Sobolev Inequalities on Complete Riemannian Manifolds

Master Thesis

Chatzikaleas Athanasios
Advisor: Professor Stathis Filippas
Department of Mathematics
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Contents

Preface

In the following pages, I present an introduction to Sobolev inequalities on complete Riemannian manifolds. The study of Sobolev inequalities started at the 1930's. By now, it is still a field of great undergoing development because of the critical role that plays in many branches of mathematics and specifically in the theory of nonlinear partial differential equations, geometry, spectral theory, mathematical physics, Markov processes, and potential theory. This survey presupposes a basic knowledge of Analysis and Riemannian Geometry. Concerning Riemannian Geometry, we refer the reader to [7] or [11].

The structure of this thesis is as follows:

- The purpose of Chapter 1 is merely to give some notation and basic definitions.
- Chapter 2 is a quick review of Sobolev inequalities in the Euclidean space.
- Chapter 3 is concerned with some geometrical preliminaries that we will constantly use in the following pages.
- In Chapter 4 we study Buser's isoperimetric inequality and, as a consequence, the validity of the L^1-Poincare inequality which will be crucial in the following chapter.
- In part I, chapter 5, we combine all the previously acquired results in order to characterize the complete Riemannian manifolds with Ricci curvature bounded from below which support a Sobolev embedding.
- In the second part of the thesis, in chapters 6 and 7, we study the connection of the optimal value of the Sobolev constant in Euclidean-type Sobolev inequalities to the geometry and the topology of a manifold having non negative or asymptotically non negative Ricci curvature.

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Chapter 1

Introduction

The main goal of this master thesis is to study complete Riemannian manifolds with Ricci curvature bounded from below supporting a Sobolev embedding. We will consider only smooth, connected, complete Riemannian manifolds (M, g) without boundary and of dimension $n \geq 3$. The $n-$dimensional Riemannian manifold (M, g) is said to support a Sobolev embedding if there exists

$$p, q \in \mathbb{R} \text{ and } m, k \in \mathbb{Z} : 1 \leq q < p \ , \ 0 \leq m < k \ , \ \frac{1}{p} = \frac{1}{q} - \frac{k-m}{n}$$

such that

$$H_k^q(M) \hookrightarrow H_m^p(M), \tag{1.1}$$

which by definition will include the continuity of the identity map

$$id : H_k^q(M) \longrightarrow H_m^p(M).$$

The function space $H_k^p(M)$ stands for the Sobolev space of M, that is the completion of

$$\mathcal{C}_k^p(M) := \Big\{ u \in C^\infty(M) : \nabla^j u \in L^p(M) \ , \ \forall j = 0, 1, \ldots, k \Big\}$$

with respect to the norm

$$\|u\|_{H_k^p(M)} := \sum_{j=0}^{k} \|\nabla^j u\|_{L^p(M)} := \sum_{j=0}^{k} \Big(\int_M |\nabla^j u|^p d\mu_g \Big)^{1/p}.$$

From this definition, $H_k^p(M)$ is the space of functions which are the limit in $H_k^p(M)$ of a Cauchy sequence in $\mathcal{C}_k^p(M)$. Notice that a Cauchy sequence in $\mathcal{C}_k^p(M)$ is also a Cauchy sequence in $L^p(M)$, and that a Cauchy sequence in $\mathcal{C}_k^p(M)$ which converges to 0 in $L^p(M)$ converges to 0 in $\mathcal{C}_k^p(M)$, and so, we can interpret the Sobolev spaces $H_k^p(M)$ as subspaces of $L^p(M)$. Namely, if $u \in H_k^p(M)$, then

(1) $u \in L^p(M)$, and

(2) $\exists$ a Cauchy sequence $\{u_m\}_{m=1}^\infty \subseteq \mathcal{C}_k^p(M)$: $\lim_{m \longrightarrow \infty} \|u_m - u\|_{L^p(M)} = 0$.

Throughout this thesis, $d\mu_g$ will denote the Riemannian volume element which is given in any chart by

$$d\mu_g = \sqrt{det(g_{ij})} dx,$$

where g_{ij} are the components of the metric g in the chart, and dx is the Lebesgue's volume element of $\mathbb{R}^n$. Furthermore, $\nabla^k u$ stands for the kth covariant derivative of the smooth function $u : M \longrightarrow \mathbb{R}$ which is a $(k, 0)$-tensor under the convention $\nabla^0 u = u$. Specifically, $\nabla u : TM \longrightarrow \mathbb{R}$ is the $(1, 0)$-tensor:

$$\nabla u(X) \equiv du(X) := X(u).$$

Locally,

$$\nabla u = \sum_{i=1}^{n} (\nabla u)_i dx_i,$$

where

$$(\nabla u)_i = \nabla u(\frac{\partial}{\partial x_i}) = \frac{\partial u}{\partial x_i}.$$

In addition, $\nabla^2 u : TM \times TM \longrightarrow \mathbb{R}$ is the $(2, 0)$-tensor:

$$(\nabla^2 u)(X, Y) := \Big(\nabla\big((\nabla u)(X)\big)\Big)(Y) = \nabla_X(\nabla_Y u) - \nabla_{\nabla_X Y} u.$$

Locally,

$$\nabla^2 u = \sum_{i,j=1}^{n} (\nabla^2 u)_{ij} dx_i \otimes dx_j,$$

where

$$(\nabla^2 u)_{ij} = \nabla^2 u(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial u}{\partial x_k}.$$

Here, Γ_{ij}^k are the Christoffel symbols of the Levi-Civita connection ∇ in the chart. Recall that, in the Riemannian context, the partial derivatives of a smooth function $u : M \longrightarrow \mathbb{R}$ are defined by

$$\frac{\partial}{\partial x_i}\Big|_p u := \frac{\partial}{\partial x_i}\Big|_{\varphi(p)} (u \circ \varphi^{-1}), \quad \frac{\partial^2}{\partial x_i \partial x_j}\Big|_p u := \frac{\partial^2}{\partial x_i \partial x_j}\Big|_{\varphi(p)} (u \circ \varphi^{-1}),$$

for any given chart $(U, \varphi = (x_1, \ldots, x_n))$, where, in the right hand side, $\frac{\partial}{\partial x_i}$ and $\frac{\partial^2}{\partial x_i \partial x_j}$ are the usual (weak) derivative operators. Finally, the length of ∇u is defined by

$$|\nabla u|^2 := g^{-1}(\nabla u, \nabla u) = \sum_{i,j=1}^{n} g^{ij} (\nabla u)_i (\nabla u)_j = \sum_{i,j=1}^{n} g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$$

and generally, for any $k \in \mathbb{N}$,

$$|\nabla^k u|^2 := \sum_{1 \le i_1, \ldots, i_k \le n} g^{i_1 j_1} \ldots g^{i_k j_k} (\nabla^k u)_{i_1 j_1} \ldots (\nabla^k u)_{i_k j_k}.$$

