Cube surroundings and tiling

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Abstract

In this work we show that the union of an axis-aligned cube and rectangle in $\mathbb{R}^d$ whose intersection has even codimension does not constitute a tile by translations in the special case where all sides of the rectangle have length greater than one. This complements a result of M. Kolountzakis who showed that in the odd codimension case, the union tiles the space by translations.

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1 Introduction

A theorem of Stein [3] states that if a small axis-aligned rectangle is removed from a corner of an axis-aligned cube in $\mathbb{R}^d$, the resulting notched cube T tiles $\mathbb{R}^d$ by translations; this means that there is a discrete set L of translation vectors such that $T + L = \mathbb{R}^d$ and any two distinct translates of T intersect only at the boundary. Kolountzakis gave in [1] a Fourier-analytic proof of the fact; in the process, he discovered that the same result holds if one attaches

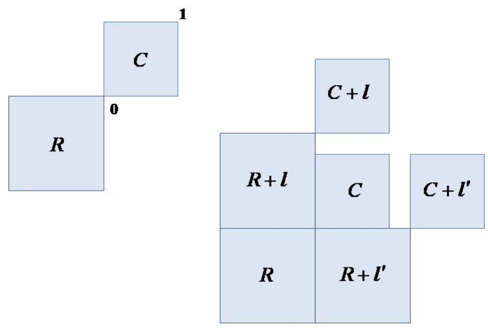

Figure 1: A union of rectangles in dimension 2, codimension 2 and "proof" of non-tilership.

a rectangle on the corner[1] of the original instead of removing it; the only essential condition was that the codimension of the supporting subspace of their intersection be odd.

It is easy to see in $\mathbb{R}^2$ that when this codimension is two, the resulting T does not tile for general sidelenghts; the same holds in $\mathbb{R}^3$ as well for codimension two. Therefore, it was natural to conjecture that it holds for all dimensions and even codimensions.

Observe that even in dimension two, there has to be a condition on the relative sizes of the two rectangles in order for their union *not* to tile the plane. For example, if the two rectangles have equal sidelengths, they definitely constitute a tile. Even if only one side of R is equal to the corresponding side of C, the resulting union is still a tile; see Figure 2. In all other cases, it is not a tile. The proof is immediate: if one tries to surround C completely with translates of T, one will immediately get a non-trivial intersection; see Figure 1. This is the intuition we generalize to higher dimensions.

Our main result is the following

Theorem 1. *Let C be the unit cube $[0,1]^d \in \mathbb{R}^d$ and R a rectangle $[l_1, r_1] \times \cdots \times [l_d, r_d]$ where $r_i - l_i > 1$ for all $i = 1 \cdots d$. Assume that the intersection $C \cap R$ contains a common vertex of C and R and has even codimension. Then $T := C \cup R$ does not tile $\mathbb{R}^d$ by translations.*

The notion of T tiling $\mathbb{R}^d$ by translations means that translates of T cover $\mathbb{R}^d$ and the intersection of any two has Lebesgue measure zero. In

[1] This means that the rectangles will have at least one common vertex.

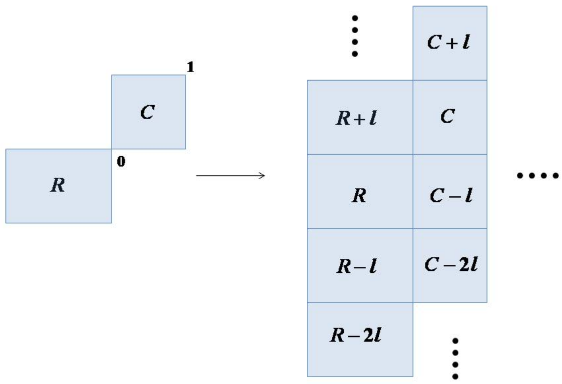

Figure 2: This union tiles: the vertical sides of the two constituents are equal.

our case, of course, since T is a piecewise linear object, intersections either contain open subsets of $\mathbb{R}^d$ or are contained in a finite union of hyperplanes, thus definitely having zero Lebesgue measure.

Using the assumption that all sides of R have length greater than one, we can reduce the tiling to a local result, where we can argue combinatorially. So far, we have not been able to adapt the argument to work in the general case, but we believe that the methods can be made to work in the general case by some modification of the reduction step from a tiling to a surrounding.

The rest of the paper is organized as follows: first, we give a sketch of the proof of Kolountzakis's result that in the odd codimension case T tiles $\mathbb{R}^d$ by translations; we follow [1] in the exposition. Then, after introducing some necessary notation, we outline the proof of Theorem 1 and we proceed with the proofs in the last section.

2 The case of odd codimension - Kolountzakis's result

In this section, we will center the unit cube at 0 since it makes computing the Fourier transforms more convenient. From the next section onwards, the unit cube will be centered at $\left(\frac{1}{2}, \cdots, \frac{1}{2}\right)$.

The proof of the theorem below is based on the following lemma, which can be proved using the Poisson summation formula:

Lemma 2. *Let $\Lambda = A\mathbb{Z}^d$ be a lattice in $\mathbb{R}^d$, $\Lambda^* = \{x \in \mathbb{R}^d : \forall \lambda \in \Lambda, \langle x, \lambda \rangle \in \mathbb{Z}\}$ be the dual lattice, and χ_T the characteristic function of a Lebesgue* *measurable set $T \subset \mathbb{R}^d$ of finite volume $m(T)$. Then T tiles $\mathbb{R}^d$ with Λ as the set of translations if and only if $\widehat{\chi_T}$ vanishes on $\Lambda^* \setminus \{0\}$ and the volume of Λ equals $m(T)$, the Lebesgue measure of T.*

The proof of more general versions of the lemma where χ_T is replaced by any L^1 function (with the conclusion appropriately modified) can be found in [1]. This observation allows us to reduce the question of tiling to identifying lattices in the zero set of the Fourier transform of χ_T. Note that in general, T may tile $\mathbb{R}^d$ but not by translations. In the case we consider here, however, we will find a *lattice* tiling. Of course, in the even codimension case, we will see that there is no tiling, lattice or not.

