\mathcal{S}^N\phi\right)(n+1) = \lambda\left(\mathcal{S}^N\phi\right)(n)$$

But this says that $\mathcal{S}^N\phi_1$ and $\mathcal{S}^N\phi_2$ satisfy the *discrete Hill's equation* and so they can be written in terms of ϕ_1 and ϕ_2. Thus

$$\begin{cases} \mathcal{S}^N\phi_1 = m_{11}\phi_1 + m_{12}\phi_2 \\ \mathcal{S}^N\phi_2 = m_{21}\phi_1 + m_{22}\phi_2 \end{cases}$$

Or else

$$\begin{bmatrix} \mathcal{S}^N\phi_1 \\ \mathcal{S}^N\phi_2 \end{bmatrix} = M \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix},$$

where M is the 2×2 *matrix* with real entries

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}.$$

Equivalently

$$\begin{bmatrix} \phi_1(n+N) \\ \phi_2(n+N) \end{bmatrix} = M \begin{bmatrix} \phi_1(n) \\ \phi_2(n) \end{bmatrix} \tag{2.13}$$

Evaluating this at $n = 0$ and $n = 1$ using (2.10), we can determine the elements of M as:

$$M = \begin{bmatrix} \phi_1(N) & \phi_1(N+1) \\ \phi_2(N) & \phi_2(N+1) \end{bmatrix}. \tag{2.14}$$

M is called a **monodromy matrix**.

On the other hand, ϕ_1 and ϕ_2 satisfy

$$\begin{cases} a_{n-1}\phi_1(n-1) + b_n\phi_1(n) + a_n\phi_1(n+1) = \lambda\phi_1(n) \\ a_{n-1}\phi_2(n-1) + b_n\phi_2(n) + a_n\phi_2(n+1) = \lambda\phi_2(n). \end{cases}$$

Eliminating λ from these, we have

$$\begin{aligned} W &\equiv a_n\Big(\phi_1(n)\phi_2(n+1) - \phi_1(n+1)\phi_2(n)\Big) \\ &= a_{n-1}\Big(\phi_1(n-1)\phi_2(n) - \phi_1(n)\phi_2(n-1)\Big) \end{aligned}$$

This is a relationship concerning a **discrete version of the Wronskian W** for the *differences* $\phi_1(n+1)-\phi_1(n)$ and $\phi_2(n+1)-\phi_2(n)$. *Lifting* n to N on one side and *lowering* n to 1 on the other, we obtain

$$\begin{aligned} W &\equiv a_N\Big(\phi_1(N)\phi_2(N+1)-\phi_1(N+1)\phi_2(N)\Big) \\ &= a_0\Big(\phi_1(0)\phi_2(1)-\phi_1(1)\phi_2(0)\Big) \\ &= a_0 \end{aligned} \tag{2.15}$$

where we have used (2.10). Further, since $a_0 = a_N$ we have

$$\det M = \phi_1(N)\phi_2(N+1)-\phi_1(N+1)\phi_2(N) = 1. \tag{2.16}$$

For some very *special values* of λ the solutions $\phi(n)$ in (2.9) can be *periodic* but more generally we have solutions satisfying:

$$\phi(n+N) = \rho\phi(n)^1, \tag{2.17}$$

which means that for $n = 0, 1$, we have

$$\begin{cases} c_1\phi_1(N) + c_2\phi_2(N) = \rho c_1 \\ c_1\phi_1(N+1) + c_2\phi_2(N+1) = \rho c_2 \end{cases} \tag{2.18}$$

Multiplying these two equations we see that ρ is a *root* of the equation:

$$\rho^2 - \Delta(\lambda)\rho + 1 = 0 \tag{2.19}$$

Here,

$$\Delta(\lambda) \equiv \phi_1(N) + \phi_2(N+1) = tr\{M\} \tag{2.20}$$

which is called the **discriminant**. And solving (2.19) we have

$$\rho = \frac{1}{2}\Big(\Delta \pm \sqrt{\Delta^2 - 4}\Big) \tag{2.21}$$

2.6 The discriminant

When λ satisfies

$$\Delta^2(\lambda) \leq 4,$$

[1]Such a function is called *Bloch function.* The fact that such a function exists is known as the *Floquet Theorem* (See chapter 7 of [7])

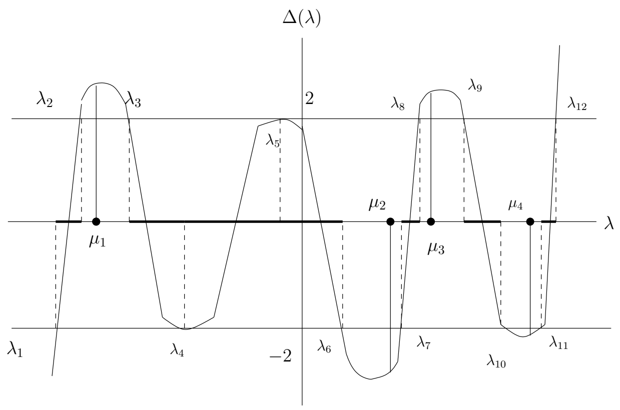

Figure 2.1: Schematic diagram of $\Delta(\lambda) \sim \lambda$ - (an example).

ρ in (2.21) is in general *complex* and $|\rho| = 1$. Thus, such λ **belongs to a *stable* region**, that is, the solution is *stable*. When $\rho = 1$, the *period* of the solution is N (since then, $\phi(n+N) = \phi(n)$) and when $\rho = -1$ the *period* is $2N$ (we have: $\phi(n+N) = -\phi(n)$).

When

$$\Delta^2(\lambda) > 4,$$

λ **belongs to an unstable region**.

The roots of the equations

$$\Delta(\lambda) - 2 = 0 \qquad \textit{and} \qquad \Delta(\lambda) + 2 = 0$$

(see Figure 2.1), belong to *eigenfunctions* with *periods* N and $2N$. It is easy to show that these are, respactively, eigenfunctions of L^+ and L^- defined by

$$L^{\pm} = \begin{bmatrix} b_1 & a_1 & & & & & & & \pm a_N \\ a_1 & b_2 & & & & & & & \\ & & \ddots & & & & 0 & & \\ & & & b_{n-1} & a_{n-1} & & & & \\ & & & a_{n-1} & b_n & a_n & & & \\ & & & & a_n & b_{n+1} & & & \\ & & 0 & & & & \ddots & & \\ & & & & & & & b_{N-1} & a_{N-1} \\ \pm a_N & & & & & & & a_{N-1} & b_N \end{bmatrix}. \tag{2.22}$$

Furthermore, the eigenvalues of these *symmetric matrices* are *real* (since the entries of the matrices are real). This is an easy fact. Indeed

$$\lambda \overline{u} \cdot u = \overline{u} \cdot \lambda u = \overline{u} \cdot Lu, \qquad \textit{and}$$

$$\overline{\lambda u} \cdot \overline{u} = u \cdot \overline{\lambda u} = u \cdot \overline{Lu} = \overline{Lu \cdot \overline{u}} = \overline{u} \cdot Lu$$

and then

$$(\lambda - \overline{\lambda})\overline{u} \cdot u = 0.$$

Therefore, λ is *real.* All the eigenvalues of (2.22) are thus real.

Now, equations (2.11), (2.12) and (2.20) yield:

$$\Delta(\lambda) = \Big(\prod_{j=1}^{N} a_j\Big)^{-1} \bigg(\lambda^N - \Big(\sum_{j=1}^{N} b_j\Big)\lambda^{N-1} + \cdots\bigg). \tag{2.23}$$

Since $\Delta^2(\lambda) = 4$ has $2N$ *roots* λ_l, $l = 1, 2, \ldots, 2N$, we have:

$$\Delta^2(\lambda) - 4 = \Big(\prod_{j=1}^{N} a_j\Big)^{-2} \prod_{l=1}^{2N} (\lambda - \lambda_l). \tag{2.24}$$

If the λ_l's are numbered in increasing order, we have:

$$\lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4 \leq \lambda_5 < \cdots < \lambda_{2N-2} \leq \lambda_{2N-1} < \lambda_{2N}. \tag{2.25}$$

Only the *intervals* $[\lambda_{2l}, \lambda_{2l+1}]$, $l = 1, 2, \ldots, N-1$ may *degenerate* to one point yielding *double roots.* The **spectrum** (2.25) consists of *two series interlaced*; one coming from $\Delta - 2 = 0$ and the other from $\Delta + 2 = 0$.

2.7 The spectrum - continued

To prove the above *alteration* (2.25) of the spectrum we use *two solutions* of (2.8) with **different** λ:

$$\phi(n) \equiv \phi(n, \lambda), \qquad \psi(n) \equiv \psi(n, \lambda').$$

Then, from (2.8) we have

$$\begin{aligned}(\lambda - \lambda')\phi(n)\psi(n) &= \psi(n)\Big(a_n\phi(n+1) + b_n\phi(n) + a_{n-1}\phi(n-1)\Big) \\ &\quad - \phi(n)\Big(a_n\psi(n+1) + b_n\psi(n) + a_{n-1}\psi(n-1)\Big) \\ &= a_n\Big(\phi(n+1)\psi(n) - \phi(n)\psi(n+1)\Big) \\ &\quad - a_{n-1}\Big(\phi(n)\psi(n-1) - \phi(n-1)\psi(n)\Big),\end{aligned}$$

so that

$$
\begin{aligned}
&(\lambda-\lambda')\sum_{n=1}^{N}\phi(n)\psi(n)\\
&\qquad = a_N\Big(\phi(N+1)\psi(N)-\phi(N)\psi(N+1)\Big)-a_0\Big(\phi(1)\psi(0)-\phi(0)\psi(1)\Big).
\end{aligned}
$$

Applying this to two *fundamental* solutions:

$$
\phi(n)\equiv\phi_1(n,\lambda),\qquad \psi(n)\equiv\phi_1(n,\lambda')
$$

we obtain

$$
\begin{aligned}
&(\lambda-\lambda')\sum_{n=1}^{N}\phi_1(n,\lambda)\phi_1(n,\lambda')\\
&\qquad = a_N\Big(\phi_1(N+1,\lambda)\phi_1(N,\lambda')-\phi_1(N,\lambda)\phi_1(N+1,\lambda')\Big)
\end{aligned}
$$

(always remembering that the *boundary conditions* for these functions satisfy: $\phi_1(1,\lambda)=\phi_1(1,\lambda')=0$). We let λ *converge* to λ' and write the *derivative* as

$$
\phi_1'\equiv\frac{d\phi_1}{d\lambda}.
$$

Then we have the expression for the *norm*

$$
\|\phi_1\|^2=\sum_{n=1}^{N}\phi_1^2(n)=a_N\Big(\phi_1(N)\phi_1'(N+1)-\phi_1(N+1)\phi_1'(N)\Big).
$$

Similar expressions are obtained for ϕ_2 and $\phi_1\cdot\phi_2$. Namely

$$
\phi_1\cdot\phi_2=\sum_{n=1}^{N}\phi_1(n)\phi_2(n)=a_N\Big(\phi_2(N)\phi_1'(N+1)-\phi_2(N+1)\phi_1'(N)\Big)
$$

and

$$
\|\phi_2\|^2=\sum_{n=1}^{N}\phi_2^2(n)=a_N\Big(\phi_2(N)\phi_2'(N+1)-\phi_2(N+1)\phi_2'(N)\Big).
$$