Chapter 2

Sobolev Inequalities on the Euclidean Space

In this chapter we will prove the validity of the classical Sobolev inequality in the Euclidean space and we will discuss the optimal value of the Sobolev constant. Let $C_c^\infty(\mathbb{R}^n)$ denote the space of infinitely differentiable functions $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ with compact support in $\mathbb{R}^n$. In the Euclidean space the simplest Sobolev inequality reads as

$$\Big(\int_{\mathbb{R}^n} |u|^{q^\star} dx\Big)^{1/q^\star} \leq C(n,q)\Big(\int_{\mathbb{R}^n} |\nabla u|^q dx\Big)^{1/q}, \quad \forall u \in C_c^\infty(\mathbb{R}^n), \tag{2.1}$$

for some positive constant $C(n,q)$, for some $1 \leq q < n$ and $q^\star := nq/(n-q)$. Clearly, (2.1) implies

$$H_1^q(\mathbb{R}^n) \hookrightarrow H_0^{q^\star}(\mathbb{R}^n) := L^{q^\star}(\mathbb{R}^n).$$

2.1. The case $q = 1$

This case is due to Gagliardo and Nirenberg, whose proof can be found in [6], theorem 1, p. 263. The reader may also look at [19] theorem 2.A, p.193. Here, we present a proof using the coarea formula [13]. Recall that the classical isoperimetric inequality states that

Theorem 2.1.1. *For any domain $\Omega \subseteq \mathbb{R}^n$ with smooth boundary $\partial\Omega$ and compact closure,*

$$\Big(Vol(\Omega)\Big)^{\frac{n-1}{n}} \leq K(n,1)Area(\partial\Omega). \tag{2.2}$$

If b_n is the volume of the unit ball in $\mathbb{R}^n$, the optimal constant is given by $K(n,1) := 1/(nb_n^{1/n})$ and is attained when Ω is any ball.

Theorem 2.1.2. *Let $u \in C_c^\infty(\mathbb{R}^n)$. There holds the inequality*

$$\Big(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx\Big)^{\frac{n-1}{n}} \leq C(n)\int_{\mathbb{R}^n} |\nabla u| dx, \tag{2.3}$$

for some positive constant $C \equiv C(n)$. The optimal constant is given by $K(n,1) := 1/(nb_n^{1/n})$, as in the isoperimetric inequality (2.2).

Proof. Let any $u \in C_c^\infty(\mathbb{R}^n)$ and assume that all level sets of u are good. Coarea formula and the isoperimetric inequality (2.2) imply

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla u| dx &= \int_0^\infty Area\big(\{x : |u(x)| = t\}\big) dt \\ &\geq n b_n^{1/n} \int_0^\infty \Big(Vol\big(\{x : |u(x)| \geq t\}\big)\Big)^{\frac{n-1}{n}} dt\end{aligned}$$

while, by Bonaventura-Cavalieri's principle:

$$\int_{\mathbb{R}^n} u^q dx = \int_0^\infty Vol\big(\{x \in \mathbb{R}^n : u(x) > t\}\big) d(t^q), \tag{2.4}$$

we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} &= \left(\int_0^\infty Vol\big(\{x : |u(x)| \geq t\}\big) d\big(t^{\frac{n}{n-1}}\big)\right)^{\frac{n-1}{n}} \\ &= \left(\int_0^\infty \frac{n}{n-1} Vol\big(\{x : |u(x)| \geq t\}\big) t^{\frac{1}{n-1}} dt\right)^{\frac{n-1}{n}} \\ = \left(\int_0^\infty \frac{n}{n-1}\Big(Vol\big(\{x : |u(x)| \geq t\}\big)\Big)^{\frac{n-1}{n}} \right.&\left.\Big(Vol\big(\{x : |u(x)| \geq t\}\big)\Big)^{\frac{1}{n}} t^{\frac{1}{n-1}} dt\right)^{\frac{n-1}{n}} \\ \leq \int_0^\infty \Big(Vol\big(\{x : |u(x)| \geq t\}\Big)^{\frac{n-1}{n}} dt,&\end{aligned}$$

where in the last line we used the obvious inequality

$$t\Big(Vol\big(\{x : |u(x)| \geq t\}\big)\Big)^{\frac{n-1}{n}} \leq \int_0^t \Big(Vol\big(\{x : |u(x)| \geq s\}\big)\Big)^{\frac{n-1}{n}} ds.$$

Let us now prove that $K(n,1) := 1/(n b_n^{1/n})$ is the smallest constant in (2.3). Assume that (2.3) holds for some positive constant $C(n)$. Let any $\epsilon > 0$ and define a function

$$u_\epsilon(|x|) := \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 + \epsilon \end{cases}$$

which is linear with $|\nabla u_\epsilon| \leq 1/\epsilon$ across $|x| = 1$ on $|x| = 1 + \epsilon$. Notice that u_ϵ is not smooth but it can be approximated by smooth functions since it is Lipschitz. Inserting u_ϵ into (2.3) we obtain

$$\begin{aligned} b_n^{\frac{n-1}{n}} &= \left(\int_{B(0,1)} dx\right)^{\frac{n-1}{n}} \leq \left(\int_{\mathbb{R}^n} |u_\epsilon|^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \\ &\leq C(n) \int_{\mathbb{R}^n} |\nabla u_\epsilon| dx = C(n) n b_n \int_1^{1+\epsilon} \left|\frac{du_\epsilon}{dr}\right| r^{n-1} dr \\ &\leq C(n) n b_n \frac{1}{\epsilon} \int_1^{1+\epsilon} r^{n-1} dr.\end{aligned}$$