Theorem 3. *Let C be the axis-aligned unit cube in $\mathbb{R}^d$ centered at 0 and $R = \prod_{j=1}^d [\frac{1}{2} - \delta_j, \frac{1}{2}]$, where the δ_j can take any values as long as they satisfy $\delta_1 \cdots \delta_d \neq 1$. Assume $C \cap R$ has odd codimension. Then $T = C \cup R$ tiles $\mathbb{R}^d$ by translations.*

Proof. To say that $C \cap R$ has odd codimension is equivalent to $sgn(\delta_1 \cdots \delta_d) = -1$, i.e. an odd number of the δ's are negative (to see this, start with all δ's positive, then successively multiply each δ_j by -1 and observe that the codimension goes up by one at each step). In this case, we can write $\chi_T = \chi_C + \chi_R$ almost everywhere where χ_R has the form

$$\chi_R(x) = \chi_C\left(\frac{x - (1-\delta_1)/2}{|\delta_1|}, \cdots, \frac{x - (1-\delta_d)/2}{|\delta_d|}\right)$$

by the definition of R in the statement of the theorem. Taking Fourier transforms and using the form given above, we get

$$\widehat{\chi_T}(\xi) = \prod_{j=1}^d \frac{\sin \pi\xi_j}{\pi\xi_j} - F(\xi) \prod_{j=1}^d \frac{\sin \pi\delta_j\xi_j}{\pi\xi_j} \tag{1}$$

where $F(\xi) = e^{\pi i K(\xi)}$ with $K(\xi) = \sum_{j=1}^d (\delta_j - 1)\xi_j$. To find a lattice whose dual is in the zero set of the function defined above, start by defining that dual as the set of $\xi \in \mathbb{R}^d$ that satisfy

$$\begin{aligned} \xi_1 - \delta_1\xi_2 &= n_1, \\ &\cdots \\ \xi_d - \delta_d\xi_1 &= n_d \end{aligned} \tag{2}$$

for some $n_1, \cdots, n_d \in \mathbb{Z}$. It is easy to see that this set is the lattice $\Lambda^* = A^{-1}\mathbb{Z}^d$, where A is the following matrix:

$$A = \begin{pmatrix} 1 & -\delta_2 & & & \\ & 1 & -\delta_3 & & \\ & & \ddots & & \\ & & & 1 & -\delta_d \\ -\delta_1 & & & & 1 \end{pmatrix}.$$

Then we recover $\Lambda = A^\top \mathbb{Z}^d$. Note that $vol(\Lambda) = \det(A) = 1 - \delta_1 \cdots \delta_d$. This is precisely the volume of the set T, so if we can show that $\widehat{\chi_T}$ vanishes on the lattice, we are done.

Start by summing the equations (2) to get $K(\xi) = -(n_1 + \cdots + n_d)$; denote this value by K. If all the coordinates of ξ are non-zero we can rewrite (1) as

$$\widehat{\chi_T}(\xi) = \frac{1}{\pi^d \xi_1 \cdots \xi_d} \left(\prod_{j=1}^{d} \sin \pi \xi_j - (-1)^K \prod_{j=1}^{d} \sin \pi \delta_j \xi_j \right). \tag{3}$$

Observe from (2) that[2]

$$\sin \pi \xi_j = (-1)^{n_j} \sin \pi \delta_{j+1} \xi_{j+1},$$

from which we get $\widehat{\chi_T}(\xi) = 0$, since the factors in the two terms of (3) match one by one.

Now suppose some coordinate of ξ, say ξ_1, is zero. Arrange the coordinates $\xi_1, \cdots, \xi_d$ in a circle and let

$$I = \{\xi_m, \xi_{m+1}, \ldots, \xi_1, \ldots, \xi_{k-1}, \xi_k\}$$

be an interval around ξ_1 which is maximal with the property that all its elements are 0. Then $\xi_{m-1} \neq 0$ and $\xi_{k+1} \neq 0$ and from (2) we get

$$\xi_{m-1} - \delta_m \xi_m = n_m \quad \text{and} \quad \xi_k - \delta_{k+1} \xi_{k+1} = n_k. \tag{4}$$

Thus n_m and n_k are both nonzero, so that ξ_{m-1} and $\delta_{k+1}\xi_{k+1}$ are both non-zero integers and $\sin \pi \xi_{m-1} = \sin \pi \delta_{k+1} \xi_{k+1} = 0$. Therefore, both terms in (1) vanish and so does $\widehat{\chi_T}(\xi)$; this concludes the proof. □

We should note here that a much more general criterion for tiling with co-tilers that are not necessarily lattices is offered in [1], following the earlier paper [2]. The handy expression in terms of the dual lattice is replaced by the Fourier transform of a distribution $\sum_{a \in \Lambda} \delta_a$ for a discrete $\Lambda \subset \mathbb{R}^d$ of bounded density.

Using these Fourier-theoretic criteria to prove that a given T is *not* a tile is very difficult; even asking to prove lattice (non-) tiling requires to consider an arbitrary lattice in the zero set of χ_T and show that it does not have the correct volume. Proving non-tiling using such a method would be interesting, especially since one would have to obtain first some sort of control over the kind (relative volumes etc.) of lattices that can occur in such a zero set.

The rest of the paper is devoted to proving Theorem (1); we begin with some essential notation.

[2] Subscripts are reduced modulo d.

3 Notation

The following notions will be used repeatedly in subsequent sections. Sometimes, along with the definition we give some basic properties that can be trivially verified.

1. The coordinates of a vector $v \in \mathbb{R}^d$ are always denoted by $v_1, \cdots, v_d$ and the members of the standard basis of $\mathbb{R}^d$ by $e_i, i = 1, \cdots, d$.

2. Boldface numbers such as $\mathbf{0}$ stand for vectors with all coordinates equal to that number.

3. For a set $S \subset \mathbb{R}^d$ we write $\dim(S)$ for the largest dimension n in which S contains a homeomorphic image of an open set of $\mathbb{R}^n$.

4. In the sequel, a rectangle R in $\mathbb{R}^d$ will always refer to an axis-aligned rectangle of full dimension, i.e. a set of the form $\prod_{i=1}^d [a_i, b_i]$ with $a_i < b_i$ for all i. Sometimes we use the notation $[a, a]$ for the singleton $\{a\}$ when it appears in a product of intervals (see the notation for a face below).

5. We will occasionally need to refer to the intervals whose product defines the rectangle R above. The interval lying at the $j-$th position in the definition of R is called the $j-$th factor of R. Much of the time, we will not be able to write the $j-$th factor of a rectangle in its correct position; in these cases we subscript the interval with the correct index in order to avoid confusion. We also write R_j to denote $[a_j, b_j]$.

6. In the rectangle $R := [v_1^0, v_1^1] \times \cdots \times [v_d^0, v_d^1]$, the $d-1$-face or *facet* f_k^0 is, by definition, the closed set

$$f_k^0 := [v_1^0, v_1^1] \times \cdots \times \{v_k^0\} \times \cdots \times [v_d^0, v_d^1]$$

   and respectively

$$f_k^1 := [v_1^0, v_1^1] \times \cdots \times \{v_k^1\} \times \cdots \times [v_d^0, v_d^1]$$

   is the face f_k^1; of course the interval replaced by the singleton lies in the $k-$th position.

7. More generally, any $(d-s)-$tuple of directions $S \subset [d]$ (here and below, $[d]$ stands for $\{1, \cdots, d\}$) and ordered set $\epsilon := \{\epsilon_i \in \{0,1\} | i \in S\}$ determine a unique $s-$face of R

$$f_S^\epsilon := \bigcap_{i \in S} f_i^{\epsilon_i}.$$

Therefore f_S^ϵ is derived from R by replacing each interval corresponding to the position i in the direct product by the singleton v_i^0 or v_i^1 according to whether ϵ_i equals 0 or 1. It is well-known and immediately seen by the above that the number of k-faces of a rectangle R in $\mathbb{R}^d$ is precisely

$$2^{d-k}\binom{d}{k}.$$

We sometimes equivalently write S for the set of direction vectors $\{e_i | i \in S\}$ and write ϵ for the $d-s$-vector of zeros and ones corresponding to the directions in S; so $f_{\{1,2\}}^{\{0,1\}}$ may be written $f_{\{e_1,e_2\}}^{(0,1)}$.