Now, if we solve these equations for $\phi_1'(N+1)$, $\phi_1'(N)$ etc. and substitute them into the **derivative of $\Delta(\lambda)$ with respect to λ**, we have

$$
\begin{aligned}
\frac{d\Delta}{d\lambda}&=\phi_1'(N)+\phi_2'(N+1)\\
&=\frac{1}{a_N}\sum_{n=1}^{N}\Big(\phi_2(N)\phi_1^2(n)-(\phi_1(N)-\phi_2(N+1))\phi_1(n)\phi_2(n)-\phi_1(N+1)\phi_2^2(n)\Big),
\end{aligned}
$$

or by using (2.16):

$$\frac{d\Delta}{d\lambda} = -\frac{\phi_1(N+1)}{a_N}\sum_{n=1}^{N}\Bigg(\Big(\phi_2(n) + \frac{\phi_1(N)-\phi_2(N+1)}{2\phi_1(N+1)}\phi_1(n)\Big)^2 - \frac{\Gamma}{4\phi_1^2(N+1)}\phi_1^2(n)\Bigg),$$

where by (2.16) and (2.20):

$$\begin{aligned}\Gamma &\equiv \Big(\phi_1(N)-\phi_2(N+1)\Big)^2 + 4\phi_1(N+1)\phi_2(N)\\ &= \Big(\phi_1(N)+\phi_2(N+1)\Big)^2 - 4\\ &= \Delta^2 - 4. \end{aligned} \tag{2.26}$$

Therefore, as long as $\Delta^2 - 4 < 0$, $\frac{d\Delta}{d\lambda}$ will have the same sign as $-\phi_1(N+1)$. If $\phi_1(N+1)$ were to vanish when $\Delta^2 - 4 < 0$, (2.16) would indicate that $\phi_1(N)\phi_2(N+1) = 1$ and that $|\Delta| = |\phi_1(N) + \frac{1}{\phi_1(N)}| \geq 2$, which would be a *contradiction*. So, $\phi_1(N+1)$ cannot vanish as long as $\Delta^2 - 4 < 0$. In other words, $\frac{d\Delta}{d\lambda}$ **can only change sign in the region where** $\Delta^2 - 4 > 0$. This proves the *alteration of the spectral points.*

Chapter 3

Analyzing the problem

3.1 The spectrum - revisited

Though the total number of the roots λ_j (i.e. the *spectrum*) of $\Delta^2(\lambda) - 4 = 0$ is $2N$ and is the same as the total number of the *dynamical variables* a_n, b_n, we cannot determine these dynamical variables even if all the λ_j's are given. Indeed, it can be shown that λ_j's **are not all independent**. Therefore, to solve the *inverse problem* we have to have further information. As such, we may use the **auxiliary spectrum** μ_j which is defined under *boundary conditions different from those for* λ_j.

But first we shall establish the above assertion about the *spectrum*. Let λ_j^+ be the roots of $\Delta(\lambda) = 2$ and λ_j^- be those of $\Delta(\lambda) = -2$. Then if we denote $a_1 a_2 \cdots a_N$ by A, we have

$$\Delta(\lambda) - 2 = A^{-1} \prod_{j=1}^{N} (\lambda - \lambda_j^+) = A^{-1} \prod_{j=1}^{N} (\lambda - \lambda_j^-) - 4,$$

which means that **if the λ_j^+'s are given then they determine** λ_j^- and vice versa.

Furhtermore, we shall show that **the simple roots of $\Delta^2(\lambda) = 4$ determine all the roots**. For this, let λ_i°, $i = 1, 2, \ldots, 2g+2$ be the *simple* roots of $\Delta^2(\lambda) = 4$ and λ_j be the remaining ones. Then, we may write

$$4 - \Delta^2(\lambda) = c_1 \prod_{i=1}^{2g+2} \left(1 - \frac{\lambda}{\lambda_i^\circ}\right) \prod_{j=1}^{N-g-1} \left(1 - \frac{\lambda}{\lambda_j}\right)^2$$

with a certain constant c_1 (where for brevity we already assumed that λ_j°, λ_j are *different from zero*).

On the other hand, let λ'_j be *the* root of $\Delta'(\lambda) \equiv \frac{d\Delta(\lambda)}{d\lambda} = 0$ which lies *between* λ°_{2j} and λ°_{2j+1}. This is *simple.* Since λ_j is a *double* root, it is also a root of $\Delta'(\lambda) = 0$. Therefore, we may write

$$\Delta'(\lambda) = c_2 \prod_{j=1}^{g} \Big(1 - \frac{\lambda}{\lambda'_j}\Big) \prod_{k=1}^{N-g-1} \Big(1 - \frac{\lambda}{\lambda_k}\Big)$$

with a certain constant c_2 (remember that since $\Delta(\lambda)$ is a polynomial of degree N with respect to λ, $\Delta'(\lambda)$ is a polynomial of degree $N-1$).

Thus, we have

$$\frac{\Delta'(\lambda)}{\sqrt{4 - \Delta^2(\lambda)}} = c_3 \frac{\prod_{j=1}^{g}(\lambda - \lambda'_j)}{\sqrt{\prod_{i=1}^{2g+2}(\lambda - \lambda^\circ_i)}},$$

where c_3 is a *constant.* Now, let

$$\Delta(\lambda) \equiv 2\cos\psi(\lambda),$$

where $\psi(\lambda)$ is a specific function. Then, we have

$$\psi'(\lambda) = \frac{d\psi}{d\lambda}(\lambda) = -\frac{\Delta'(\lambda)}{2\sin\psi(\lambda)} = \pm\frac{\Delta'(\lambda)}{\sqrt{4-\Delta^2}} = c_4 \frac{\prod_{j=1}^{g}(\lambda - \lambda'_j)}{\sqrt{\prod_{i=1}^{2g+2}(\lambda - \lambda^\circ_i)}},$$

with $c_4 = \pm c_3$. *Integrating*, we take

$$\psi(\lambda) = c_4 \int_{\lambda_1}^{\lambda} \frac{\prod_{j=1}^{g}(\widetilde{\lambda} - \lambda'_j)}{\sqrt{\prod_{i=1}^{2g+2}(\widetilde{\lambda} - \lambda^\circ_i)}} d\widetilde{\lambda}.$$

Therefore, for *sufficiently large* λ we have $\psi(\lambda) \sim c_4 \log\lambda + \cdots$. But we have $\Delta(\lambda) \sim A^{-1}(\lambda^N + \cdots)$. Thus, we see that

$$c_4 = \pm iN.$$

Also, since $\Delta^2(\lambda) < 4$ and $\Delta'(\lambda)$ is *real* for $\lambda \in [\lambda^\circ_{2j-1}, \lambda^\circ_{2j}]$, ψ' is *purely imaginary* in this region and we have

$$\psi(\lambda^\circ_{2j-1}) - \psi(\lambda^\circ_{2j}) = i\varepsilon,$$

where ε is a certain *real* number. However, λ°_{2j} is a *root* of $\Delta^2 = 4$ which implies that: $\cos\psi(\lambda^\circ_{2j}) = \pm 1$, or

$$\psi(\lambda^\circ_{2j}) = \pm m_j \pi$$

for some integer m_j. Therefore, we have

$$\pm 1 = \cos\psi(\lambda^\circ_{2j-1}) = \cos(\pm m_j\pi + i\varepsilon) = \pm\cosh\varepsilon.$$

So, we obtain $\varepsilon = 0$ and

$$\psi(\lambda^\circ_{2j-1}) = \psi(\lambda^\circ_{2j}).$$

Hence, we have the *simultaneous equations*:

$$\int_{\lambda^\circ_{2j-1}}^{\lambda^\circ_{2j}} \frac{\prod_{k=1}^{g}(\lambda - \lambda'_k)}{\sqrt{\prod_{i=1}^{2g+2}(\lambda - \lambda^\circ_i)}} d\lambda = 0, \quad j = 1, 2, \ldots, g.$$

These equations *determine* the λ_k's as functions of λ°_j which means that the function $\psi(\lambda)$ and then **all the roots of $\Delta^2(\lambda) = 4$ are determined by the simple roots λ°_i**.

3.2 The auxiliary spectrum (I)

Now, we define **the auxiliary spectrum μ_j** by:

(3.1) $$\phi_1(N+1, \mu_j) = 0, \quad j = 1, 2, \ldots, N-1.$$

Since $\phi_1(N+1)$ is *polynomial* of order $N-1$, there are $N-1$ roots μ_j. According to the definition of ϕ_1, we have: $\phi_1(1, \mu_j) = 0$ (remember that it also holds: $\phi_1(0, \mu_j) = 1$, but this doesn't affect the previous definition). Since by (3.1) and (2.16):

$$\phi_1(N, \mu_j)\phi_2(N+1, \mu_j) = 1,$$

we have

$$\Delta(\mu_j) = \phi_1(N, \mu_j) + \frac{1}{\phi_1(N, \mu_j)}$$

by (2.20). Hence

$$|\Delta(\mu_j)| \geq 2$$

which means that the μ_j **lie in the unstable regions** (or else, **spectral gaps**). All the μ_j are *simple* and each μ_j lies between λ_{2j} and λ_{2j+1}:

$$\lambda_{2j} \leq \mu_j \leq \lambda_{2j+1}, \quad j = 1, 2, \ldots, N-1.$$

We are going to show below the reason why the *auxiliary spectrum μ_j is simple.*

The equation for ϕ_1 is

(3.2) $$a_n\phi_1(n+1) + b_n\phi_1(n) + a_{n-1}\phi_1(n-1) = \lambda\phi_1(n).$$

Differentiating with respect to λ and writing $\phi_1' = \frac{d\phi_1}{d\lambda}$, we have

$$a_n\phi_1'(n+1) + b_n\phi_1'(n) + a_{n-1}\phi_1'(n-1) = \phi_1(n) + \lambda\phi_1'(n). \tag{3.3}$$

By *eliminating* λ from (3.2), (3.3) and then taking the sum we obtain

$$\sum_{n=1}^{N}\Big(a_n\phi_1(n+1)\phi_1'(n) - a_{n-1}\phi_1(n)\phi_1'(n-1)\Big)$$
$$- \sum_{n=1}^{N}\Big(a_n\phi_1(n)\phi_1'(n+1) - a_{n-1}\phi_1(n-1)\phi_1'(n)\Big)$$
$$= -\sum_{n=1}^{N}\phi_1^2(n)$$

But since the *auxiliary spectrum* satisfies

$$\begin{cases} \phi_1(N+1,\mu_j) = 0 \\ \phi_1'(0,\mu_j) = \phi_1'(1,\mu_j) = 0, \end{cases}$$

we have

$$a_N\phi_1(N,\mu_j)\cdot\phi_1'(N+1,\mu_j) = \sum_{n=1}^{N}\phi_1^2(n,\mu_j) \neq 0.$$

Therefore:

$$\begin{cases} \phi_1(N,\mu_j) \neq 0 \\ \dfrac{d\phi_1(N+1,\lambda)}{d\lambda}\Bigg|_{\lambda=\mu_j} \neq 0 \end{cases} \tag{3.4}$$

and the second equation of (3.4) implies that μ_j are **simple**.