Letting $\epsilon \longrightarrow 0$ we get $b_n^{\frac{n-1}{n}} \leq C(n) n b_n \Longrightarrow C(n) \geq \frac{1}{n b_n^{1/n}} := K(n,1)$. □

2.2. The case $1 < q < n$

For this case we will give an outline proof of G.Talenti's work using symmetric rearrangements in the sense of Hardy-Littlewood. For details we refer to [18] or [19] . Before we proceed to the proof, let us give some preliminaries. We will denote by m the Lebesque's measure in $\mathbb{R}^n$. Let any $u \in C_c^\infty(\mathbb{R}^n)$, that is smooth with

$$m\big(\{x \in \mathbb{R}^n : |u(x)| > t\}\big) < +\infty,\ \forall t > 0.$$

Consider the functions

(1) The distribution function of u: $\mu(t) := m\big(\{x \in \mathbb{R}^n : |u(x)| > t\}\big),\ t \geq 0.$

(2) The decreasing rearrangement of u: $u^*(s) := \min\{t \geq 0 : \mu(t) < s\},\ s \geq 0.$

(3) The symmetric rearrangement of u: $u^\bigstar(x) := u^*\big(b_n|x|^n\big),\ x \in \mathbb{R}^n,$

where b_n denotes the volume of the unit ball in $\mathbb{R}^n$. It can proved (see [19]) that

(i) $u^\bigstar$ is non-negative, radial, and radially decreasing,

(ii) $u^\bigstar$ and u have the same distribution function

(iii) For any increasing and convex function $\Phi : [0,\infty) \longrightarrow [0,\infty)$ with $\Phi(0) = 0$,

$$\int_{\mathbb{R}^n} \Phi\big(|\nabla u|\big)dx \geq \int_{\mathbb{R}^n} \Phi\big(|\nabla u^\bigstar|\big)dx.$$

Theorem 2.2.1. *Let any $1 < q < n$ and $q^\star := nq/(n-q)$. Then, there exists a positive constant $C(n,q)$ such that*

$$\Big(\int_{\mathbb{R}^n} |u|^{q^\star}dx\Big)^{1/q^\star} \leq C(n,q)\Big(\int_{\mathbb{R}^n} |\nabla u|^q dx\Big)^q,\quad \forall u \in C_c^\infty(\mathbb{R}^n). \tag{2.5}$$

The optimal constant is given by

$$\frac{1}{K(n,q)} = \sqrt{\pi}n^{1/q}\Big(\frac{n-q}{q-1}\Big)^{1-1/q}\Big(\frac{\Gamma(n/q)\Gamma(1+n-n/q)}{\Gamma(n)\Gamma(1+n/2)}\Big)^{1/n} \tag{2.6}$$

and is attained for any

$$u(x) := \big(\alpha + \beta|x|^{\frac{q}{q-1}}\big)^{1-n/q}, \tag{2.7}$$

where α and β are arbitrary positive constants.

Proof. By (ii) and (2.4) we have

$$\int_{\mathbb{R}^n} |u|^{q^\star}dx = \int_{\mathbb{R}^n} |u^\bigstar|^{q^\star}dx,$$

while by (iii),

$$\int_{\mathbb{R}^n} |\nabla u|^q dx \geq \int_{\mathbb{R}^n} |\nabla u^\bigstar|^q dx.$$

Let $\mathcal{R}$ be the set all of all non-negative, radial, radially decreasing and Lipschitz continuous functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $\mathcal{L}$ the set of all functions in $\mathcal{R}$ with $\|f\|_{L^{q^\star}(\mathbb{R}^n)} = 1$. Then, according to (i) and what we said above,

$$\begin{aligned}
\frac{\Big(\int_{\mathbb{R}^n}|\nabla u|^q dx\Big)^{1/q}}{\Big(\int_{\mathbb{R}^n}|u|^{q^\star}dx\Big)^{1/q^\star}} \geq \frac{\Big(\int_{\mathbb{R}^n}|\nabla u^\bigstar|^q dx\Big)^{1/q}}{\Big(\int_{\mathbb{R}^n}|u^\bigstar|^{q^\star}dx\Big)^{1/q^\star}} &\geq \inf_{f\in\mathcal{R}} \frac{\Big(\int_{\mathbb{R}^n}|\nabla f|^q dx\Big)^{1/q}}{\Big(\int_{\mathbb{R}^n}|f|^{q^\star}dx\Big)^{1/q^\star}} \\
&= (nb_n)^{1/n} \inf_{f\in\mathcal{R}} \frac{\Big(\int_0^\infty\big(-f'(r)\big)^q r^{n-1}dr\Big)^{1/q}}{\Big(\int_0^\infty\big(f(r)\big)^{q^\star} r^{n-1}dr\Big)^{1/q^\star}} \\
&= (nb_n)^{1/n} \inf_{f\in\mathcal{L}} \Big(\int_0^\infty\big(-f'(r)\big)^q r^{n-1}dr\Big)^{1/q}. \qquad (2.8)
\end{aligned}$$

Now, let any $g \in C_c^\infty(\mathbb{R})$ and consider the function

$$F(\epsilon) := \Big(\int_0^\infty\big(-f'(r)-\epsilon g'(r)\big)^q r^{n-1}dr\Big)^{1/q}, \quad \epsilon > 0. \qquad (2.9)$$

If f attains the infimum in (2.8), then we must have

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} F(\epsilon) = 0,$$

which leads to

$$-|f'(t)|^{q-2}\left((q-1)f''(t) + \frac{n-1}{t}f'(t)\right) = \frac{1}{\big(K(n,q)\big)^q} f^{q^\star-1}(t). \qquad (2.10)$$

The last equation can be written as

$$-\Delta_q f := div\big(|\nabla u|^{q-2}\nabla u\big) = \frac{1}{\big(K(n,q)\big)^q} f^{q^\star-1},$$

where Δ_q stands for the $q-$Laplacian operator in $\mathbb{R}^n$. The solutions of (2.10) are given by (2.7). Finally, to obtain the best value of the Sobolev constant $K(n,q)$ one has to substitute (2.7) into

$$\frac{1}{\big(K(n,q)\big)^q} = \int_{\mathbb{R}^n}|\nabla f(x)|^q dx.$$

□

Part I

Sobolev Inequalities on Complete Riemannian Manifolds with Ricci Curvature Bounded from Below

The goal of this part is to characterize the complete Riemannian manifolds with Ricci curvature bounded from below which support a Sobolev embedding. We will first give some geometrical preliminaries such as the Coarea formula on manifolds, a useful formula for the volume of the balls in geodesic spherical coordinates, Bishop-Gromov's volume comparison theorem and a very useful covering lemma.