8. When we have more than one rectangle, in order to distinguish between faces of different rectangles, we subscribe the rectangle before the face; for example, the facet of a rectangle R normal to e_1 and containing $\mathbf{1}$ will be denoted by ${}_Rf_1^1$.

9. We define the positive and negative closed i-halfspaces H_i^+ and H_i^- by

$$\begin{aligned} H_i^+ &= \{x \in \mathbb{R}^d | x_i \geq 0\} \\ H_i^- &= \{x \in \mathbb{R}^d | x_i \leq 0\} \end{aligned}$$

10. A rectangle $R := [v_1^0, v_1^1] \times \cdots \times [v_d^0, v_d^1]$ can be written as the intersection

$$\begin{aligned} R = \left(v_1^0 e_1 + H_1^+\right) \quad &\cap \quad \left(v_1^1 e_1 + H_1^-\right) \cap \\ &\cdots \quad \cap \left(v_d^0 e_d + H_d^+\right) \cap \left(v_d^1 e_d + H_d^-\right), \end{aligned}$$

of the $i-$slabs $\left(v_i^0 e_i + H_i^+\right) \cap \left(v_i^1 e_i + H_i^-\right)$ bounded by the boundaries of the corresponding halfspaces. In order to relieve some of the weight of this notation, we write

$$\begin{aligned} {}_RH_i^1 &:= v_i^1 e_i + H_1^- \\ {}_RH_i^0 &:= v_i^0 e_i + H_1^+ \end{aligned}$$

As the notation implies, ${}_RH_i^1$ has the face ${}_Rf_i^1$ at its boundary and similarly ${}_RH_i^0$ has the face ${}_Rf_i^0$ at its boundary. The intersections of various ${}_RH_i^\epsilon$ give 'octants' with lower-dimensional faces of R at their corners.

11. Given two axis-aligned rectangles R, R' in $\mathbb{R}^d$, we say that R and R' *touch* if they have a nonempty intersection of dimension at most $d-1$. Given any $s-$face ${}_Rf_S^\epsilon$ of R, we say that R' *partially covers* ${}_Rf_S^\epsilon$ if ${}_Rf_S^\epsilon \cap R'$ has dimension s, i.e. its s-interior is contained in R'. If R' contains the entire face and not only a ($s-$dimensional) part of it, we

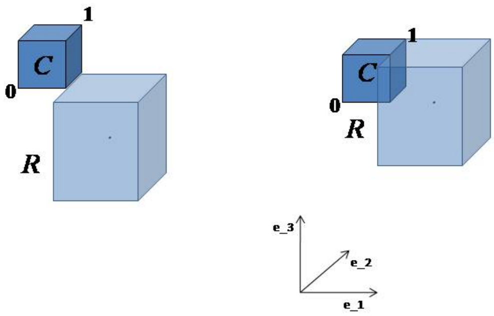

Figure 3: The leftmost tile has R attached to $f_{\{1,3\}}^{\{1,0\}}$; the rightmost one has R covering that face but is not attached to it; it is attached to f_1^1.

say it *fully* covers the face. If R' covers an $s-$face ${}_Rf_S^\epsilon$ of R but no face of dimension $t > s$, we say that R' is *attached* to ${}_Rf_S^\epsilon$. If it fully covers the face but no face of higher dimension, we say it is *completely attached* to the face. See figure 3 for clarification.

It is easy to see (see lemma 4 below) that if R' is partially or completely attached to ${}_Rf_S^\epsilon$, this is then the unique s-face R' covers and in fact the rectangles R' and R touch at an s-face of R', more specifically the face ${}_{R'}f_S^{\epsilon'}$; here $\epsilon' := (\epsilon'_1, \cdots, \epsilon'_s)$, where $\epsilon'_i + \epsilon_i = 1$. So if R' is attached to ${}_Rf_S^\epsilon$ the following must be true:

$$R \cap R' = {}_Rf_S^\epsilon \cap {}_{R'}f_S^{\epsilon'}$$

12. A local picture of tiling by T is given by the notion of a non-overlapping surrounding. Given rectangles R,R' in $\mathbb{R}^d$ we say we have a non-overlapping surrounding (from now on abbreviated as n.o.s) of R by translates of R' if the following hold: there is a finite set of $d-$vectors L such that $R \subset int(R \cup (R' + L))$ and for distinct $l_1, l_2 \in L$, $l_1 + R'$ and $l_2 + R'$ intersect possibly only at their boundaries.

13. Let $L + R$ be a surrounding of the cube and R_1, R_2 translates of R. R_2 is called a j-follower of R_1 (and R_1 a j-leader of R_2) if ${}_{R_1}f_j^0 \cap_{R_2} f_j^1$

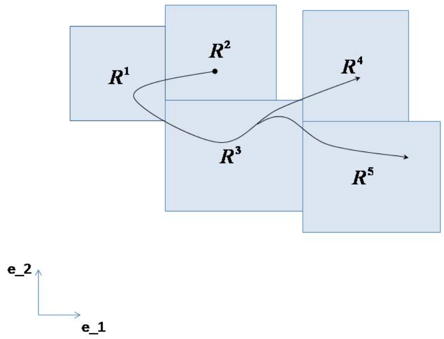

Figure 4: The positive 1-tube of R^2 is just R^2 while the negative 1-tube contains R^1 as well. The 1-tube of R^2 consists of all the rectangles in the figure.

has dimension $d-1$. The translates are called j-adjacent if one is a follower of the other. This extends to an equivalence relation we call j-connectedness so that R_1, R_2 are j-connected if either $R_1 = R_2$ or there is a $k \geq 2$ and a sequence of k translates $R'_1, \cdots, R'_k$ such that $R'_1 = R_1, R'_k = R_2$ and R'_j is adjacent to R'_{j+1} for $j = 1, \cdots, k-1$. The equivalence class under j-connectedness of an $l + R$ is called the j-tube of $l + R$, written $\tau_j(l + R)$. We usually identify translates of R in a particular surrounding with translation vectors and view $\tau_j(l)$ as a subset of L.