3.3 The first results

Since by (2.11) and (3.1):

$$\phi_1(N+1,\lambda) = -a_0\Big(\prod_{j=1}^{N}a_j\Big)^{-1}\left(\lambda^{N-1} - \Big(\sum_{j=2}^{N}b_j\Big)\lambda^{N-2} + \cdots\right)$$
$$= -a_0\Big(\prod_{j=1}^{N}a_j\Big)^{-1}\prod_{j=1}^{N-1}(\lambda-\mu_j),$$

we obtain the relation

$$\sum_{j=2}^{N} b_j = \sum_{j=1}^{N-1} \mu_j$$

which can be written as

$$\sum_{j=1}^{N} b_j - b_1 = \sum_{j=1}^{N-1} \mu_j .$$

On the other hand, comparing (2.23), (2.24) we have

$$\widetilde{\Lambda} \equiv \sum_{j=1}^{N} b_j = \frac{1}{2} \sum_{j=1}^{2N} \lambda_j = const. \quad (independent \quad of \quad time). \tag{3.5}$$

Thus, if all the μ_j are *known*, b_1 is given as:

$$b_1 = \widetilde{\Lambda} - \sum_{j=1}^{N-1} \mu_j = \frac{1}{2} \sum_{j=1}^{2N} \lambda_j - \sum_{j=1}^{N-1} \mu_j . \tag{3.6}$$

When the *curve* $\Delta(\lambda)$ "cuts" the lines $y = \pm 2$, we have a *simple* root, and when it "touches" $y = \pm 2$ we have a *double* root. When λ_{2j} and λ_{2j+1} *coincide*, (i.e. we have a *double root*), μ_j also *coincides* with them ($\lambda_{2j} = \mu_j = \lambda_{2j+1}$). When λ_{2j} and λ_{2j+1} *differ* (so we are in the case of *simple* roots), μ_j lies *between* them (actually, μ_j **oscillates with time between** λ_{2j} **and** λ_{2j+1} as will be shown later). In the following discussion the *simple* roots λ_j play a central role. Changing a little bit the previously used notation for λ_j we henceforth denote the *simple* roots by:

$$\lambda_1, \lambda_2, \ldots, \lambda_{2g+2},$$

assuming that their number is $2g + 2$. For the *double* ones we write:

$$\lambda_{2j+1} = \lambda_{2j+2}, \quad j = g+1, \ldots, N-1.$$

We also change the numbering of μ_j so that:

$$\lambda_{2j} < \mu_j < \lambda_{2j+1}, \quad j = 1, 2, \ldots, g$$

and of course for the remaining *double* roots:

$$\lambda_{2j+1} = \mu_j = \lambda_{2j+2}, \quad j = g+1, \ldots, N-1.$$

Using these definitions, (3.6) is rewritten now as:

$$\begin{cases} b_1 = \Lambda - \sum_{j=1}^{g} \mu_j \\ \Lambda = \frac{1}{2} \sum_{j=1}^{2g+2} \lambda_j = const. \end{cases}$$

Therefore, b_1 **is obtained** by the **auxiliary spectrum** μ_j, $j = 1, 2, \ldots, g$.

3.4 Generalizing the procedure

A *similar argument* to the above will lead us to a formula for all b_k if we *shift all the suffixes n by a constant k.* Thus, let $\phi_1(n|k) \equiv \phi_1(n, \lambda|k)$ and $\phi_2(n|k) \equiv \phi_2(n, \lambda|k)$ denote the solutions of

$$a_{n+k-1}\phi(n-1) + b_{n+k}\phi(n) + a_{n+k}\phi(n+1) = \lambda\phi(n),$$

subject to the *boundary conditions*

$$\begin{cases} \phi_1(0|k) = 1, & \phi_1(1|k) = 0 \\ \phi_2(0|k) = 0, & \phi_2(1|k) = 1 \end{cases}$$

In terms of $\phi_1(n)$ and $\phi_2(n)$ we can *express* $\phi_1(n|k)$, $\phi_2(n|k)$ as

$$\begin{cases} \phi_1(n|k) = \alpha_1\phi_1(k+n) + \beta_1\phi_2(k+n) \\ \phi_2(n|k) = \alpha_2\phi_1(k+n) + \beta_2\phi_2(k+n) \end{cases}$$

For $n = 0$ and $n = 1$, we have

$$\begin{cases} 1 = \alpha_1\phi_1(k) + \beta_1\phi_2(k) \\ 0 = \alpha_1\phi_1(k+1) + \beta_1\phi_2(k+1) \\ 0 = \alpha_2\phi_1(k) + \beta_2\phi_2(k) \\ 1 = \alpha_2\phi_1(k+1) + \beta_2\phi_2(k+1) \end{cases}$$

Therefore, *eliminating* β_1 from the first two equations we have

$$\phi_2(k+1) = \alpha_1\Big(\phi_1(k)\phi_2(k+1) - \phi_1(k+1)\phi_2(k)\Big) = \frac{W}{a_k}\alpha_1 = \frac{a_0}{a_k}\alpha_1$$

where we have used the *Wronskian W* (take a look at (2.15)). Similarly

$$\begin{cases} \phi_1(k+1) = -\frac{W}{a_k}\beta_1 = -\frac{a_0}{a_k}\beta_1 \\ \phi_2(k) = -\frac{W}{a_k}\alpha_2 = -\frac{a_0}{a_k}\alpha_2 \\ \phi_1(k) = \frac{W}{a_k}\beta_2 = \frac{a_0}{a_k}\beta_2 \end{cases}$$

Thus:

$$\begin{cases} \phi_1(n|k) = \frac{a_k}{a_0}\Big(\phi_2(k+1)\phi_1(k+n) - \phi_1(k+1)\phi_2(k+n)\Big) \\ \phi_2(n|k) = \frac{a_k}{a_0}\Big(-\phi_2(k)\phi_1(k+n) + \phi_1(k)\phi_2(k+n)\Big) \end{cases} \tag{3.7}$$

It is easy to show now that the *discriminant* remains *invariant*

$$\Delta(\lambda|k) = \phi_1(N|k) + \phi_2(N+1|k) = \Delta(\lambda),$$

which means that **the roots** $\lambda_j(k)$ **of** $\Delta^2(\lambda|k) - 4 = 0$ **are invariant**:

$$\lambda_j(k) = \lambda_j(0) \equiv \lambda_j$$

We define now $\mu_j(k)$ by:

$$\phi_1(N+1, \mu_j(k)|k) = 0.$$

By an argument similar to the above, we see that:

$$\lambda_{2j} \leq \mu_j(k) \leq \lambda_{2j+1},$$

but in general:

$$\mu_j(k) \neq \mu_j \equiv \mu_j(0).$$

Similarly to (2.11), we have

$$\begin{aligned}
\phi_1(N+1, \lambda|k) &= -a_k \Big(\prod_{j=1}^{N} a_{j+k} \Big)^{-1} \Bigg(\lambda^{N-1} - (\sum_{j=2}^{N} b_{j+k}) \lambda^{N-2} + \cdots \Bigg) \\
&= -a_k \Big(\prod_{j=1}^{N} a_{j+k} \Big)^{-1} \prod_{l=1}^{N-1} (\lambda - \mu_l(k)) \qquad (3.8)
\end{aligned}$$

Therefore, by virtue of

$$\begin{aligned}
\sum_{j=2}^{N} b_{j+k} &= b_{k+2} + b_{k+3} + \cdots + b_N + b_{N+1} + \cdots + b_{k+N-1} + b_{k+N} \\
&= b_{k+2} + b_{k+3} + \cdots + b_N + b_1 + \cdots + b_{k-1} + b_k \\
&= \sum_{l=1}^{N} b_l - b_{k+1},
\end{aligned}$$

we have

$$\sum_{l=1}^{N} b_l - b_{k+1} = \sum_{j=1}^{N-1} \mu_j(k).$$

On the other hand, we obtain (3.5) or

$$\widetilde{\Lambda} \equiv \frac{1}{2} \sum_{j=1}^{2N} \lambda_j = \sum_{j=1}^{N} b_j.$$

Thus, we are led to the *formula*:

$$b_{k+1} = \widetilde{\Lambda} - \sum_{j=1}^{N-1} \mu_j(k).$$

By rearranging the numbering so that $\mu_j(k)$, $j = 1, 2, \ldots, g$ is between simple roots, we have:

$$\begin{cases} \lambda_{2j} < \mu_j(k) < \lambda_{2j+1}, & j = 1, 2, \ldots, g \\ \lambda_{2j+1} = \mu_j(k) = \lambda_{2j+2}, & j = g+1, \ldots, N-1. \end{cases}$$

Using the definitions we thus obtain the important formula:

$$\begin{cases} b_{k+1} = \Lambda - \sum_{j=1}^{g} \mu_j(k) \\ \Lambda = \frac{1}{2} \sum_{j=1}^{2g+2} \lambda_j \end{cases} \tag{3.9}$$

3.5 Riemann surfaces: a few facts

Remember that our task is to solve an *Inverse Problem* and since - as will be clear in the following pages - this is related to differentials on a *Riemann surface* [1], called **Abelian differentials**, we shall investigate them in this section providing a useful formula.

Our **Riemann surface** consists of *two sheets of complex planes* joined along the *branch cuts* $[\lambda_1, \lambda_2]$, $[\lambda_3, \lambda_4]$,. . .,$[\lambda_{2g+1}, \lambda_{2g+2}]$. We first make two complex spheres by *stereographic projection* of each sheet. Along the *banks of the cuts* we put $+$ and $-$ signs: the $+$ signs refer to the positive side of the imaginary axis and the $-$ signs to the negative side - take a look at Figure 3.1.

We *place* these spheres so that the corresponding *branch cuts* of the two spheres face each other: the $+$ banks facing the $-$ banks of the other sphere and vice versa. We *open the cuts* widely and join the facing banks by $g+1$ **tubes** and *paste*. (See Figure 3.2)

By *topological deformation* we have a surface consisting of *handles* (*tubes*) and *a* sphere which is made from the two facing spheres and the *tube* for the branch cut $[\lambda_1, \lambda_2]$. Then, we *cut* along the curves $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ on the surface passing from a point O as shown in Figure 3.3. Thus, we obtain a *simply connected region* S_0 as shown in the same Figure. We specify the *edges* of this region by the arrows $a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1}$, and *flatten the surface* to the **normal form**: $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$ (also called **the fundamental polygon of the Riemann surface** - take a look at Figure 3.4). In this case we say that our *Riemann surface* is of **genus** g.

Consider α_j, a *closed contour which surrounds the cut* $[\lambda_{2j+1}, \lambda_{2j+2}]$, $j = 1, 2, \ldots, g$ **on the upper sheet** of the *Riemann surface*. Also take β_j to be a

[1] for example see [3], [4], or [5]. Some introductory results can be found also in the appendix of [7].