Finally we prove that for complete Riemannian manifolds with Ricci curvature bounded from below, all Sobolev embeddings are valid for (M, g) if and only if there is a uniform lower bound for the volume of balls which is independent of their center, namely if and only if

$$\inf_{x\in M} Vol_g B(x,r) > 0, \quad \forall r > 0. \tag{2.11}$$

The exposition follows [8],[9]. It is a classical fact (see theorem 5.1.3) that, under a scaling argument, the embedding

$$H_1^1(M) \hookrightarrow L^{\frac{n}{n-1}}(M)$$

implies all other Sobolev embeddings, that is for all $p, q \in \mathbb{R}$ and $m, k \in \mathbb{Z}$ satisfying

$$1 \le q < p \ , \ 0 \le m < k \ , \ \frac{1}{p} = \frac{1}{q} - \frac{k-m}{n},$$

we have

$$H_k^q(M) \hookrightarrow H_m^p(M).$$

Using the same argument as in this theorem we can prove that if

$$\exists q_0 \in (1, n) : H_1^{q_0}(M) \hookrightarrow L^{p_0}(M), \tag{2.12}$$

then for all $q \in [q_0, n)$,

$$H_1^q(M) \hookrightarrow L^p(M),$$

where $1/p = 1/q - 1/n$. We will show (see theorems 5.1.5 and 5.1.9) that under the conditions (2.11) and (2.12) the embedding

$$H_1^q(M) \hookrightarrow L^p(M),$$

holds for any exponent $q \in [1, n)$ even smaller than q_0 and therefore condition (2.12) implies all other embeddings for (M, g).

Chapter 3

Geometrical Preliminaries

In this chapter we present some geometrical preliminaries such as the Coarea formula on manifolds, a useful formula for the volume of the balls in geodesic spherical coordinates, Bishop-Gromov's volume comparison theorem and a very useful covering lemma.

3.1. Coarea formula on manifolds

Coarea formula extends naturally from the Euclidean space to Riemannian manifolds as follows:

Theorem 3.1.1. *Let (M, g) be a Riemannian manifold and $f \in C_c^\infty(M)$ a fixed function. Then, for each $h \in L^1(M)$ we have*

$$\int_M h d\mu_g = \int_{\mathbb{R}} \int_{\Gamma(t)} \frac{h}{|\nabla f|} dA(t) dt, \tag{3.1}$$

where $\Gamma(t) := f^{-1}(\{t\}) = \{p \in M : f(p) = t\}$ and $dA(t)$ is the induced measure on $\Gamma(t)$.

Proof. By Sard's theorem, the critical values of f have measure zero, so $\Gamma(t)$ is an immersed $(n-1)-$dimensional submanifold of M for almost all $t \in f(M) \subseteq \mathbb{R}$. We also have that $Reg(f)$, the set of all regular values of f, is an open subset of $\mathbb{R}$. Let $(\alpha, \beta) \subseteq Reg(f)$ and pick any fixed number $c \in (\alpha, \beta)$. Let φ_{t-c} be the flow determined by the vector field $\frac{\nabla f}{|\nabla f|^2}$ restricted to $f^{-1}\big((a, b)\big)$, that is

$$\frac{d}{dt}\varphi_{t-c}(p) = \frac{\nabla f}{|\nabla f|^2} \text{ and } \varphi_{t-c}(p)\Big|_{t=c} = p \ , \text{ for any } p \in f^{-1}\big((\alpha, \beta)\big).$$

Clearly, the mapping

$$\Phi : f^{-1}(c) \times (\alpha, \beta) \longrightarrow f^{-1}\big((\alpha, \beta)\big) \ , \ (p, t) \longmapsto \Phi(p, t) := \varphi_{t-c}(p),$$

defines a diffeomorphism onto $f^{-1}\big((\alpha, \beta)\big)$. For all $(p, t) \in f^{-1}(c) \times (\alpha, \beta)$ we have

$$f\big(\Phi(p, c)\big) = f(\varphi_0(p)) = f(p) = c,$$
$$\frac{d}{dt} f\big(\Phi(p, c)\big) = \nabla f\Big|_{\Phi(p,c)} \cdot \frac{d}{dt}\big(\varphi_{t-c}(p)\big) = \Big(\nabla f \cdot \frac{\nabla f}{|\nabla f|^2}\Big)\Big|_{\Phi(p,c)} = 1.$$

Thus, for any $(p,t) \in f^{-1}(c) \times (\alpha, \beta)$,

$$f(\Phi(p,t)) = t \Longrightarrow \Phi(p,t) \in \Gamma(t).$$

Furthermore, $\frac{\partial \Phi(t)}{\partial t} \perp \Gamma(t)$ since $\frac{\partial \Phi(t)}{\partial t} = \frac{\nabla f}{|\nabla f|^2}$ is parallel to ∇f and ∇f is perpendicular to $\Gamma(t)$. Since Φ is diffeomorphism we can choose local coordinates (p,t) on the open $f^{-1}\big((\alpha,\beta)\big) \subseteq M$ where $p \in \Gamma(t)$ and $t \in (\alpha, \beta)$. Specifically, if $G_{ij}(t)$ $(i,j = 1,2,\dots,n-1)$ are the components of the metric on $\Gamma(t)$, we may write

$$g\Big|_{f^{-1}\big((\alpha,\beta)\big)} = \begin{pmatrix} \text{metric on } \Gamma(t) & 0 \\ 0 & \text{metric on } \big(\Gamma(t)\big)^{\perp} \end{pmatrix} = \begin{pmatrix} G_{ij}(t) & 0 \\ 0 & |\frac{\partial \Phi}{\partial t}|^2 \end{pmatrix}.$$