One also considers the positive j-tube $\tau_j^+(R_1)$ of all consecutive follower-leader chains starting with R_1, and respectively the negative j-tube $\tau_j^-(R_1)$ of all consecutive follower-leader chains ending with R_1. Thus R_k is in the positive j-tube if there is a chain $R_1, \cdots, R_k$ such that R_{i-1} is a follower of R_i (and *not* just adjacent). Note that it is not true in general that $\tau_j(R_1) = \tau_j^+(R_1) \cup \tau_j^+(R_1)$. For example, one can take an R in $\mathbb{R}^2$, attach a translate R^1 on top of R and slide it halfway across, then attach an R^2 at the exposed part of the bottom of R^1, which will then have a part of its top exposed and proceed similarly. This will make an infinite tube in the vertical direction by our definition, while for any translate the vertical plus or minus tube will consist of at most two translates. We will encounter this zigzag picture in the proof of the main lemma. Refer to figure 4 for a typical situation.

14. Suppose S is a possibly empty subset of $[d]$. For $r_i > 1$, we will en-

counter the intervals $I_i^+ = [0, r_i]$ and $I_i^- = [-r_i, 0]$ very often. Define $I(S)_i = I_i^+$ if $i \notin S$ and $I(S)_i = I_i^-$ if $i \in S$. We will abuse notation and write the rectangle $R = \prod_{i=1}^d I(S)_i$ as $\prod_{i\in S}[-r_i, 0]_i \times \prod_{i\notin S}[0, r_i]_i$; the subscript i under the interval indicates that the interval is in the $i-$th position. We chose this notation so that the reader can immediately see the positive and negative constituents of R.

15. The following table defines quantities dependent on a symbol ϵ_i and a rectangle $R := \prod_{i\in S}[-r_i, 0] \times \prod_{i\notin S}[0, r_i]$:

ϵ_i	ϵ_i'	$\dot{\epsilon_i}$	$\widehat{\epsilon_i}$	$\bar{\epsilon_i}$
0	1	0	$-$	1
1	0	$1 + r_i$	$+$	r_i

16. For an $s-$face f of the unit cube $C = [0, 1]^d$, f° denotes the $s-$dimensional interior of f, or equivalently the interior of f in the supporting hyperplane. For $\zeta > 0$, the interior $\zeta-$fattening F_f^ζ of a face f_S^ϵ is defined by

$$\prod_{i\notin S}(0, 1)_i \times \prod_{i\in S}(\epsilon_i e_i + (-\zeta, \zeta))_i\,;$$

visually, it is a $\zeta-$fattening of the interior but only in the directions normal to the supporting hyperplane of the face. Note again here the abuse of notation in the definition of the fattening.

For a general rectangle R, the ζ-fattening of a face is defined completely analogously.

17. Any s-face $f := f_S^\epsilon$ defines 2^{d-s} quadrants and respective *compartments.* The quadrant Q_f^δ, where $\delta = \{\delta_i \in \{+, -\} | i \in S\}$, is defined by

$$Q^\delta := \bigcap_{i\in S}\left(\epsilon_i e_i + H_i^{\delta_i}\right);$$

the compartment $C_f^{\delta,\zeta}$ is defined by $Q_f^\delta \cap F_f^\zeta$ for some suitable small ζ. For our purposes it will be enough to take $\zeta = \frac{1}{2}\min_{z\in Z}(z)$ where Z is the set of all $\{r_i - 1 | i = 1, \cdots, d\}$; recall that $r_i > 1$ for all i. We then drop the superscript ζ since it will not change throughout the paper. One can also write the compartment C_f^δ as

$$C_f^\delta = \prod_{i\notin S}(0, 1)_i \times \prod_{i\in S}\widehat{\delta_i}\cdot[0, \zeta) \tag{5}$$

where the obvious notation $+[0, \zeta) = [0, \zeta)$, $-[0, \zeta) = (-\zeta, 0]$ is used.

18. Of all compartments defined by a face $f := f_S^\epsilon$, the most important one is the exterior compartment $C_f^{\widehat{\epsilon}}$; it is the only compartment that translates of R *attached* to the face can occupy. In figure 5, we highlight the exterior compartments of the faces f_1^1 and $f_{\{1,2\}}^{\{1,0\}}$; it is obvious that

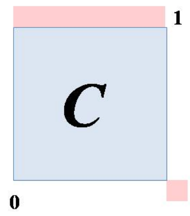

Figure 5: The 2-cube and exterior compartments of some faces; here $\zeta = \frac{1}{8}$.

the exterior compartments are precisely those occupied by rectangles attached to the face. If an exterior compartment is fully occupied by a translate attached to a higher-dimensional face (containing the given one), one cannot attach a translate to that face.

4 Outline of proof

In this section we outline the steps we will execute to prove the lack of tiling property. The data we are given are the unit cube $C = [0,1]^d$ centered at $\frac{1}{2}\mathbf{1}$, a set of directions S of even cardinality and a rectangle

$$R = \prod_{i \in S} [-r_i, 0]_i \times \prod_{i \notin S} [0, r_i]_i \tag{6}$$

with all $r_i > 1$. We assume $T := R \cup C$ tiles $\mathbb{R}^d$ by a set of translations L'. Without loss of generality, L' contains the zero vector. We show that $0+T$ cannot even be completely surrounded by translates of itself. A trivial lemma shows that in any tiling of $\mathbb{R}^d$ by translates of T, the C-constituent of T will only touch translates of R; this is because all r_i are greater than one, and if two translates of C touched, the corresponding translates of R would intersect nontrivially. Therefore, the tiling gives for each translate of C an n.o.s by a set of translates L of R; i.e. $l \in L$ if and only if $l+R$ touches the given translate of C. The choice of the translate of C to consider is irrelevant as any one will give a contradiction, so we look at $0+C$; all cube faces from now on will belong to that translate. Note that in this n.o.s, the translate $0+R$ is attached to the even codimension face $f^{\mathbf{0}}_{[d]\setminus S}$.

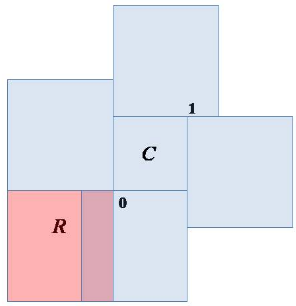

Figure 6: C is surrounded by translates of R, where translates attached to edges already cover the exterior compartments of vertices.

After that, we descend from $\mathbb{R}^d$ to $\mathbb{R}^{|S|}$ by taking successive intersections of the n.o.s with hyperplanes normal to the directions in $[d] \setminus S$. In the resulting n.o.s the translate $0 + R$ (we will denote the $|S'|-$dimensional intersections of C and R again by C and R to reduce notation) will be attached to the vertex $\mathbf{0}$, a face of even codimension $|S| - 0 = |S|$.

Next, from that n.o.s we derive a new n.o.s with the following properties:

1. The vertex $\mathbf{0}$ still has the copy $0 + R$ attached to it.

2. Each translate attached to some k-face f fully occupies the exterior compartments of precisely $\binom{k}{l}$ l-subfaces and does not occupy any part of the exterior compartment of any other face.

We then define two combinatorial quantities (basically enumerating the number of k-faces with an attached translate) and use the two properties stated above to derive a contradiction by showing that this number is zero for k even, in particular $k = |S|$. Thus, the vertex $\mathbf{0}$ (compare figure 6) will be covered by a translate of R attached to some higher dimensional face (a 1-face in the figure); equivalently, its exterior compartment will be already occupied by an R attached to a higher-dimensional face.