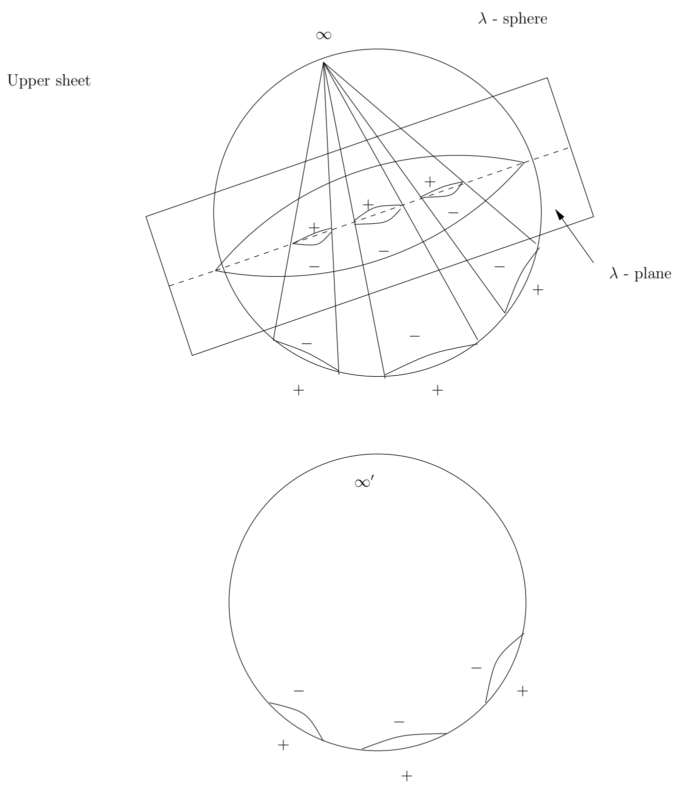

Figure 3.1: Stereographic projection of two Riemann sheets.

closed contour which starts at λ_2*, goes* **on the lower sheet** *as far as* λ_{2j+1}*, crosses* **back to the upper sheet** *and ends where it started at* λ_2.

So, a_1 gives the *path of integration* α_1, b_1 gives the *path* β_1, a_1^{-1} gives the *path* α_1^{-1} - which is the *reverse* of α_1, b_1^{-1} is the *reverse* of β_1 and so forth.

Let ω, η be *meromorphic differentials* on the *Riemann surface* [2] such that

$$\begin{cases} \int_{\alpha_i} \omega = A_i, & \int_{\beta_i} \omega = B_i \\ \int_{\alpha_i} \eta = A'_i, & \int_{\beta_i} \eta = B'_i \end{cases}$$

The A_i's are called the α-**periods** and the B_i's are named the β-**periods** of ω. The same holds for the **periods** of η.

Let Q' be a point on the curve a_i^{-1} which *corresponds* to Q on a_i as shown in Figure 3.5 and let

$$\omega = df,$$

[2]they are meromorphic functions on the Riemann surface times $d\lambda$.

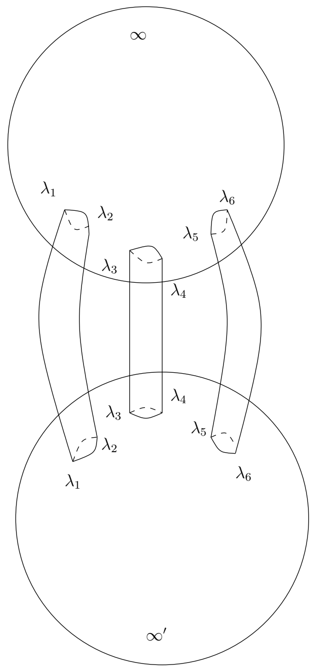

Figure 3.2: Topological mapping of the Riemann surface.

for a function f on the *Riemann surface*. Then, we have

$$\int_{QQ'} \omega = \int_{QO} df + \int_{OO'} df + \int_{O'Q'} df = \int_{OO'} df = \int_{\beta_i} df = B_i,$$

where we just used the fact that

$$\int_{QO} df = -\int_{O'Q'} df.$$

But this says exactly that

$$B_i = f(Q') - f(Q).$$

Similarly, we obtain

$$-A_i = f(R') - f(R)$$

where R' is the point on b_i^{-1} *corresponding* to R on b_i.

So, since it holds

$$\eta(Q') = -\eta(Q)$$

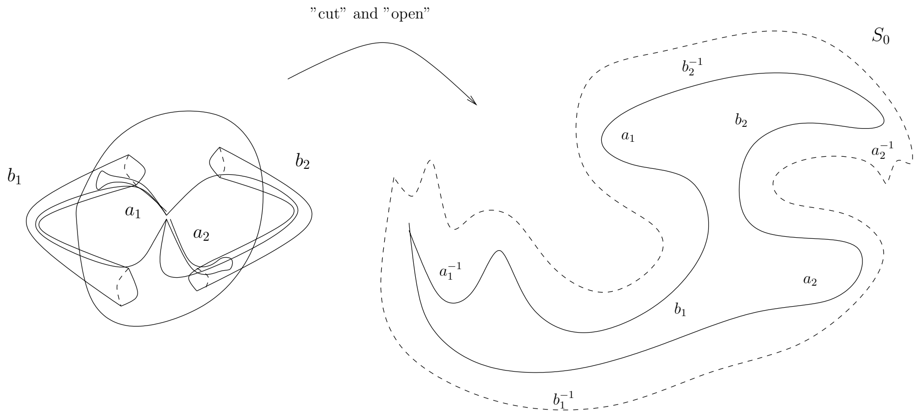

Figure 3.3: Canonical dissection (g=2).

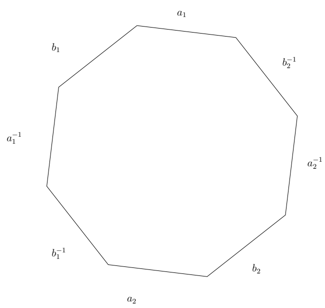

Figure 3.4: Normal form (g=2).

too, we have

$$\begin{aligned}\int_{\alpha_j+\beta_j+\alpha_j^{-1}+\beta_j^{-1}} f\eta &= \int_{\alpha_j} f\eta + \int_{\beta_j} f\eta - \int_{\alpha_j}(f+B_j)\eta - \int_{\beta_j}(f-A_j)\eta \\ &= -B_j\int_{\alpha_j}\eta + A_j\int_{\beta_j}\eta \\ &= A_jB_j' - B_jA_j'.\end{aligned}$$

Therefore

$$\int_C f\eta = \sum_{j=1}^{g}(A_jB_j' - B_jA_j'),$$

where the *contour* C *encircles* the *polygon* $a_1b_1\cdots a_g^{-1}b_g^{-1}$.

Some *definitions* now: meromorphic differentials on a Riemann surface are called **Abelian differentials**. The ones that have **no poles** are called **Abelian differentials of the first kind**. Differentials **with poles** but **vanishing residues**

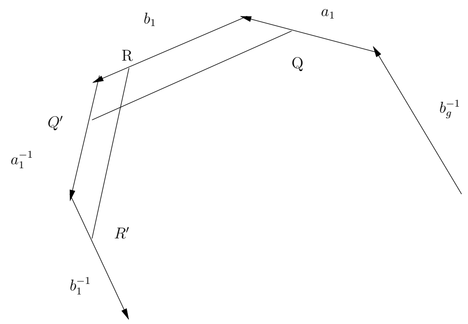

Figure 3.5: Paths of integration.

are called **Abelian differentials of the second kind** and the remaing ones, i.e. differentials **with nonvanishing residues** are **Abelian differentials of the third kind**.

Let $\omega = df = \omega^1$ be a *differential of the first kind* and $\eta = \omega^3$ be one of *the third kind*. Especially, assume that the latter has *poles* at P_l, $l = 1, 2, \ldots, m$ with *residues* C_l, $l = 1, 2, \ldots, m$. Then, by the *Residue Theorem*, we have

$$\sum_{j=1}^{g}(A_j B'_j - B_j A'_j) = \int_C f\omega^3 = 2\pi i \sum_{l=1}^{m} f(P_l)C_l = 2\pi i \sum_{l=1}^{m} C_l \int_{P_0}^{P_l} \omega^1 \tag{3.10}$$

where P_0 is a point on S_0 (such that $f(P_0) = 0$) and $f(P_l) = \int_{P_0}^{P_l} df = \int_{P_0}^{P_l} \omega^1$.

Furthermore, let ω^3 be an *Abelian differential of the third kind* such that it has residues $+1$ and -1 at two points P and Q respectively and furthermore, $\int_{\alpha_j} \omega^3 = 0$. Such a *differential* is called **a normal differential of the third kind**.

Also, consider $\omega^1 = \omega_k$ to be an *Abelian differential of the first kind* such that $\int_{\alpha_j} \omega^1 = \delta_{ij}$, δ_{ij} being *Kronecker's delta*. Such *differentials* are called **normalized Abelian differentials of the first kind**.

So, with all the above in mind we have

$$A'_j = 0 \quad and \quad A_j = \delta_{jk}, \quad j = 1, 2, \ldots.$$

Then by (3.10), we obtain

$$B'_k = 2\pi i \sum_{l=1}^{m} C_l \int_{P_0}^{P_l} \omega_k$$

or

$$\int_{\beta_k} \omega^3 = 2\pi i \sum_{l=1}^{m} C_l \int_{P_0}^{P_l} \omega_k. \tag{3.11}$$

If ω^3 has a *pole* at $P \equiv \mu_j(k)$ with *residue* $C_P = 1$ and at $Q \equiv \mu_j(0)$ with *residue* $C_Q = -1$, then (3.11) reduces to:

$$\int_{\beta_k} \omega^3 = 2\pi i \Big(\int_{P_0}^{P} \omega_k - \int_{P_0}^{Q} \omega_k \Big),$$

or

$$\int_{\beta_k} \omega^3 = 2\pi i \int_{Q}^{P} \omega_k. \tag{3.12}$$

This is the *formula* we promished from the begining.

Chapter 4

The role of Riemann surfaces

4.1 The auxiliary spectrum (II)

While the roots λ_j of $\Delta^2(\lambda) - 4 = 0$ are *independent of time*, each of $\mu_j(k)$ will *oscillate* in the interval $[\lambda_{2j+1}, \lambda_{2j+2}]$, $j = 1, 2, \ldots, g$. Before *solving the system* we shall note an important *relation* between $\mu_j(k)$ and $\mu_j(0)$.