Thus, locally

$$\begin{aligned} d\mu_g &= \sqrt{det(g_{ij})} dx_1 \wedge \dots dx_{n-1} \wedge dt \\ &= \sqrt{|\frac{\partial \Phi}{\partial t}|^2 det(G_{ij}(t))} dx_1 \wedge \dots dx_{n-1} \wedge dt \\ &= |\frac{\partial \Phi}{\partial t}| \sqrt{det(G_{ij}(t))} dx_1 \wedge \dots dx_{n-1} \wedge dt \\ &= \Big|\frac{\nabla f}{|\nabla f|^2}\Big| \Big(\sqrt{det(G_{ij}(t))} dx_1 \wedge \dots dx_{n-1}\Big) \wedge dt \\ &= \frac{1}{|\nabla f|} dA(t) \wedge dt. \end{aligned}$$

and so, for any function $h \in L^1(M)$, we have

$$h d\mu_g = \frac{h}{|\nabla f|} dA(t) \wedge dt \Longrightarrow \int_M h d\mu_g = \int_{\mathbb{R}} \int_{\Gamma(t)} \frac{h}{|\nabla f|} dA(t) \wedge dt.$$

□

3.2. Geodesic spherical coordinates

In this section, we will present a very useful formula for the volume of balls $B(p,r)$ in (M,g) with the only assumption that on $B(p,r)$ all geodesic curves starting at p are minimizing. Let (M,g) be a complete Riemannian manifold. For any point $p \in M$, we denote by $Cut(p)$ the cut locus of M at p, that is the set of all points in the manifold where the geodesics starting at p stop being minimizing. It is a classical fact that the set $Cut(p)$ has Riemannian measure zero for any $p \in M$. More about cut locus can be found in [14]. First, we need the following lemma.

Lemma 3.2.1. *Let (M,g) be a Riemannian manifold, $p \in M$, $w \in T_pM$ and let $\gamma(t) := exp_p(tw)$, $t \in [0,l]$. Then, the unique Jacobi field $Y(t)$ along γ satisfying the initial conditions $Y(0) = 0, Y'(0) = w$ can be written as:*

$$Y(t) = \Big(d(exp_p)\Big)_{\gamma'(0)}(tw) \ , \ t \in [0,l].$$

Proof. Let $\gamma : [0,l] \longrightarrow M$ be the unique geodesic with $\gamma(0) = p$, $\gamma'(0) = w$, that is $\gamma(t) := \exp_p(tw)$, $s \in [0,l]$. We define the variation

$$f : (-\epsilon,\epsilon) \times [0,l] \longrightarrow M, \quad f(s,t) := exp_p(tv(s)),$$

of γ where $v \equiv v(s)$, $s \in (-\epsilon,\epsilon)$ stands for any curve on T_pM with $v(0) = \gamma'(0)$. Notice that f is well defined because for small values of ϵ the points $tv(s) \in T_pM$ belong in the domain of exp_p. By definition, the Jacobi field of the geodesic

$$t \longmapsto exp_p(tv(s)),$$

is given by

$$\begin{aligned} Y(t) := \frac{\partial f}{\partial s}(0,t) = \frac{\partial}{\partial s}\Big|_{s=0}\Big(exp_p(tv(s))\Big) = \Big(d(exp_p)\Big)_{tv(0)}\Big(\frac{d}{ds}\Big|_{s=0}(tv(s))\Big) \\ = \Big(d(exp_p)\Big)_{tv(0)}\Big(t\frac{dv}{ds}(0)\Big) = t\Big(d(exp_p)\Big)_{tv(0)}\Big(\frac{dv}{ds}(0)\Big). \end{aligned}$$

Clearly, $Y(0) = 0$. We calculate

$$\begin{aligned} Y'(0) &= \frac{\nabla}{dt}\Big|_{t=0} Y(t) = \frac{\nabla}{dt}\Big|_{t=0}\left(t\Big(d(exp_p)\Big)_{tv(0)}\Big(\frac{dv}{ds}(0)\Big)\right) \\ &= 1 \cdot \Big(d(exp_p)\Big)_{tv(0)}\Big(\frac{dv}{ds}(0)\Big)\Big|_{t=0} + \Big(t \cdot (\cdots)\Big)\Big|_{t=0} \\ &= \Big(d(exp_p)\Big)_0\Big(\frac{dv}{ds}(0)\Big) = \frac{dv(0)}{ds}. \end{aligned}$$

Setting $\frac{dv(0)}{ds}(0) := w$ the result follows. □

With such a result we are now in position to prove the following.

Theorem 3.2.2. *Let (M,g) be a complete Riemannian manifold. Assume that the ball $B(p,r) \subseteq M$ does not intersect the cut-locus of M at p. For any $v \in T_pM$ with $\|v\| = 1$ and any orthonormal base $\{e_1,\dots,e_{n-1},v\}$ of T_pM, let Y_i be the Jacobi field along the geodesic $\gamma(t) := exp_p(tv)$ satisfying the initial conditions $Y_i(0) = 0$, $Y_i'(0) = e_i$ $(i = 1,\dots,n-1)$. Then*

$$Vol_g(B(p,r)) = \int_{\mathbb{S}^{n-1}(1)}\int_0^r \sqrt{det\Big(g(Y_i(s), g(Y_j(s)))_{1\le i,j\le n-1}\Big)}dsdv$$

where dv stands for the canonical $(n-1)$-measure on the unit tangent sphere $\mathbb{S}_p^{n-1}(1) := \{\xi \in T_pM : \|\xi\| = 1\} \subseteq T_pM$ at p. The function $J(v,s)$ in the integral above is called volume element in the geodesic spherical coordinates.

Proof. Since the ball $B(p,r)$ does not intersect the cut-locus of M at p, the map $exp_p^{-1}\Big|_{B(p,r)}$ is diffeomorfism and the pair $(B(p,r),\varphi)$ with

$$\varphi : B(p,r) \subseteq M \longrightarrow \mathbb{R}^n, \quad \varphi(x) := (Q \circ exp_p^{-1})(x)$$

can be used as a chart in the definition of the volume form $d\mu|_{B(p,r)}$. Here, the function Q is defined by

$$Q : T_pM \longrightarrow \mathbb{R}^n, \quad w = \sum_{j=1}^{n} w_j e_j \longmapsto Q(w) := (w_1, \dots, w_n),$$

for some base $\{e_1, \dots, e_n\}$ of T_pM. Let $v \in T_pM$ with $\|v\| = 1$ and $\{e_1, \dots, e_{n-1}, v\}$ an orthonormal base for T_pM. Each point $x \in B(p,r)$ can be written with respect to these coordinates as

$$\begin{aligned}\varphi(x) = (x_1, \dots, x_n) &\Longrightarrow x = \varphi^{-1}(x_1, \dots, x_n) \Longrightarrow x = exp_p\big(Q^{-1}(x_1, \dots, x_n)\big)\\ &\Longrightarrow x = exp_p\big(x_1e_1 + \dots + x_{n-1}e_{n-1} + x_n v\big).\end{aligned}$$