Schematically, the following reductions will lead to the desired contradiction:

1. From a tiling $L' + T$ we isolate an n.o.s of C by translates of R.

2. By taking slices of the n.o.s along hyperplanes normal to directions in $[d] \setminus S$ we get an n.o.s in $\mathbb{R}^{|S|}$ with the same properties as above and

furthermore we have an attached translate of R at the vertex $\mathbf{0}$, which is now an even codimension face of C.

3. From the given n.o.s we derive a new n.o.s by pushing the various translates of R to remove redundant translates and make the number of faces covered by each attached translate constant as a function of the dimension of the face the translate is attached to.

5 Proofs

5.1 Attached copies

Here we prove the assertion in item 11 of the notation section about how attached translates of R look and what they cover. The set S below is not related to the S defining R.

Lemma 4. *Let $R = [v_1^0, v_1^1] \times \cdots \times [v_d^0, v_d^1]$ and $R' = [w_1^0, w_1^1] \times \cdots \times [w_d^0, w_d^1]$ be rectangles in $\mathbb{R}^d$ and suppose R' is partially or completely attached to the s-face ${}_Rf_S^\epsilon$ of R. Then for each $e_{i_j} \in S$, if $\epsilon_j = 0$ then $w_{i_j}^1 = v_{i_j}^0$ and if $\epsilon_j = 1$ then $w_{i_j}^0 = v_{i_j}^1$. Furthermore, the intersection $R \cap R'$ equals the intersection ${}_Rf_S^\epsilon \cap {}_{R'}f_S^{\epsilon'}$ where as usual the vector ϵ' comes from ϵ by switching the zeros and ones in the coordinates (ϵ'_j is defined by $\epsilon_j + \epsilon'_j = 1$).*

Proof. It is obvious that the intersection of R and R' is a rectangle

$$R \cap R' = [\max(v_1^0, w_1^0), \min(v_1^1, w_1^1)] \times \cdots \\ \times [\max(v_d^0, w_d^0), \min(v_d^1, w_d^1)]$$

To say that R' is (perhaps partially) attached to ${}_Rf_S^\epsilon$ means that the intersection must have dimension s; therefore, exactly $d - s$ factors in the rectangle above must be singletons. However, the face ${}_Rf_S^\epsilon$ is the rectangle derived from R by replacing directions in S by singletons $v_{i_j}^0$ if $e_j = 0$ and $v_{i_j}^1$ if $e_j = 1$. Since $R \cap R'$ and ${}_Rf_S^\epsilon$ must have s-dimensional intersection, and precisely $d - s$ factors in the rectangle representation of ${}_Rf_S^\epsilon$ are singletons, exactly the same factors of $R \cap R'$ must be singletons (if there is a position where one of the two, $R \cap R'$ or ${}_Rf_S^\epsilon$ has a singleton while the other does not, the number of singletons in their intersection goes above $d - s$ and therefore the intersection is less than s-dimensional). Obviously then, the singletons in ${}_Rf_S^\epsilon$ must be equal to the singletons in $R \cap R'$. This gives us the equations:

$$\max(v_{i_j}^0, w_{i_j}^0) = \min(v_{i_j}^1, w_{i_j}^1) = v_{i_j}^{\epsilon_j}, \quad e_{i_j} \in S. \tag{7}$$

This immediately implies, as $v_{i_j}^0 \neq v_{i_j}^1$ and $w_{i_j}^0 \neq w_{i_j}^1$, that $v_{i_j}^{\epsilon_j} = w_{i_j}^{\epsilon'_j}$ as the lemma demands. These equations immediately show that $R \cap R'$ is also contained in the face ${}_{R'}f_S^{\epsilon'}$. □

5.2 Reduction to a n.o.s.

The following trivial lemma will reduce the question of tiling to the question of surrounding.

Lemma 5. *Let $C = [0,1]^d$, $S' \subset [d]$ with even cardinality and*

$$R = \prod_{i \in S'} [-r_i, 0] \times \prod_{i \notin S'} [0, r_i];$$

assume $T := R \cup C$ tiles $\mathbb{R}^d$ by a set of translations L containing 0. Then $0+C$ (denoted simply by C from now on) is completely surrounded by translates of R.

Proof. Since $L + T$ is a tiling, C is surrounded by translates of T. Suppose $C \cap (l + C) \neq \emptyset$. Since C is the unit cube, this means that $||l||_\infty \leq 1$. For each factor $R_j = [a,b]$ of R we have $|R_j| > 1$. Therefore, $[a,b] \cap ([a,b] + l_j) = [a + l_j, b]$ if $l_j > 0$ and $[a, b + l_j]$ if $l_j < 0$. In any case, since $|l_j| < 1$ while $b - a > 1$, we get that $R_j \cap (l_j + R_j)$ is a proper interval for every j, and therefore $R \cap (l + R)$ intersect nontrivially in $\mathbb{R}^d$, contradicting the tiling property.

Therefore, only translates of R possibly touch C. □

By the definition of attachment, R is attached to the even codimension face $f^{\epsilon}_{[d] \setminus S'}$ of C, with ϵ having all elements equal to zero. Together with this R, the rest of the copies touching C constitute an n.o.s $L' + R$ of C for some finite $L' \subset L$.

5.3 Reduction to the case $0 + R$ is attached to 0.

Lemma 6. *Suppose $L + R$ is an n.o.s of C in $\mathbb{R}^d$, as usual with $R := \prod_{k \in S} [-r_k, 0]_k \times \prod_{k \in [d] \setminus S} [0, r_k]_k$. Also suppose $i \notin S$. If L' is the set of $l \in L$ such that $l + R \cap (H_i^+)^\circ \neq \emptyset$ and simultaneously $l + R \cap \langle e_i \rangle^\perp \neq \emptyset$ then $L' + \pi_i(R)$ is an n.o.s of $\pi_i(C)$ (in an isomorphic copy of $\mathbb{R}^{d-1}$, where π_i is the projection to the $d-1$-dimensional subspace $\langle e_i \rangle^\perp$.*

Figure 7 exemplifies most of the details found in the proof.