The **Bloch function** defined by: $\phi(n+N) = \rho\phi(n)$ in (2.17) may be written except for a constant factor as

$$\phi^{\pm}(n) = \frac{c_1}{c_2}\phi_1(n) + \phi_2(n), \tag{4.1}$$

where $\frac{c_1}{c_2}$ is given by solving (2.18) as

$$\frac{c_1}{c_2} = \frac{\rho - \phi_2(N+1)}{\phi_1(N+1)} = \frac{\phi_1(N) - \phi_2(N+1) \pm \sqrt{\Delta^2 - 4}}{2\phi_1(N+1)}. \tag{4.2}$$

Alternatively, we may write

$$\frac{c_1}{c_2} = \frac{\phi_2(N)}{\rho - \phi_1(N)} = \frac{2\phi_2(N)}{-\phi_1(N) + \phi_2(N+1) \pm \sqrt{\Delta^2 - 4}}.$$

Equating these two expressions we once again obtain (2.26). On the other hand, using (4.1), (4.2) and (2.26) we have

$$\begin{aligned}\phi^+(n)\phi^-(n) &= \frac{1}{4\phi_1^2(N+1)}\Bigg(\Big(\phi_1(N)-\phi_2(N+1)+\sqrt{\Delta^2-4}\Big)\phi_1(n) \\ &\qquad + 2\phi_1(N+1)\phi_2(n)\Bigg) \\ &\quad \cdot\Bigg(\Big(\phi_1(N)-\phi_2(N+1)-\sqrt{\Delta^2-4}\Big)\phi_1(n)+2\phi_1(N+1)\phi_2(n)\Bigg) \\ &= \frac{1}{\phi_1(N+1)}\Bigg(\Big(\phi_1(N)-\phi_2(N+1)\Big)\phi_1(n)\phi_2(n) \\ &\qquad + \phi_1(N+1)\phi_2^2(n)-\phi_2(N)\phi_1^2(n)\Bigg) \\ &= \frac{a_0/a_{n-1}}{\phi_1(N+1)}\phi_1(N+1|n-1),\end{aligned}$$

where for the last line we have used (3.7) and (2.13), (2.14). *Changing* n to $n+1$, we have

$$\phi^+(n+1)\phi^-(n+1) = \frac{a_0}{a_n}\frac{\phi_1(N+1|n)}{\phi_1(N+1)}. \tag{4.3}$$

Now, by the definition of $\mu_j \equiv \mu_j(0)$ we have $\phi_1(N+1,\mu_j)=0$ where - by (2.26) - the *numerator* of (4.2) for the + sign is

$$\phi_1(N,\mu_j)-\phi_2(N+1,\mu_j)+\sqrt{\Delta^2(\mu_j)-4} = 2\sqrt{\Delta^2(\mu_j)-4},$$

which does not vanish in general. Therefore, $\phi^+(n+1)$ has a *pole* at $\lambda=\mu_j$. On the other hand, $\phi_1(N+1,\lambda|n)$ vanishes at $\lambda=\mu_j(n)$ so that by (4.3) we see that $\phi^-(n+1,\lambda)$ has a *zero* at $\lambda=\mu_j(n)$. Thus writing k for n, we have

$$\phi^+(k+1,\mu_j)=\infty, \qquad \phi^-(k+1,\mu_j(k))=0.$$

We can consider $\phi^+(k+1,\lambda)$ and $\phi^-(k+1,\lambda)$ as values of the function $\phi(k+1,\lambda)$ on the *two sheets* of the **Riemann surface S** with *branch cuts along the intervals between* λ_{2j-1} *and* λ_{2j}, $j=1,2,\ldots,g$, (the zeros of $(\Delta^2(\lambda)-4)^{1/2}$). On the **upper sheet** $(\Delta^2(\lambda)-4)^{1/2}$ has the value: $\sqrt{\Delta^2(\lambda)-4}$ and on the **lower sheet** has the value: $-\sqrt{\Delta^2(\lambda)-4}$. Thus, the *Bloch function* $\phi(k+1,\lambda)$ has **simple zeros at** $\mu_j(k)$ and **simple poles at** $\mu_j(0)$.

Moreover, $\phi(k+1,\lambda)$ **has a zero and a pole at infinity on the Riemann surface**. In fact, for *sufficiently large* λ since $\phi_1(N) \sim \lambda^{N-2}$, $\phi_2(N) \sim \lambda^{N-1}$, $\phi_1(N+1) \sim \lambda^{N-1}$, $\phi_2(N+1) \sim \lambda^N$ and $\Delta(\lambda) = \phi_1(N) + \phi_2(N+1) \sim \lambda^N$,

$$\phi^+(k+1) \sim \phi_2(k+1) \sim \lambda^k \tag{4.4}$$

and by (4.3), (4.4) and (3.8) we have

$$\phi^-(k+1) \sim \frac{\phi_1(N+1|k)}{\phi_1(N+1)\phi^+(k+1)} \sim \lambda^{-k}.$$

Therefore, $\phi(k+1)$ **has a pole of k^{th}-order at ∞ on the upper sheet and a zero of k^{th}-order at ∞' on the lower sheet**.

Consider now the *differential*:

$$\omega(k) = \Big(\frac{d}{d\lambda}\log\phi(k+1,\lambda)\Big)d\lambda, \tag{4.5}$$

which **has poles at $\mu_j(0)$ and $\mu_j(k)$ with residues $+1$ and -1 respectively and poles at ∞ and ∞' with residues $+k$ and $-k$ respectively**.

Since there are $2g+2$ *simple roots* among the $2N$ roots of $\Delta^2(\lambda) - 4 = 0$, we may write

$$\Big(\Delta^2(\lambda) - 4\Big)^{1/2} = (polynomial \quad of \quad \lambda)\cdot\Big(R(\lambda)\Big)^{1/2}$$

with

$$R(\lambda) = \prod_{j=1}^{2g+2}(\lambda - \lambda_j).$$

We introduce the *differentials*:

$$\omega_{(s)} = \frac{\lambda^s d\lambda}{\Big(R(\lambda)\Big)^{1/2}}, \quad s = 0,1,2,\dots,g-1$$

and define the following **base** $\{\omega_l\}$ for the space of holomorphic differentials:

$$\omega_l = \sum_{s=0}^{g-1} c_{l,s}\omega_{(s)}, \quad l = 1,2,\dots,g. \tag{4.6}$$

Such *differentials* **have no pole** and so are *Abelian differentials of the first kind.* The base $\{\omega_l\}$ consists of *normalized differentials of the first kind* when the *coefficients* $c_{l,s}$ are *normalized* in such a way that

$$\int_{\alpha_j}\omega_l = \delta_{jl}, \quad j,l = 1,2,\dots,g. \tag{4.7}$$

Also, we put

$$\int_{\beta_j} \omega_l = \tau_{jl}, \quad j, l = 1, 2, \ldots, g.$$

Here, α_j and β_j are the previously introduced curves on the *Riemann surface* so that the above two integrals on the *Riemann surface* are the α and β-**periods** (See Figure 4.1).

The function $\Big(R(\lambda)\Big)^{1/2}$ is **real** as $\lambda \to +\infty$ on the real axis. Thus, $\Big(R(\lambda)\Big)^{1/2}$ is **purely imaginary between** $[\lambda_{2j+1}, \lambda_{2j+2}]$ on the real axis and therefore the *coefficients* $c_{l,s}$ **are purely imaginary too**. Consequently, τ_{jk} are also **purely imaginary**. Finally, one can easily show that: $\tau_{jk} = \tau_{kj}$.

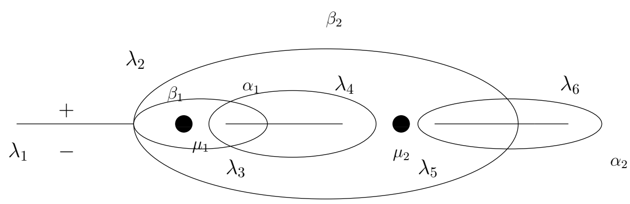

Figure 4.1: α-periods and β-periods. (g=2)

4.2 Jacobi's inversion problem

By the *Residue Theorem*, the sum of the residues vanishes. Let $\omega(P, Q)$ denote the *Abelian differential of the third kind* with residue $+1$ and -1 at P and Q respectively. We can add certain differentials of the first kind to the differential of the third kind so that all the α-periods vanish:

$$\int_{\alpha_j} \omega(P, Q) = 0.$$

Hence, $\omega(P, Q)$ is a normal differential of the third kind.

Let now Ω be an *Abelian differential of the second kind.* The differential (4.5) can be expressed as a linear combination of the differential Ω, the normal differential $\omega(P, Q)$ of the third kind and the normalized differential ω_j of the first kind. Being carereful with the poles, we can express (4.5) as

$$\omega(k) = \Omega + k\omega(\infty', \infty) + \sum_{j=1}^{g} \omega\Big(\mu_j(k), \mu_j(0)\Big) + \sum_{j=1}^{g} c_j \omega_j, \tag{4.8}$$

where c_j are *complex numbers.*

Since in (4.5), $\phi(k+1,\lambda)$ is a *single-valued* function on the *Riemann surface* S, we have

$$\begin{cases} \int_{\alpha_l} \omega(k) = 2\pi n_l i \\ \int_{\beta_l} \omega(k) = 2\pi m_l i \end{cases}$$

where n_l and m_l are certain *integers.* And since $\omega\Big(\mu_j(k), \mu_j(0)\Big)$ is *normalized* so that

$$\int_{\alpha_l} \omega\Big(\mu_j(k), \mu_j(0)\Big) = 0,$$

(4.8) yields (by virtue of (4.7))

$$c_j = 2\pi n_j i.$$

Therefore, the β_l-integral of (4.8) gives

$$k \int_{\beta_l} \omega(\infty', \infty) + \sum_{j=1}^{g} \int_{\beta_l} \omega\Big(\mu_j(k), \mu_j(0)\Big) + 2\pi i \sum_{j=1}^{g} n_j \int_{\beta_l} \omega_j = 2\pi m_l i. \tag{4.9}$$

However, from (3.12) it holds

$$\int_{\beta_l} \omega\Big(\mu_j(k), \mu_j(0)\Big) = 2\pi i \int_{\mu_j(0)}^{\mu_j(k)} \omega_l,$$

where as usual ω_l is the normalized differential of the first kind defined by (4.6). Thus, (4.9) yields

$$k \int_{\infty}^{\infty'} \omega_l + \sum_{j=1}^{g} \int_{\mu_j(0)}^{\mu_j(k)} \omega_l = -\sum_{j=1}^{g} n_j \tau_{lj} + m_l.$$

Though the first term on the left-hand side vanishes for $k=0$, the second term depends on the path of integration and is equal - for $k=0$ - to the right-hand side which does not depend on k. We therefore have:

$$\begin{aligned} \sum_{j=1}^{g} \int_{\mu_0}^{\mu_j(k)} \omega_l = k \int_{\infty'}^{\infty} \omega_l \\ + \sum_{j=1}^{g} \int_{\mu_0}^{\mu_j(0)} \omega_l - \sum_{j=1}^{g} n_j \tau_{lj} + m_l, \quad l = 1, 2, \ldots, g, \end{aligned} \tag{4.10}$$

where μ_0 is a *fixed* point on S which can be chosen arbitrarily. This is the required relationship between $\mu_j(k)$ and $\mu_j(0)$. If the λ_j are given and if we know $\mu_j(0)$, the right-hand side of (4.10) is a known quantity.

Now comes the power of *Riemann surface* theory. The above equations indicate that we can find the $\mu_j(k)$ from the $\mu_j(0)$. In order to prove how this actually happens, we have to state some definitions first.

Consider the (discrete) subset of $\mathbb{C}^g$

$$L(S) \equiv \{\mathbf{m} + \tau\mathbf{n} | \mathbf{m}, \mathbf{n} \in \mathbb{Z}^g\} \subset \mathbb{C}^g$$

and define the **Jacobian variety** of S

$$J(S) \equiv \mathbb{C}^g / L(S).$$

(It is a compact, commutative, g-dimensional, complex Lie group). Further, we define **Abel's map** with base point μ_0

$$\begin{array}{rccc} \mathcal{U}_{\mu_0}: & S & \rightarrow & J(S) \\ & P & \mapsto & \Big[\big(\int_{\mu_0}^{P} \omega_1, \ldots, \int_{\mu_0}^{P} \omega_g \big) \Big], \end{array}$$

where $[\mathbf{z}] \in J(S)$ denotes the equivalence class of $\mathbf{z} \in \mathbb{C}^g$. It is holomorphic (since $\omega_l, l = 1, 2, \ldots, g$ *are*) and well-defined as long as we choose the same path of integration for *all* ω_l.