Pick any $s \in [0,r]$ and set $q = exp_p(sv)$. It is a classical fact that q is not a conjugate point of p. Thus, the $\big(d(exp_p)\big)_{sv}$ is non-singular. A direct calculation shows that

$$\begin{aligned}\frac{\partial}{\partial x_j}\Big|_q &= \big(d(exp_p)\big)_{sv}(e_j) \quad (j = 1, \dots, n-1),\\ \frac{\partial}{\partial x_n}\Big|_q &= \big(d(exp_p)\big)_{sv}(v).\end{aligned}$$

Indeed; for any $f \in C^\infty(B(p,r))$ and for all $j = 1, \dots, n-1$,

$$\begin{aligned}\frac{\partial}{\partial x_j}\Big|_q f &:= \frac{\partial}{\partial x_j}\Big|_{\varphi(q)} \big(f \circ \varphi^{-1}\big)(x_1, \dots, x_n)\\ &= \frac{\partial}{\partial x_j}\Big|_{\varphi(exp_p(sv))} \big(f \circ exp_p \circ Q^{-1}\big)(x_1, \dots, x_n)\\ &= \frac{\partial}{\partial x_j}\Big|_{Q(sv)} \Big(\Big(f \circ exp_p\Big)\big(Q^{-1}(x_1 \dots, x_n)\big)\Big)\\ &= \frac{\partial}{\partial x_j}\Big|_{(0,\dots,0,s)} \Big(\Big(f \circ exp_p\Big)(x_1e_1, \dots, x_{n-1}e_{n-1} + x_n v)\Big)\\ &= \Big(d(f \circ exp_p)\Big)_{sv}\Big(\frac{\partial}{\partial x_j}\Big|_{(0,\dots,0,s)}(x_1e_1 + \dots + x_{n-1}e_{n-1} + x_n v)\Big)\\ &= \Big((df)_{exp_p(sv)} \circ (d(exp_p))_{sv}\Big)(e_j)\\ &= (df)_{exp_p}\Big(\big(d(exp_p)\big)_{sv}(e_j)\Big)\\ &= \Big(\big(d(exp_p)\big)_{sv}(e_j)\Big)[f].\end{aligned}$$

Similarly, one can prove the other equality. For any $i = 1, 2, \dots n-1$, let Y_i be the Jacobi field along $\gamma(s) := exp_p(sv),\ s \in [0,r]$ satisfying $Y_i(0) = 0$, $Y_i'(0) = e_i$. Using the previous theorem, the fact that the mapping $d(exp)_p(.)$ is linear and what we said above, we obtain

$$Y_i(s) = \big(d(exp_p)\big)_{sv}(se_i) = s\big(d(exp_p)\big)_{sv}(e_i) = s\frac{\partial}{\partial x_i}\Big|_q \quad (i = 1, 2, \dots, n-1).$$

Also, we may write

$$\frac{\partial}{\partial x_n}\Big|_q = \gamma'(s),$$

since, for all $f \in C^\infty(B(p,r))$,

$$\begin{aligned}\gamma'(s)[f] &= (d\gamma)_s\big(\frac{d}{dt}\Big|_s\big)[f] = \frac{d}{dt}\Big|_s[f\circ\gamma] = \frac{d}{dt}\Big|_s[\big(f\circ exp_p\big)(tv)] = \\ &= \big(d(exp_p)\big)_{sv}\big(\frac{d}{dt}\Big|_s(tv)\big)[f] = \Big(\big(d(exp_p)\big)_{sv}(v)\Big)[f] = \frac{\partial}{\partial x_n}\Big|_q[f].\end{aligned}$$

As a consequence, the canonical basis for T_qM can be written as

$$\big\{\frac{\partial}{\partial x_1}\Big|_q,\ldots,\frac{\partial}{\partial x_n}\Big|_q\big\} = \big\{\frac{1}{s}Y_1(s),\ldots,\frac{1}{s}Y_{n-1}(s),\gamma'(s)\big\}.$$

Now, let $(v,s) \in \{v \in T_pM : \|v\| = 1\}\times(0,r)$ be the spherical coordinates of $(x_1,\ldots,x_n) \in \mathbb{R}^n$. Then, the volume element $d\mu|_{B(p,r)}$ at q can be written as

$$\begin{aligned}
d\mu|_{B(p,r)} &= \left(det\Big(g(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j})\Big)_{1\le i,j\le n}\right)^{1/2} dx_1\wedge\cdots\wedge x_n \\
&= \left(det\begin{pmatrix} \frac{1}{s^2}g(Y_1(s),Y_1(s)) & \cdots & \frac{1}{s^2}g(Y_1(s),Y_{n-1}(s)) & \frac{1}{s^2}g(Y_1,\gamma') \\ \vdots & & \vdots & \vdots \\ \frac{1}{s^2}g(Y_{n-1}(s),Y_1(s)) & \cdots & \frac{1}{s^2}g(Y_{n-1}(s),Y_{n-1}(s)) & \frac{1}{s^2}g(Y_{n-1},\gamma') \\ \frac{1}{s}g(Y_1,\gamma') & \cdots & \frac{1}{s}g(Y_{n-1},\gamma') & g(\gamma',\gamma') \end{pmatrix}\right)^{1/2} dx_1\wedge\cdots\wedge dx_n \\
&= \left(det\begin{pmatrix} \frac{1}{s^2}g(Y_1(s),Y_1(s)) & \cdots & \frac{1}{s^2}g(Y_1(s),Y_{n-1}(s)) & 0 \\ \vdots & & \vdots & \vdots \\ \frac{1}{s^2}g(Y_{n-1}(s),Y_1(s)) & \cdots & \frac{1}{s^2}g(Y_{n-1}(s),Y_{n-1}(s)) & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}\right)^{1/2} dx_1\wedge\cdots\wedge dx_n \\
&= \left(\frac{1}{s^{2(n-1)}}det\Big(g(Y_i(s),Y_j(s)\Big)_{1\le i,j\le n-1}\right)^{1/2} dx_1\wedge\cdots\wedge dx_n \\
&= \left(det\Big(g(Y_i(s),Y_j(s)\Big)_{1\le i,j\le n-1}\right)^{1/2} \frac{1}{s^{n-1}}dx_1\wedge\cdots\wedge dx_n \\
&= \left(det\Big(g(Y_i(s),Y_j(s)\Big)_{1\le i,j\le n-1}\right)^{1/2} ds\wedge dv
\end{aligned}$$