Proof. Consider the intersection of the n.o.s with $\langle e_i \rangle^\perp$. $\pi_i(C)$ is obviously in this intersection. Define the set $\mathfrak{T}$ to be those translates $l + R$ that satisfy

$$((l + R) \cap H_i^+)^\circ \neq \emptyset$$

and additionally

$$(l + R) \cap \langle e_i \rangle^\perp \neq \emptyset.$$

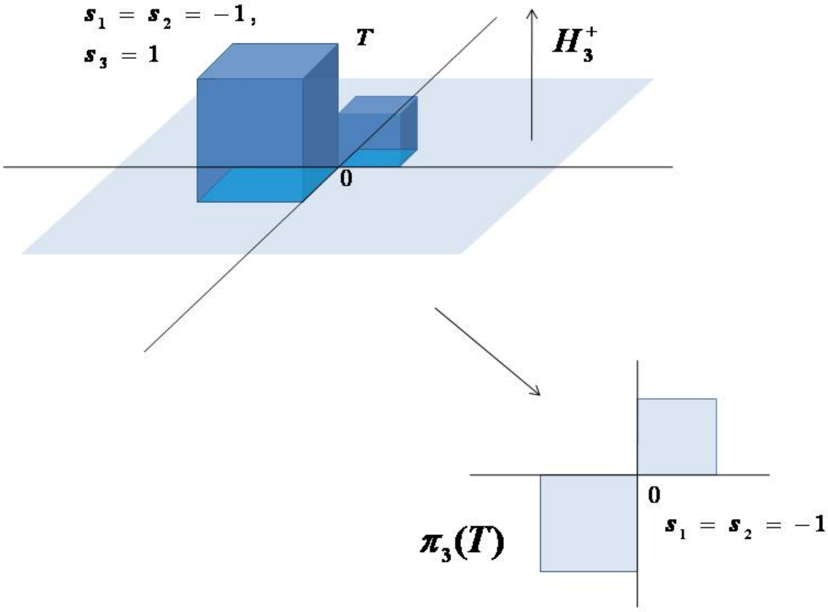

Figure 7: Going from dimension 3, codimension 2 to dimension 2, codimension 2.

Denote the set of the above l by $\mathfrak{L}$. Note that 0 is one of those translation vectors. Furthermore, for two $l^1, l^2 \in \mathfrak{L}$, the intersection

$$S := \left(\left((l^1+R)\cap\langle e_i\rangle^\perp\right)\cap\left((l^2+R)\cap\langle e_i\rangle^\perp\right)\right)$$

is trivial in $\langle e_i\rangle^\perp$. For, suppose it was not. Both l^1+R and l^2+R have full intersection with H_i^+; in fact, the first contains a set of the form

$$\left((l^1+R)\cap\langle e_i\rangle^\perp\right)\times I$$

and respectively the second contains a

$$\left((l^2+R)\cap\langle e_i\rangle^\perp\right)\times I'$$

for small intervals $I=[0,a]$ and $I'=[0,b]$ depending again on l_i^1, l_i^2. If S contained an open set in $\langle e_i\rangle^\perp$, this would mean that $(l^1+R\cap(l^2+R)$ would contain $S\times[0,\min(a,b)]$ and therefore an open set in $\mathbb{R}^d$, contradicting the nonoverlapping property.

Finally, it is obvious that $\pi_i(C)$ is covered by these translates (if there was any gap, it would extend to a gap in $\mathbb{R}^d$ because the translates are rectangles, and would contradict the surrounding property). The above says that in $\mathbb{R}^{d-1}\simeq\langle e_i\rangle^\perp$, $C'=[0,1]^{d-1}$ is covered by nonoverlapping translates of $R'=\prod_{j\in S}[-r_j,0]_j\times\prod_{j\notin S, j\neq i}[0,r_j]$ that do not intesect nontrivially. Thus we get an n.o.s in one dimension down, with one $i\notin S$ eliminated. □

The above shows that we can reduce the tiling to one dimension lower eliminating one $i\notin S$. In fact, checking the proof above shows that we can iterate this procedure, going in the same way from an n.o.s in $\mathbb{R}^d$ to an n.o.s in $\mathbb{R}^{|S|}$ by successively eliminating all $i\notin S$. There, the resulting R is simply $R=\prod_{i=1}^{|S|}[-r_i,0]$ and $C=[0,1]^{|S|}$ so $C\cap R=\{\mathbf{0}\}$, which means we have reduced the problem to the case of even dimension and $0+R$ attached to $\mathbf{0}$.

Thus from now on d will be even, $R=\prod_{i=1}^{d}[-r_i,0]$, L will consist of $d-$vectors, $L+R$ will be an n.o.s of $C=[0,1]^d$ and $0+R$ will be attached to $\mathbf{0}$. Finally, since S will not appear anymore in relation to R, we reserve this symbol to describe faces as defined in item 7 of the notation section.

5.4 Properties of the surrounding

Here we list some properties of the surrounding that will be used both as a 'visual' aide and in the proof proper in the result of the next section. The following lemma describes the entries of the translation vectors of translates attached to particular faces.

Lemma 7. *Let $l\in L$ so that $l+R$ is attached to a face f_S^ϵ for some S,ϵ. Then*

1. $0 \leq l_i \leq 1 + r_i$ *for* $i = 1, \cdots, d$.

2. *For* $i \in S$, $l_i = \dot{\epsilon}_i$.

3. *For* $i \notin S$, $0 < l_i < r_i + 1$.

Proof. By definition, $(l + R) \cap C \neq \emptyset$ and for this to hold, since C is in the positive quadrant, one cannot slide R in a negative direction. If $l_i < 0$ for some i, then $l + R \subset (H_i^-)^\circ$, while $C \subset H_i^+$. Therefore, $l_i \geq 0$ for all i, giving the left hand inequality in 1.

The other inequality follows similarly: if $l_i > 1 + r_i$ for some i, then $l + R \subset (e_i + H_i^+)^\circ$ while of course $C \subset (e_i + H_i^-)$.

The second item follows direcly from lemma 4 by replacing R with $l + R$ and R' by C and working out the equations. The third one follows from that lemma as well since if either of the two extremes held, the translate would be attached to a lower-dimensional face. □

5.5 Getting the new n.o.s from the old.

In this section we describe how to slide translates in the n.o.s to get a surrounding whose incidence combinatorics are immediately apparent. We start with translates attached to facets and alter the surrounding one dimension at a time.

Lemma 8. *Suppose* $L + R$ *is an n.o.s of* C. *Then there is* L' *such that* $L' + R$ *is an n.o.s of* C *and the following holds: for all* $l \in L'$, *if* $l + R$ *is attached to* f_S^ϵ, *then* $l_i = \dot{\epsilon}_i$ *for* $i \in S$, *and* $l_i = 1$ *or* $l_i = r_i$ *for* $i \notin S$.

This condition gives a large number of implications described and proved in the next lemma.

Proof. We describe the procedure to get L' from L in an algorithmic format. Each step is easily seen to accomplish its stated functions. After the first iteration, all facets of C will have a single translate attached to them, all lower-dimensional faces covered by translates attached to a facet will in fact be completely covered by them (i.e. their exterior compartment of each lower-dimensional face is either unoccupied or completely contained in a unique translate attached to some face) and the configuration $L + R$ will still be an n.o.s. Subsequent iterations do the same for the lower-dimensional faces. Below we give the pseudocode implementing the entire procedure.