We need to extend the Abel map to the set $Div(S)$ of **divisors** of S. A divisor $D \in Div(S)$ is a linear combination of points on S, i.e. $D = \sum_{P \in S} n_P P$, where only finitely many of the integers n_P are nonzero. The **degree** of the divisor D is the *integer*: $deg(D) = \sum_{P \in S} n_P$.

Now, the extension of Abel's map is defined by

$$\begin{array}{rccc} \mathcal{A}_{\mu_0}: & Div(S) & \rightarrow & J(S) \\ & D & \mapsto & \sum_{P \in S} D(P) \mathcal{U}_{\mu_0}(P). \end{array}$$

Note that the above sum is to be understood in $J(S)$.

Also, if we consider the set S_g (g being the genus of S) to be the set of all $D \in Div(S)$ such that $deg(D) = g$, we have **Jacobi's inversion problem** which is to *invert* $\mathcal{A}_{\mu_0}: \quad S_g \quad \rightarrow \quad J(S)$.

So, 4.10 can be seen in the above spirit. The left-hand side is the image of a divisor on S (with degree g) under the Abel map and the right-hand side is a point in $J(S)$.

In our case it is not necessary to obtain each $\mu_j(k)$ but we have to find $\sum_{j=1}^{g} \mu_j(k)$ in (3.9) to express b_{k+1} as a function of k.

4.3 The Riemann ϑ function

The *multidimensional ϑ function* (the *Riemann ϑ function*) is defined by:

$$\vartheta(\mathbf{u}) = \sum_{m_1,\dots,m_g=-\infty}^{+\infty} \exp\Big(2\pi i \sum_{j=1}^{g} m_j u_j + \pi i \sum_{j,k=1}^{g} \tau_{jk} m_j m_k\Big).$$

This can be also written as

$$\vartheta(\mathbf{u}) = \sum_{\mathbf{m}} \exp\Big(2\pi i \mathbf{m}\cdot\mathbf{u} + \pi i \mathbf{m}\cdot\tau\mathbf{m}\Big),$$

where $\mathbf{u}$ and $\mathbf{m}$ are *vectors* and τ is a *matrix*. More presicely

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_g \end{bmatrix}, \mathbf{m} = \begin{bmatrix} m_1 \\ \vdots \\ m_g \end{bmatrix}, \tau = \begin{bmatrix} \tau_{11} & \cdots & \tau_{1g} \\ \vdots & \ddots & \vdots \\ \tau_{g1} & \cdots & \tau_{gg} \end{bmatrix}$$

Now, we introduce the notation

$$e_k = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \qquad \tau_k = \begin{bmatrix} \tau_{1k} \\ \vdots \\ \tau_{kk} \\ \vdots \\ \tau_{gk} \end{bmatrix}.$$

We easily see that

$$\left\{ \begin{array}{l} \vartheta(\mathbf{u}+e_k) = \vartheta(\mathbf{u}) \\ \vartheta(\mathbf{u}+\tau_k) = e^{-2\pi i u_k - \pi i \tau_{kk}} \vartheta(\mathbf{u}) \end{array} \right. \tag{4.11}$$

The first equality is clear. The second one is derived by replacing m_k by $m_k + 1$. Indeed, we have

$$\begin{aligned} \vartheta(\mathbf{u}) &= \sum_{\mathbf{m}} e^{2\pi i(\mathbf{m}+e_k)\cdot\mathbf{u} + \pi i(\mathbf{m}+e_k)\cdot\tau(\mathbf{m}+e_k)} \\ &= e^{2\pi i e_k\cdot\mathbf{u} + \pi i e_k\cdot\tau e_k} \sum_{\mathbf{m}} e^{2\pi i\mathbf{m}\cdot\mathbf{u} + \pi i\mathbf{m}\cdot\tau\mathbf{m} + 2\pi i\mathbf{m}\cdot\tau e_k} \\ &= e^{2\pi i e_k\cdot\mathbf{u} + \pi i e_k\cdot\tau e_k} \sum_{\mathbf{m}} e^{2\pi i\mathbf{m}\cdot(\mathbf{u}+\tau e_k) + \pi i\mathbf{m}\cdot\tau\mathbf{m}} \end{aligned}$$

But this tells us that

$$\vartheta(\mathbf{u}+\tau_k) = e^{-2\pi i u_k - \pi i \tau_{kk}} \vartheta(\mathbf{u}).$$

We consider the function

$$F(\mathbf{u}) = \vartheta(\mathbf{u} - \overline{\mathbf{c}})$$

where $\overline{\mathbf{c}}$ is a *constant vector*; namely

$$\overline{\mathbf{c}} = \begin{bmatrix} \overline{c}_1 \\ \vdots \\ \overline{c}_g \end{bmatrix}.$$

Therefore

$$\begin{cases} F(\mathbf{u} + e_k) = F(\mathbf{u}) \\ F(\mathbf{u} + \tau_k) = e^{-2\pi i(u_k - \overline{c}_k) - \pi i \tau_{kk}} F(\mathbf{u}) \end{cases} \tag{4.12}$$

Now, let

$$u_l(P) = \int_{P_0}^{P} \omega_l \tag{4.13}$$

or

$$du_l = \omega_l = \sum_{s=0}^{g-1} c_{l,s} \frac{\lambda^s d\lambda}{\Big(R(\lambda)\Big)^{1/2}},$$

where P is a point on the *Riemann surface*. Moreover, we write

$$f(P) \equiv F(\mathbf{u}(P)) = \vartheta\Big(\mathbf{u}(P) - \overline{\mathbf{c}}\Big),$$

and let P_j be the *zeros* of $f(P)$:

$$f(P_j) = 0.$$

We are going to find the *number of zeros* of f. In the *vicinity* of P_j we have: $f \to re^{i\theta}$ and $\int \frac{df}{f} = i \int d\theta$. Therefore, the integral

$$n(f) = \frac{1}{2\pi i} \int_C \frac{df}{f} \tag{4.14}$$

over the *contour* C gives the *number of zeros* of f.

Let u_j^+ and u_j^- be *the values at the corresponding points* on a_k and a_k^{-1}, or b_k and b_k^{-1}. If P is on a_k, then

$$u_j^- = u_j^+ + \int_{\beta_k} \omega_j = u_j^+ + \tau_{kj}, \tag{4.15}$$

and

$$\text{(4.16)} \qquad \begin{cases} f^- = F(\mathbf{u}^-) = F(\mathbf{u}^+ + \tau_k) = e^{-2\pi i(u_k - \bar{c}_k) - \pi i \tau_{kk}} f^+ \\ df^- = e^{-2\pi i(u_k - \bar{c}_k) - \pi i \tau_{kk}} \Big(df^+ - 2\pi i f^+ du_k \Big). \end{cases}$$

Similarly, if P is on b_k

$$u_j^- = u_j^+ + \int_{\alpha_k^{-1}} \omega_j = u_j^+ - \delta_{jk},$$

and

$$f^- = F(\mathbf{u}^-) = F(\mathbf{u}^+ - e_k) = f^+.$$

Then, we have

$$\text{(4.17)} \qquad \begin{cases} \frac{df^-}{f^-} = \frac{df^+}{f^+} - 2\pi i \omega_k, \quad on \quad a_k \\ \frac{df^-}{f^-} = \frac{df^+}{f^+}, \quad on \quad b_k \end{cases}$$

and therefore (4.14) gives

$$\begin{aligned} n(f) &= \frac{1}{2\pi i} \sum_{k=1}^{g} \Big(\int_{\alpha_k} + \int_{\beta_k} \Big) \Big(\frac{df^+}{f^+} - \frac{df^-}{f^-} \Big) \\ &= \frac{1}{2\pi i} \sum_{k=1}^{g} 2\pi i \int_{\alpha_k} \omega_k \\ &= \sum_{k=1}^{g} \delta_{kk} = g \end{aligned}$$

Hence, we see that **the number of zeros** P_j **of** f **is equal to the genus** g.

4.4 An important formula

Since the P_j are *the zeros* of $f(P)$, we may write

$$\text{(4.18)} \qquad \sum_{j=1}^{g} u_l(P_j) = \frac{1}{2\pi i} \int_C u_l \frac{df}{f},$$

by the *Residue Theorem*. Using (4.15), (4.16) and (4.17) we rewrite the *integral* on the right-hand side as

$$\begin{aligned}\int_C u_l \frac{df}{f} &= \sum_{k=1}^{g}\Big(\int_{\alpha_k} + \int_{\beta_k}\Big)\Big(u_l^+ \frac{df^+}{f^+} - u_l^- \frac{df^-}{f^-}\Big)\\ &= \sum_{k=1}^{g}\int_{\alpha_k}\left(u_l^+\frac{df^+}{f^+} - (u_l^+ + \tau_{lk})\Big(\frac{df^+}{f^+} - 2\pi i\omega_k\Big)\right)\\ &\quad + \sum_{k=1}^{g}\int_{\beta_k}\left(u_l^+\frac{df^+}{f^+} - (u_l^+ - \delta_{lk})\frac{df^+}{f^+}\right)\\ &= 2\pi i\sum_{k=1}^{g}\tau_{lk}\int_{\alpha_k}\omega_k - \sum_{k=1}^{g}\tau_{lk}\int_{\alpha_k}\frac{df^+}{f^+}\\ &\quad + 2\pi i\sum_{k=1}^{g}\int_{\alpha_k}u_l^+\omega_k + \int_{\beta_l}\frac{df^+}{f^+}.\end{aligned} \tag{4.19}$$

On α_k if we start from $Q_0^{(k)}$ and finish at $Q_1^{(k)}$, then

$$f^+\Big(Q_1^{(k)}\Big) = F\Big(\mathbf{u}^+(Q_0^{(k)}) + e_k\Big) = f^+\Big(Q_0^{(k)}\Big)$$

by (4.12). Therefore

$$\int_{\alpha_k}\frac{df^+}{f^+} = \log f^+(Q_1^{(k)}) - \log f^+(Q_0^{(k)}) = 0.$$

Similarly, on β_l if we start from $\overline{Q}_0^{(l)}$ and finish at $\overline{Q}_1^{(l)}$, then

$$f^+\Big(\overline{Q}_1^{(l)}\Big) = F\Big(\mathbf{u}^+(\overline{Q}_0^{(l)}) + \tau_l\Big) = e^{-2\pi i\left(u_l(\overline{Q}_0^{(l)}) - \overline{c}_l - \pi i\tau_{ll}\right)}\cdot f^+\Big(\overline{Q}_0^{(l)}\Big).$$

Hence

$$\begin{aligned}\int_{\beta_l}\frac{df^+}{f^+} &= \log f^+\Big(u(\overline{Q}_1^{(l)})\Big) - \log f^+\Big(u(\overline{Q}_0^{(l)})\Big)\\ &= -2\pi i\Big(u_l(\overline{Q}_0^{(l)}) - \overline{c}_l\Big) - \pi i\tau_{ll}.\end{aligned}$$

Thus, using (4.18) and (4.19) we have the important *formula*:

$$\sum_{j=1}^{g} u_l(P_j) = \overline{c}_l - K_l + \sum_{k=1}^{g}\tau_{lk}, \tag{4.20}$$

where the K_l (called *Riemann* constants) are given by:

$$K_l = -\sum_{k=1}^{g}\int_{\alpha_k}u_l^+\omega_k + \frac{1}{2}\tau_{ll} + u_l(\overline{Q}_0^{(l)}).$$