where in the 3rd line we used the fact that the geodesic γ is parametrised by its arc length and that

$$Y_i \perp \gamma' \quad (i = 1,2,\ldots,n-1) \tag{3.2}$$

which can be easily proved by observing that the function $g(\gamma'(s), Y_i(s))$ is a solution to the initial value problem

$$\begin{aligned}
\frac{d^2}{ds^2}g(\gamma'(s), Y_i(s)) =& \frac{d}{ds}\Big(\frac{d}{ds}g(\gamma'(s), Y_i(s))\Big)\\
=& \frac{d}{ds}\Big(g(\nabla_{\gamma'(s)}\gamma'(s), Y_i(s)) + g(\gamma'(s), \nabla_{\gamma'(s)}Y_i(s)\Big)\\
=& \frac{d}{ds}\Big(g(\gamma'(s), \nabla_{\gamma'(s)}Y_i(s)\Big)\\
=& g(\nabla_{\gamma'(s)}\gamma'(s), \nabla_{\gamma'(s)}Y_i(s)) + g(\gamma'(s), \nabla_{\gamma'(s)}\nabla_{\gamma'(s)}Y_i(s))\\
=& g(\gamma'(s), \nabla_{\gamma'(s)}\nabla_{\gamma'(s)}Y_i(s))\\
=& g\big(\gamma'(s), -R(Y_i(s), \gamma'(s))\gamma'(s)\big) = 0,\\
g(\gamma'(0), Y_i(0)) =& g(\gamma', 0) = 0,\\
\frac{d}{ds}\Big|_{s=0} g(\gamma'(s), Y_i(s)) =& g(\gamma'(0), Y_i'(0)) = g(\gamma'(0), e_i) = 0.
\end{aligned}$$

□

Remark 3.2.3. *In theorem above we do not really need the assumption that the ball $B(p,r)$ does not intersects the cut-locus of M at p. For given $v \in T_pM$ with $\|v\| = 1$ we define the distance along the geodesic γ_v from $p \in M$ to its cut-point along γ_v by*

$$\begin{aligned}
c(v) :=& \sup\{s > 0 : exp_p(tv) \notin Cut(p)\ ,\ \forall 0 < t < s\}\\
=& \sup\{s > 0 : sv \in TM\ ,\ d_g(p, \gamma_v) = s\}.
\end{aligned} \tag{3.3}$$

Here, γ_v denotes the geodesic parametrised by its arc length having $\gamma_v(0) = p$, $\gamma_v'(0) = v$. Observe that, if there exists $s_1 > 0$ such that $d_g(p, \gamma_v(s_1)) = s_1$, then $d_g(p, \gamma_v(s)) = s$, for all $0 \ \leq s \leq s_1$. Indeed; if there exists $0 \leq s_0 \leq s_1$ such that $d_g(p, \gamma_v(s_0)) < s_0$, then the triangle inequality implies

$$\begin{aligned}
d_g(p, \gamma_v(s_1)) \leq& d_g(p, \gamma_v(s_0)) + d_g(\gamma_v(s_0), \gamma_v(s_1))\\
<& s_0 + (s_1 - s_0) = s_1\ ,
\end{aligned}$$

a contradiction. Thus, the geodesic minimizes distance between p and $\gamma_v(s)$ for all $0 \leq s < c(v)$ but fails to minimize distance for all $c(v) \leq s < +\infty$. In this general case we will have

$$Vol_g(B(p,r)) = \int_{\|v\|=1}\int_0^{min\{c(v),r\}} \sqrt{det\Big(g\big(Y_i(s), Y_j(s)\big)\Big)_{1\leq i,j\leq n-1}} ds \wedge dv. \tag{3.4}$$

We note that, in case where $c(v) < r$, the portion of the geodesic γ_v from $s = c(v)$ towards $s = r$ does not contribute to the integral in (3.4). But, since the set of all these points

$$\bigcup \big\{exp_p(sv) : v \in T_pM \ \text{with}\ \|v\| = 1\ ,\ c(v) < r\ ,\ c(v) \leq s \leq r\big\},$$

has $n-$dimensional measure zero, formula (3.4) gives the volume of the entire ball $B(p,r) \subseteq M$.

Let us now present an application of the theorem above to Riemannian manifolds of constant sectional curvatures.

Proposition 3.2.4. *If the Riemannian manifold (M,g) has constant sectional curvatures equal to $k \in \mathbb{R}$, then the volume of the geodesic ball of center $p \in M$ and radius r is given by*

$$Vol_g\Big(B(p,r)\Big) = \begin{cases} b_n \int_0^r (sinh(s\sqrt{-k}))^{n-1} ds, & k<0 \\ b_n \int_0^r s^{n-1} ds, & k=0 \\ b_n \int_0^r (sin(s\sqrt{k}))^{n-1} ds, & k>0 \end{cases}$$

where b_n is the volume of the unit ball in the tangent space T_pM.