Begin Modify
Input: R,L{Output is going to be the modified L.}
for $k = d - 1$ to 0 step -1 **do**
  {Traverse through the k-faces of C, so $|S| = d - k$.}
  for all $f_S^\epsilon \subset C$ **do**
    if there exists at least one translate $l + R$ attached to f_S^ϵ **then**

{In the $k = d - 1$ case this is always true. We pick a random such translate.}
for all $j \notin S$ **do**
if $l_i \in (0, 1)$ **then**
for all $l' \in \tau_i(l + R)$ **do**
$l' \leftarrow l' + (1 - l_i)e_j$ {This is the sliding operation. No translate attached to a higher-dimensional face is affected because by the previous iterations such a translate would fully occupy the exterior compartment of the present face and there would be no attached translates to it.}
end for
else if $l_i \in (r_i, 1 + r_i)$ **then**
for all $l' \in \tau_i(l + R)$ **do**
$l' \leftarrow l' + (r_i - l_i)e_j$
end for
else if $l_i \in (1, r_i)$ **then**
{In this case, the entire j-tube is empty; we opt to slide it back and then append a new translate to cover the lower-dimensional faces the original translate abandons.}
$l \leftarrow l + (1 - l_i)e_j${$l_i$ becomes 1.}
$\widehat{l} \leftarrow l + (r_i + 1)e_i${$\widehat{l_i} = 1 + r_i$ and is otherwise aligned with $l + R$.}
$L \leftarrow \widehat{l}$ {Now we pulled l back and covered what was left uncovered with a new translate $\widehat{l} + R$. Nothing overlaps since if $\widehat{l} + R$ overlapped with anything, so would the original $l + R$.}
end if
end for
end if
end for
end for
return L
End

Now the returned L is our L'. Note that we opted to *push* translates with $l_i < 1$ to $l_i = 1$ and *pull* translates with $l_i > r_i$ to $l_i = r_i$. We could have done the opposite resulting in $l_i = 0$, $l_i = 1 + r_i$; this would not change things since translates in the $i-$tube would come to cover the gaps left by the displaced translate, but our choice of operation makes it clearer that we do not introduce any gaps during the operation.

There are three points we want to prove:

1. $L' + R$ is an n.o.s.

2. Given $l \in L'$ attached to f_S^ϵ, $l_i = \dot{\epsilon_i}$ for $i \in S$, while $l_i = 1$ or r_i for $i \notin S$.

3. $0 + R$ is still attached to $\mathbf{0}$.

To prove item 1 we need to show that sliding the tubes does not introduce gaps; the nonoverlapping propery is obvious because we slide entire tubes and these are by definition the only (nontrivial) blocks to sliding a translate.

Item 2 seems to be immediate by the way we chose our slidings. However, we must guarantee that after we slide a translate to its 'correct' position in all directions, subsequent slidings of other faces of the same dimension or lower dimension do not affect the translate (if this were to happen, we would definitely lose the covering property along with destroying the nice form we had obtained for our translation vector). To show this it suffices to prove that in all cases where sliding occurs, the j-tube of the face consists only of translates attached to faces of dimension equal to or lower than the given one, and no l in the j-tube has already the j-th coordinate equal to 0, 1, r_j or $1 + r_j$. This last fact will follow immediately by a simple description of the j-tube as a tree of "zigzags" of leader-follower pairs. It also implies that no translate $l + R$ attached to a higher-dimensional face f_S^ϵ is disturbed, since all entries $j \notin S$ are either 1 or r_j and all entries in S are either 0 or $1 + r_j$ by the effects of a previous iteration.

We begin by providing the required description of the j-tube. Suppose $l + R$ is attached to f_S^ϵ, $j \notin S$ and $l_j \neq 1$ or r_j. First suppose $l_j \in (0, 1)$. Then the set of followers of $l+R$ is empty, because any follower $l^1 + R$ must satisfy $l_j^1 = l_j - r_j$ in order for ${}_{l^1+R}f_j^1$ to touch ${}_{l^1+R}f_j^0$. Of course then $l_j^1 < 0$ so $(l^1 + R) \cap C = \emptyset$, a contradiction. Similarly, if $l_j \in (r_j, 1 + r_j)$, the set of *leaders* is empty. Finally, if $l_j \in (1, r_j)$, both $\tau_j^{\pm}(l)$ are empty so the entire j-tube is empty.

So start with a $l + R$, $l_j \in (0, 1)$. There are only leaders and each leader $l^1 + R$ satisfies $l_j^1 - r_j = l_j$, so in particular $l_j^1 \in (r_j, 1 + r_j)$. Thus the leader $l^1 + R$ has only followers, and each follower $l^2 + R$ must satisfy $l_j^2 = l_j^1 - r_j = l_j + r_j - r_j = l_j$. Thus we are back to the first case (although not necessarily to the initial $l + R$) and we can iterate this to get the above descriptions for all elements of $\tau_j(l)$. In particular, we see that no element of $\tau_j(l)$ has $l'_j = 0, 1, r_j$ or $1 + r_j$. In fact, *every* element in the j-tube of $l + R$, $l \in (0, 1)$ has either $l'_j = l_j$ or $l'_j = l_j + r_j$ and if $l \in (r_j, 1 + r_j)$, either $l'_j = l_j$ or $l'_j = l_j - r_j$. This will be used in proving item 2 below.

Now we prove item 1. As mentioned above, nonoverlapping is trivially established. At each sliding operation, we only affect the j-tube of a translate with $j-$th entry say $l_j < 1$, by sliding it forward by $1 - l_j$. Covering is a local property so it is sufficient that this sliding operation does not throw away anything the original j-tube covered in C_ζ, the $\zeta-$fattening of C for some $\zeta < \min_k(1 - r_k)$. In other words, we must have $\tau_j(l) \cap C_\zeta = ((1 - l_j)e_j + \tau_j(l)) \cap C_\zeta$. Consider the set of rays

$$Y := \{\rho + te_j | \rho \in \langle e_j \rangle^\perp \cap \tau_j(l), \quad t \in [l_j - r_j, l_j + r_j]\}. \tag{8}$$

Then $Y \subset \tau_j l$ by the fact that $L + R$ is a covering (simply take a relevant ρ and see how far the ray can extend in the two directions; use the fact that $l_j \neq 1$ or r_j). Also since ζ is small, $\tau_j(l) \cap C_\zeta = Y \cap C_\zeta$. Obviously $Y \cap C_\zeta$ is the set of rays restricted to the interval $(-\zeta, 1+\zeta)$ but so is $((1-l_j)e_j + Y) \cap C_\zeta$ since $\zeta < \min_m(r_m - 1)$. Thus

$$\tau_j(l) \cap C_\zeta = Y \cap C_\zeta = (Y + (1-l_j)e_j) \cap C_\zeta = ((1-l_j)e_j + \tau_j(l)) \cap C_\zeta$$

as we needed. The case $l_j \in (r_j, 1 + r_j)$ is completely analogous.