4.5 The integral $\int_C \lambda \frac{df}{f}$.

Here, we consider the *integral*: $\int_C \lambda \frac{df}{f}$. By (4.17) we have

$$\frac{1}{2\pi i}\int_C \lambda \frac{df}{f} = \frac{1}{2\pi i}\sum_{k=1}^{g}\Big(\int_{\alpha_k} + \int_{\beta_k}\Big)\lambda\Big(\frac{df^+}{f^+} - \frac{df^-}{f^-}\Big) = \sum_{k=1}^{g}\int_{\alpha_k}\lambda\omega_k. \tag{4.21}$$

On the other hand, by a *Residue Calculation* we have

$$\frac{1}{2\pi i}\int_C \lambda \frac{df}{f} = \sum_{j=1}^{g}\lambda(P_j) + Res(\infty) + Res(\infty'), \tag{4.22}$$

where $\lambda(P_j)$ is the projection of the zero P_j of $f(P) = \vartheta\Big(\mathbf{u}(P) - \bar{\mathbf{c}}\Big)$ on $\mathbb{C}\cup\{\infty\}$. The *residue* at infinity can be calculated as follows. Let $\zeta^{-1} = \lambda$; then

$$\frac{1}{2\pi i}\int \lambda \frac{d}{d\lambda}\log\vartheta d\lambda = -\frac{1}{2\pi i}\int \lambda \frac{d}{d\zeta}\log\vartheta d\zeta.$$

Noting that the *direction* of the integration with respect to ζ is the *reverse* of the integration with respect to λ, we have

$$Res(\infty) \equiv Res\Big(\lambda\frac{d}{d\lambda}\log\vartheta, \lambda = \infty\Big) = Res\Big(\lambda\frac{d}{d\zeta}\log\vartheta, \zeta = 0\Big). \tag{4.23}$$

However, since $\frac{d}{d\zeta} = -\frac{1}{\zeta^2}\frac{d}{d\lambda}$ if we use the notation

$$D_l = \frac{\partial}{\partial u_l}$$

for the *upper sheet*, we have

$$\begin{aligned}
\frac{d}{d\zeta}\log\vartheta(\mathbf{u} - \bar{\mathbf{c}}) &= -\frac{1}{\zeta^2}\sum_{l=1}^{g}\frac{du_l}{d\lambda}D_l\log\vartheta(\mathbf{u} - \bar{\mathbf{c}}) \\
&= -\frac{1}{\zeta^2}\sum_{l=1}^{g}\sum_{j=0}^{g-1}c_{l,j}\frac{\lambda^j}{\sqrt{R(\lambda)}}D_l\log\vartheta(\mathbf{u} - \bar{\mathbf{c}}) \\
&= -\frac{1}{\zeta^2}\sum_{l=1}^{g}\frac{c_{l,g-1}\lambda^{g-1} + c_{l,g-2}\lambda^{g-2} + \cdots}{\sqrt{\prod_{j=1}^{2g+2}(\lambda - \lambda_j)}}D_l\log\vartheta(\mathbf{u} - \bar{\mathbf{c}}) \\
&= -\sum_{l=1}^{g}\Big(c_{l,g-1} + O(\zeta)\Big)D_l\log\vartheta(\mathbf{u} - \bar{\mathbf{c}}),
\end{aligned}$$

for "small" ζ. Thus

$$\lim_{\lambda\to\infty} \lambda \frac{d}{d\zeta} \log \vartheta(\mathbf{u} - \overline{\mathbf{c}}) = -\lim_{\zeta\to 0} \frac{1}{\zeta} \sum_{l=1}^{g} c_{l,g-1} D_l \log \vartheta(\mathbf{u} - \overline{\mathbf{c}})$$

and the *residue* at ∞ is given by (4.23) as

$$Res(\infty) = -\sum_{l=1}^{g} c_{l,g-1} D_l \log \vartheta(\mathbf{u}(\infty) - \overline{\mathbf{c}}).$$

For the *lower sheet*, the *sign* of $\sqrt{R(\lambda)}$ is different and we have

$$Res(\infty') = +\sum_{l=1}^{g} c_{l,g-1} D_l \log \vartheta(\mathbf{u}(\infty') - \overline{\mathbf{c}}).$$

Therefore, (4.21) and (4.22) give:

$$\sum_{j=1}^{g} \lambda(P_j) = \sum_{l=1}^{g} \int_{\alpha_l} \lambda\omega_l + \sum_{l=1}^{g} c_{l,g-1} D_l \log \frac{\vartheta\Big(\mathbf{u}(\infty) - \overline{\mathbf{c}}\Big)}{\vartheta\Big(\mathbf{u}(\infty') - \overline{\mathbf{c}}\Big)}. \tag{4.24}$$

Chapter 5

Solving the problem

5.1 Solution of Jacobi's inverse problem

Let

$$\sum_{j=1}^{g} u_l(P_j) \equiv X_l.$$

Then, from (4.20) we have

$$\bar{c}_l = X_l + K_l - \sum_{j=1}^{g} \tau_{lj}, \tag{5.1}$$

where u_l is given by (4.13). Therefore, for a given X_l the solution of *Jacobi's inverse problem*:

$$\sum_{j=1}^{g} \int_{P_0}^{P_j} \omega_l = X_l$$

is given as the *zeros* $P_1, P_2, \ldots, P_g$ of $\vartheta\Big(\mathbf{u}(P) - \bar{\mathbf{c}}\Big)$ with $\bar{\mathbf{c}}$ specified by (5.1).

Equation (4.10) can be written as:

$$\sum_{j=1}^{g} \int_{\mu_0}^{\mu_j(k)} \omega_l = X_l(k)$$

where:

$$X_l(k) = k \int_{\infty'}^{\infty} \omega_l + \sum_{j=1}^{g} \int_{\mu_0}^{\mu_j(0)} \omega_l - \sum_{j=1}^{g} n_j \tau_{lj} + m_l. \tag{5.2}$$

Thus, if the last term of the right-hand side of (5.1) is absorbed in $X_l(k)$ (namely, in the third term on the right-hand side of (5.2)), $\mu_1(k), \mu_2(k), \ldots, \mu_g(k)$ are given as the values of λ at the zeros P_j, $j = 1, 2, \ldots, g$ of $\vartheta\Big(\mathbf{u}(P) - \mathbf{X}(k) - \mathbf{K}\Big)$, or

$$\mu_j(k) = \lambda(P_j).$$

Thus, by (4.24) we have:

$$\sum_{j=1}^{g} \mu_j(k) = \sum_{l=1}^{g} \int_{\alpha_l} \lambda\omega_l + \sum_{l=1}^{g} c_{l,g-1} D_l \log \frac{\vartheta\Big(\mathbf{u}(\infty) - \mathbf{X} - \mathbf{K}\Big)}{\vartheta\Big(\mathbf{u}(\infty') - \mathbf{X} - \mathbf{K}\Big)}$$

and by (5.2) we may write

$$u_l(\infty) - X_l - K_l = kc_l + d_l$$

with

$$(5.3) \qquad \begin{cases} c_l = \int_{\infty}^{\infty'} \omega_l \\ d_l = -\sum_{j=1}^{g} \int_{\mu_0}^{\mu_j(0)} \omega_l + \sum_{j=1}^{g} n_j \tau_{lj} - m_l + \int_{\mu_0}^{\infty} \omega_l - K_l \end{cases}$$

By the *periodicity* (4.11) of ϑ, the second and third terms of d_l can be omitted.

Furthermore

$$u_l(\infty') = \int_{\mu_0}^{\infty'} \omega_l = \int_{\mu_0}^{\infty} \omega_l + \int_{\infty}^{\infty'} \omega_l.$$

Therefore we have:

$$(5.4) \qquad \sum_{j=1}^{g} \mu_j(k) = \sum_{l=1}^{g} \int_{\alpha_l} \lambda\omega_l + \sum_{l=1}^{g} c_{l,g-1} D_l \log \frac{\vartheta(k\mathbf{c} + \mathbf{d})}{\vartheta\Big((k+1)\mathbf{c} + \mathbf{d}\Big)},$$

where $\sum_{l=1}^{g} \int_{\alpha_l} \lambda\omega_l = \sum_{j=1}^{g} \lambda_j$, i.e. a *constant.* Inserting (5.4) into (3.9) we finally obtain:

$$(5.5) \qquad b_{k+1} = const. - \sum_{l=1}^{g} c_{l,g-1} D_l \log \frac{\vartheta(k\mathbf{c} + \mathbf{d})}{\vartheta\Big((k+1)\mathbf{c} + \mathbf{d}\Big)}.$$

5.2 Time evolution

The *equations of motion* for the *lattice*:

$$\begin{cases} \dot{Q}_n = P_n \\ \dot{P}_n = e^{-(Q_n - Q_{n-1})} - e^{-(Q_{n+1} - Q_n)} \end{cases}$$

can be written -as stated in chapter 2- as:

$$\begin{cases} \dot{a}_n = a_n(b_n - b_{n+1}) \\ \dot{b}_n = 2(a_{n-1}^2 - a_n^2) \end{cases}$$

where:

$$a_n = \frac{1}{2}e^{-\frac{Q_{n+1}-Q_n}{2}}, \quad b_n = \frac{1}{2}P_n.$$

We may also write them as:

$$\dot{L} = BL - LB \tag{5.6}$$

with the definition:

$$\begin{cases} (B\phi)_n \equiv -a_n\phi(n+1) + a_{n-1}\phi(n-1) \\ (L\phi)_n \equiv a_n\phi(n+1) + b_n\phi(n) + a_{n-1}\phi(n-1) \end{cases} \tag{5.7}$$

In addition, we impose the *periodic conditions*

$$Q_{n+N} = Q_n, \quad P_{n+N} = P_n, \quad n \in \mathbb{Z},$$

which imply

$$A \equiv \prod_{k=1}^{N} a_k = 2^{-N}.$$

We consider $\phi \equiv \phi(n)$, $n \in \mathbb{Z}$ which satisfy (2.8) or the equation with a *time-independent* parameter λ:

$$\begin{cases} L\phi = \lambda\phi \\ \dot{\lambda} = 0. \end{cases} \tag{5.8}$$

Differentiating (5.8) with respect to time, we obtain

$$\dot{L}\phi + L\dot{\phi} = \lambda\dot{\phi}.$$

We multiply (5.6) by ϕ and then substract it from the above equation to obtain

$$L(\dot{\phi} - B\phi) = \lambda(\dot{\phi} - B\phi).$$

Since this is of the same form as (5.8), if we let ϕ_1 and ϕ_2 be the *fundamental solutions* of (5.8) we see that $\dot{\phi}_1 - B\phi_1$ and $\dot{\phi}_2 - B\phi_2$ are given as linear combinations of ϕ_1 and ϕ_2. For example

$$(\dot{\phi}_1 - B\phi_1)_n = \alpha\phi_1(n) + \beta\phi_2(n) \tag{5.9}$$

where α and β are certain constants. We let $n = 0$ and $n = 1$ in (5.8), (5.9) and make use of the conditions: $\phi_1(0) = 1$, $\phi_1(1) = 0$, $\phi_2(0) = 0$ and $\phi_2(1) = 1$. Thus, we get

$$\begin{cases} (L\phi_1)_0 = b_0 + a_{-1}\phi_1(-1) = \lambda \\ (L\phi_1)_1 = a_1\phi_1(2) + a_0 = 0 \\ -(B\phi_1)_0 = -a_{-1}\phi_1(-1) = \alpha \\ -(B\phi_1)_1 = a_1\phi_1(2) - a_0 = \beta \end{cases}$$

and therefore

$$\alpha = b_0 - \lambda, \quad \beta = -2a_0.$$

Hence, (5.9) yields

$$\dot{\phi}_1(n) = -a_n\phi_1(n+1) + a_{n-1}\phi_1(n-1) + (b_0 - \lambda)\phi_1(n) - 2a_0\phi_2(n). \tag{5.10}$$