We continue with item 2. Start with $k = d - 1$, a facet f_k^ϵ and an attached $l+R$. For any $j \neq k$, the algorithm above obviously has the stated effect on $l + R$; furthermore, if $l' \in \tau_j(l)$, by the description of the values of l'_j above, l'_j becomes either $0, 1, r_j$ or $1 + r_j$ according to the interval l_j belongs to and whether l' is a leader or a follower in the tube. Furthermore, since the new $l + R$ has all entries $j \neq k$ equal to 1 or r_j, it fully covers the (exterior compartment of) the facet. After proceeding to the next face and other translates, the algorithm never modifies the previously modified translates as other tubes that will be slided must contain only translates with coordinates $\neq 0, 1, r_j, 1 + r_j$, so they cannot contain any of the above.

Dropping dimension, the same justification as above for relevant directions gives that slidings never push previously slided translates and of course the effect of the sliding makes all relevant entries 1 or r_j and does not affect other entries. The details are easy to work out.

Finally, note that $0+R$ remains attached to $\mathbf{0}$ simply because all entries of the zero vector are already 0 so they did not belong to any j-tube of any translate that was slided, and thus no modification was done on this attached copy. □

Now that we have the n.o.s with the properties we need, a contradiction is easy to establish. We just need to justify the following list of implications.

Lemma 9. *The following hold for the modified n.o.s $L + R$:*

1. *Each face of C has a unique translate of R fully occupying its exterior compartment (not necessarily attached to the face).*

2. *Each translate $l + R$ attached to a K-face fully occupies the exterior compartments of precisely $\binom{K}{k}$ $k-$subfaces; furthermore, it does not cover any other face.*

3. *The combinatorial quantities M_k^K and M^K can be defined: M^K is the number of $K-$faces with an attached translate and M_k^K is the number of (exterior compartments of) k-faces occupied by some translate attached to a $K-$face.*

4. $M_k^K = M^K\binom{K}{k}$.

Proof. Consider any $f := f_S^\epsilon$ with an attached translate $l+R$. The exterior compartment of f can be written $\prod_{i\notin S}[0,1]\times\prod_{i\in S}\widehat{\epsilon_i}\cdot[0,\zeta)$ where $+[0,\zeta) = [0,\zeta)$ and $-[0,\zeta) = (-\zeta,0]$. Since for all $i\notin S$, $l_i = 1$ or r_i, it follows that $[0,1]\subset(l+R)_i$. For $i\in S$, $l_i = \dot{\epsilon_i}$ so if $\epsilon_i = 0$, $(l+R)_i = [-r_i,0]$ so $\widehat{\epsilon_i}[0,\zeta) = (-\zeta,0]\subset(l+R)_i$ and similarly for $\epsilon_i = 1$. Thus $C_f^{\widehat{\epsilon}}\subset(l+R)$ and thus the exterior compatment is covered by $l+R$ and no other translate is attached to the face (or there would be a nontrivial overlap between the two translates, as the exterior compartment would be a subset of both).

Let $K = d-|S|$ be the dimension of f. We show that of the $2^{K-k}\binom{K}{k}$ k-subfaces of f_S^ϵ, $l+R$ covers completely the exterior compartments of $\binom{K}{k}$ of them. Note that a k-subface f^1 of f can be written as $f_{S^1}^{\epsilon^1}$ for $S\subset S_1$, $\epsilon_1|_S = \epsilon$ and $d-|S_1| = k$ and is precisely $f^1 = \prod_{i\notin S^1}[0,1]\times\prod_{i\in S^1}\{\epsilon_i\}$. Thus its exterior compartment is $C_{f^1}^{\widehat{\epsilon_1}} = \prod_{i\notin S^1}[0,1]\times\prod_{i\in S^1}\widehat{\epsilon_1}\cdot[0,\zeta)$ and $l+R$ contains this set precisely when for all $i\in S^1\setminus S$, $\widehat{\epsilon_i}[0,\zeta)\subset(l_i+[-r_i,0])$ or equivalently, $l_i = \dot{\epsilon_i}$. Of course for each S^1 containing S there is precisely one selection of $\epsilon_i, i\in S^1\setminus S$ so that l_i agrees with $\widehat{\epsilon_i}$ and for the rest of the $0-1$ vectors, $(l+R)\cap C_{S^1}^{\widehat{\epsilon_1}} = \emptyset$. Thus $l+R$ covers precisely one subface for each S^1 and no other, for a total of precisely $\binom{K}{k}$ k-subfaces of f.

The fact that M^K and M_k^K can be well defined is now obvious. The equation in item 4 is simply the obvious statement "number of K-faces with attached translates times number of k-subfaces covered by each translate equals number of total k-subfaces covered" where we proved that "number of k-subfaces covered by each translate" equals $\binom{K}{k}$ for every translate in the paragraph above. □

Theorem 10. *With the above notation, $M^0 = 0$; this gives a contradiction since $0+R$ is attached to the dimension 0 face $\mathbf{0}$.*

Proof. The total number of k-faces is $2^{d-k}\binom{d}{k}$; the number of faces with an attached copy equals the total number of faces minus the number of ones whose exterior compartment is covered by a translate attached to some higher-dimensional face. Therefore:

$$M^k = 2^{d-k}\binom{d}{k} - \sum_{K=k+1}^{d-1} M_k^K \tag{9}$$

By point 4 in lemma 9, this equation becomes

$$M^k = 2^{d-k}\binom{d}{k} - \sum_{K=k+1}^{d-1} M^K\binom{K}{k} \tag{10}$$

Now it is only a matter of a computation to show $M^0 = 0$. Multiply the above equation by $(-1)^{d-k}$ and sum from $k=0$ to $d-1$. We get (remember

d is even)

$$\begin{aligned}
\sum_{k=0}^{d-1}(-1)^{d-k}M^k &= \sum_{k=0}^{d-1}(-2)^{d-k}\binom{d}{k} - \sum_{k=0}^{d-1}(-1)^{d-k}\sum_{K=k+1}^{d-1}M^K\binom{K}{k} \\
&= (-2+1)^d - 1 - \sum_{k=0}^{d-1}\sum_{K=1}^{d-1}(-1)^{d-k}M^K\binom{K}{k}1_{K>k} \\
&= -\sum_{K=1}^{d-1}\sum_{k=0}^{d-1}(-1)^{d-k}M^K\binom{K}{k}1_{K>k} \\
&= -\sum_{K=1}^{d-1}M^K(-1)^{d-K}\sum_{k=0}^{d-1}(-1)^{K-k}\binom{K}{k}1_{K>k} \\
&= -\sum_{K=1}^{d-1}M^K(-1)^{d-K}\sum_{k=0}^{K-1}(-1)^{K-k}\binom{K}{k} \\
&= -\sum_{K=1}^{d-1}M^K(-1)^{d-K}(-1)
\end{aligned}$$

Thus if we relabel the last sum with k instead of K and notice the initial index $k = 1$, we get from the very first and very last equations in the computation above

$$\sum_{k=0}^{d-1}(-1)^{d-k}M^k = \sum_{k=1}^{d-1}M^k(-1)^{d-k} \tag{11}$$

or equivalently

$$(-1)^d M^0 = 0 \tag{12}$$

which implies $M^0 = 0$ and completes the contradiction. □

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