In the same way, we have

$$(\dot{\phi}_2 - B\phi_2)_n = \overline{\alpha}\phi_1(n) + \overline{\beta}\phi_2(n) \tag{5.11}$$

so that

$$\begin{cases} (L\phi_2)_0 = a_0 + a_{-1}\phi_2(-1) = 0 \\ (L\phi_2)_1 = a_1\phi_2(2) + b_0 = \lambda \\ -(B\phi_2)_0 = a_0 - a_{-1}\phi_2(-1) = \overline{\alpha} \\ -(B\phi_2)_1 = a_1\phi_2(2) = \overline{\beta} \end{cases}$$

and therefore

$$\overline{\alpha} = 2a_0, \quad \overline{\beta} = \lambda - b_0.$$

Thus, (5.11) yields

$$\dot{\phi}_2(n) = -a_n\phi_2(n+1) + a_{n-1}\phi_2(n-1) + 2a_0\phi_1(n) + (\lambda - b_0)\phi_2(n).$$

If we *differentiate*

$$\Delta(\lambda) = \phi_1(N) + \phi_2(N+1)$$

with respect to time considering the periodic boundary conditions, we obtain

$$\begin{aligned} \dot{\Delta}(\lambda) &= \dot{\phi}_1(N) + \dot{\phi}_2(N+1) \\ &= -a_N\phi_1(N+1) + a_{N-1}\phi_1(N-1) + (b_N - \lambda)\phi_1(N) - 2a_N\phi_2(N) \\ &\quad - a_{N+1}\phi_2(N+2) + a_N\phi_2(N) + 2a_N\phi_1(N+1) + (\lambda - b_N)\phi_2(N+1). \end{aligned}$$

But now we see that

$$\dot{\Delta}(\lambda) = 0.$$

Namely, $\Delta(\lambda)$ does not depend on time.

Now, we put $n = N+1$ in (5.7), (5.10) to obtain

$$\begin{cases} a_1\phi_1(N+2) + b_1\phi_1(N+1) + a_0\phi_1(N) = \lambda\phi_1(N+1) \\ \dot{\phi}_1(N+1) = -a_1\phi_1(N+2) + a_0\phi_1(N) + (b_0-\lambda)\phi_1(N+1) - 2a_0\phi_2(N+1). \end{cases}$$

Eliminating $\phi_1(N+2)$, we get

$$\dot{\phi}_1(N+1) = 2a_0\phi_1(N) + (b_0+b_1-2\lambda)\phi_1(N+1) - 2a_0\phi_2(N+1).$$

The *auxiliary spectrum* μ satisfies: $\phi_1(N+1,\mu) = 0$. Therefore, by (2.26) we have

$$\dot{\phi}_1(N+1,\lambda)\Big|_{\lambda=\mu} = 2a_0\Big(\phi_1(N,\mu) - \phi_2(N+1,\mu)\Big) = \pm 2a_0\sqrt{\Delta^2(\mu)-4}. \tag{5.12}$$

From (2.24) we may rewrite the right-hand side of (5.12):

$$\sqrt{\Delta^2(\lambda)-4} = A^{-1}Q(\lambda)\sqrt{R(\lambda)} \tag{5.13}$$

with

$$Q(\lambda) = \prod_{l=g+1}^{N-1}\Big(\lambda - \mu_l(0)\Big), \quad R(\lambda) = \prod_{j=1}^{2g+2}\Big(\lambda - \lambda_j\Big).$$

On the other hand, we may write

$$\phi_1(N+1,\lambda) = -a_0A^{-1}Q(\lambda)\prod_{j=1}^{g}\Big(\lambda - \mu_j(0)\Big). \tag{5.14}$$

For $l \geq g+1$, $\mu_l(0) = \lambda_{2l+1} = \lambda_{2l+2}$ is a constant. Therefore, if we differentiate (5.14) with respect to time keeping λ constant, we have

$$\dot{\phi}_1(N+1,\lambda) = A^{-1}a_0Q(\lambda)\sum_{j=1}^{g}\dot{\mu}_j(0)\prod_{l\neq j}\Big(\lambda - \mu_l(0)\Big) - \dot{a}_0A^{-1}Q(\lambda)\prod_{j=1}^{g}\Big(\lambda - \mu_j(0)\Big).$$

We then let $\lambda \to \mu_k(0)$ to obtain

$$\dot{\phi}_1(N+1,\lambda)\Big|_{\lambda=\mu_k(0)} = A^{-1}a_0Q\Big(\mu_k(0)\Big)\dot{\mu}_k(0)\prod_{l\neq k}\Big(\mu_k(0) - \mu_l(0)\Big). \tag{5.15}$$

Equating (5.12) with (5.15) and using (5.13), we have:

$$\dot{\mu}_k(0) = \mp\frac{2\sqrt{R(\mu_k(0))}}{\prod_{l\neq k}\Big(\mu_k(0) - \mu_l(0)\Big)}. \tag{5.16}$$

5.3 Lagrange's interpolation formula

Continuing our study, we prove the famous *Lagrange's Interpolation Formula*. We are going to use it in a while, so it is necessary to explain it first.

For that, consider a polynomial of degree $n+1$, namely

$$P(x) = \prod_{l=0}^{n}(x - x_l),$$

with constants $x_l \neq 0$. By *Cauchy's Integral Theorem*, for a *contour* C which encircles all the x_l, we have

$$\frac{1}{2\pi i}\oint_C \frac{z^m}{P(z)(z-x)}dz = \frac{x^m}{P(x)} + \sum_{j=0}^{m}\frac{x_j^m}{P'(x_j)(x_j - x)}.$$

On the other hand, this is the same as the integral over a circle of center O and "large enough" radius R. Hence

$$\frac{1}{2\pi i}\oint_C \frac{z^m}{P(z)(z-x)}dz = \lim_{R\to+\infty}\frac{R^m}{P(R)}.$$

Therefore we have

$$\frac{x^m}{P(x)} + \sum_{j=0}^{n}\frac{x_j^m}{P'(x_j)(x_j - x)} = \begin{cases} 0, & 0 \le m \le n \\ 1, & m = n+1 \end{cases}$$

where

$$P'(x_j) = \prod_{\substack{l=1 \\ l\neq j}}^{n}(x_j - x_l).$$

If we put $x = 0$, we have

$$\sum_{j=0}^{n}\frac{x_j^{m-1}}{\prod_{l=1,l\neq j}^{n}(x_j - x_l)} = \begin{cases} 0, & m < n+1 \\ 1, & m = n+1 \end{cases}$$

or equivalently:

$$\sum_{j=1}^{g}\frac{x_j^{s}}{\prod_{l=1,l\neq j}^{g}(x_j - x_l)} = \begin{cases} 0, & s < g-1 \\ 1, & s = g-1 \end{cases} \tag{5.17}$$

So, if $f(x)$ denotes a *polynomial* of $n^{th}-$*degree*, then since

$$f(x) = \sum_{m=0}^{n} c_m x^m,$$

we have:

$$f(x) = \sum_{j=0}^{n} \frac{f(x_j)P(x)}{P'(x_j)(x-x_j)}$$

which is exactly **Lagrange's Interpolation Formula**.

5.4 Solving the problem

We return now to the problem discussed earlier and go back to (5.3) and ask for the *rate of change of d_l with respect to time* which turns out to be:

$$\dot{d}_l(t) = -\sum_{j=1}^{g} \dot{\mu}_j(0,t) \frac{\omega_l}{d\lambda}\Big|_{\lambda=\mu_j(0)} = -\sum_{j=1}^{g} \dot{\mu}_j(0,t) \frac{\sum_{s=0}^{g-1} c_{l,s}\mu_j^s(0,t)}{\pm\sqrt{R\Big(\mu_j(0,t)\Big)}}. \tag{5.18}$$

On the right-hand side, using (5.16) we may write

$$\sum_{j=1}^{g} \frac{\dot{\mu}_j(0,t)\mu_j^s(0,t)}{\pm\sqrt{R\Big(\mu_j(0,t)\Big)}} = 2\sum_{j=1}^{g} \frac{\mu_j^s(0,t)}{\prod_{l\neq j}\Big(\mu_j(0,t)-\mu_l(0,t)\Big)} \tag{5.19}$$

which can be *simplified* by the use of **Lagrange's Interpolation Formula**.

Indeed, putting $x_j = \mu_j(0,t)$ in (5.17), from (5.19) we have

$$\sum_{j=1}^{g} \frac{\dot{\mu}_j(0,t)\mu_j^s(0,t)}{\mp\sqrt{R\Big(\mu_j(0,t)\Big)}} = \begin{cases} 0, & s < g-1 \\ 2, & s = g-1 \end{cases} \tag{5.20}$$

Equations (5.16), (5.20) were first derived by *Kac* and *van Moerbeke*. Substituting (5.20) into (5.18) we obtain:

$$\dot{d}_l(t) = -2c_{l,g-1}t$$

or

$$d_l(t) = d_l(0) - 2c_{l,g-1}t. \tag{5.21}$$

Thus, using (5.21) we can write (5.5) as:

$$b_{n+1}(t) = const. - \sum_{l=1}^{g} c_{l,g-1} D_l \log \frac{\vartheta\Big(n\mathbf{c}+\mathbf{d}(t)\Big)}{\vartheta\Big((n+1)\mathbf{c}+\mathbf{d}(t)\Big)}$$

where we note from (5.21) that

$$2c_{l,g-1}D_l = -\frac{d}{dt}.$$

Finally, since $b_n = \frac{P_n}{2} = \frac{\dot{Q}_n}{2}$ we have the final results:

$$\begin{cases} P_{n+1}(t) = \overline{P}_0 + \frac{d}{dt}\log\frac{\vartheta\left(n\mathbf{c}-\mathbf{c}'t+\delta'\right)}{\vartheta\left((n+1)\mathbf{c}-\mathbf{c}'t+\delta'\right)} \\ Q_{n+1}(t) = \overline{Q}_{n+1}(0) + \overline{P}_0 t + \log\frac{\vartheta\left(n\mathbf{c}-\mathbf{c}'t+\delta'\right)}{\vartheta\left((n+1)\mathbf{c}-\mathbf{c}'t+\delta'\right)} \end{cases}$$

where $\overline{P}_0$ and $\overline{Q}_{n+1}(0)$ are some *constants*. These results were first obtained by *Date* and *Tanaka*. In these *formulas*

$$c_l = \int_{\infty'}^{\infty} \omega_l, \qquad c'_l = 2c_{l,g-1} \qquad \textit{and} \quad \delta' = (\delta'_1, \ldots, \delta'_g) \equiv (d_1(0), \ldots, d_g(0))$$

are *phase constants* determined by the *initial conditions*.

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