Proof. Pick any $v \in T_pM$ with $\|v\| = 1$, $\{e_1, \dots, e_{n-1}, v\}$ an orthonormal base for T_pM and let Y_i be the Jacobi field along $\gamma(t) := exp_p(tv)$, $t \in [0,l]$ satisfying the initial conditions $Y_i(0) = 0$, $Y_i'(0) = e_i$ $(i = 1, \dots, n-1)$. Using (3.2) and the fact that (M,g) has constant sectional curvature equal to $k \in \mathbb{R}$, we have

$$\begin{aligned} &g\big(R(Y_i,\gamma')\gamma', X\big) = -k\big\{g(Y_i,\gamma')g(\gamma',X) - g(\gamma',\gamma')g(Y_i,X)\big\},\ \forall X \in T_pM \Longrightarrow \\ &g\big(R(Y_i,\gamma')\gamma', X\big) = kg(Y_i,X),\ \forall X \in T_pM \Longrightarrow \\ &g\big(R(Y_i,\gamma')\gamma' - kY_i, X\big) = 0,\ \forall X \in T_pM \Longrightarrow R(Y_i,\gamma')\gamma' = kY_i \end{aligned}$$

Hence,

$$Y_i'' + R(Y_i,\gamma')\gamma' = 0 \Longrightarrow Y_i'' + kY = 0.$$

Solving this initial value problem for Y_i, we find

$$Y_i(s) = \begin{cases} sinh(s\sqrt{-k}), & k<0 \\ s, & k=0 \\ sin(s\sqrt{k}), & k>0 \end{cases}$$

and

$$g(Y_i(s), Y_j(s)) = \begin{cases} sinh(s\sqrt{-k}))^2, & k<0 \\ s^2, & k=0 \\ (sin(s\sqrt{k}))^2, & k>0 \end{cases}$$

The desired result follows immediately from the definition of $J_k(s)$. □

3.3. Bishop-Gromov's volume comparison theorem

We will now introduce a volume comparison theorem due to Gromov which was later improved by Bishop. This theorem will help us find upper bounds for the balls $B(p,r) \subseteq M$ on complete Riemannian manifolds (M,g) with the only assumption that on $(B(p,r),g)$ the Ricci curvature is bounded from below, that is

$$\exists k \in \mathbb{R} : Ric_{(M,g)} \geq (n-1)kg, \quad \text{on } B(p,r).$$

Since the Ricci tensor is a symmetric bilinear form, the above inequality reads in terms of the eigenvalues:

$$\begin{aligned}\exists k \in \mathbb{R} &: \text{ eigenvalues of } Ric_{(M,g)} - (n-1)kg \geq 0 \Longleftrightarrow \\ \exists k \in \mathbb{R} &: Ric_{(M,g)}(X,X) \geq (n-1)kg(X,X), \text{ for all } X \in TM \Longleftrightarrow \\ \exists k \in \mathbb{R} &: Ric_{(M,g)}(X,X) \geq (n-1)k, \text{ for all } X \in TM \text{ with } \|X\| = 1.\end{aligned}$$

Theorem 3.3.1. *Let (M,g) be a complete Riemannian manifold, $p \in M$ and $r > 0$ and assume that, on $B(p,r)$, $Ric_{(M,g)} \geq (n-1)kg$, for some real constant k. Furthermore, let $M_{0,k}$ be the complete simply connected Riemannian manifold of constant curvature k. Then,*

(i) $Vol_g(B(p,r)) \leq V_k(r)$,

(ii) The function $J(s,v)\big/J_k(s)$ is decreasing with respect to s,

(iii) The function $Vol_g(B(p,r))\big/V_k(r)$ is decreasing with respect to r,

where $Vol_g(B(p,r))$ denotes the volume of the geodesic ball of center $p \in M$ and radius r in M, $V_k(r)$ denotes the volume of a ball of radius r in $M_{0,k}$ and $J(s,v), J_k(s)$ are the volume elements in the geodesic spherical coordinates in M and $M_{0,k}$ respectively.

Proof. Pick any $x \in B(p,r)$ and $v \in T_pM$ such that $\|v\| = 1$, $x = exp_p(\alpha v)$ for $\alpha = d(p,x)$. Also, let $\gamma(s) := exp_p(sv)$, $s \in [0,\alpha]$ be the geodesic curve with $\gamma(0) = p$, $\gamma(\alpha) = x$, $\gamma'(0) = v$ and let $\{e_1, \ldots, e_{n-1}, v\}$ be an orthonormal base for T_pM. Clearly, the set $\{e_1(s), \ldots, e_{n-1}(s), \gamma'(s)\}$ forms an orthonormal base for $T_{\gamma(s)}M$ where $e_j(s)$ is the parallel transport of e_j along γ. Let Y_j be the Jacobi field along γ satisfying the initial conditions $Y_j(0) = 0$, $Y_j'(0) = e_j$ $(j = 1,2,\ldots,n-1)$. By (3.2), there exist functions $\alpha_{ji} = \alpha_{ji}(s) \in C^\infty([0,\alpha])$ such that

$$Y_i(s) = \sum_{j=1}^{n-1} \alpha_{ji}(s) e_j(s). \tag{3.5}$$

Set $A(s) := (\alpha_{ij}(s))$, $s \in [0,\alpha]$. For any $i,j = 1,2,\ldots,n-1$, we calculate

$$g(Y_i(s), Y_j(s)) = \sum_{k=1}^{n-1} \alpha_{ki}(s)\alpha_{\lambda j}(s) = \left(A^T(s)A(s)\right)_{i,j}$$

Thus, the volume element $J(s,v)$ can be written as

$$J(s,v) = detA(s). \tag{3.6}$$

For all $s \in [0,\alpha]$ and $i = 1,2,\ldots,n-1$, we have

$$Y_i'(s) := \nabla_{\gamma'(s)} Y_i(s) = \sum_{j=1}^{n-1} \left(\alpha_{ji}'(s)e_j(s) + \alpha_{ji}(s)\nabla_{\gamma'(s)}e_j(s)\right) = \sum_{j=1}^{n-1} \alpha_{ji}'(s)e_j(s),$$

$$Y_i''(s) := \nabla_{\gamma'(s)}\nabla_{\gamma'(s)} Y_i(s) = \sum_{j=1}^{n-1} \left(\alpha_{ji}''(s)e_j(s) + \alpha_{ji}'(s)\nabla_{\gamma'(s)}e_j(s)\right) = \sum_{j=1}^{n-1} \alpha_{ji}''(s)e_j(s),$$

since the vector field $e_j(s)$ is parallel along γ. Using the Jacobi equation, we may write

$$\sum_{j=1}^{n-1}\alpha''_{ji}(s)e_j(s)=Y''_i(s)=-R(Y_i(s),\gamma'(s))\gamma'(s)=-\sum_{j=1}^{n-1}\alpha_{ji}(s)R(e_j(s),\gamma'(s))\gamma'(s).$$

Taking the inner product with $e_k(s)$ we find

$$\alpha''_{ki}(s)=\sum_{j=1}^{n-1}\alpha_{ji}(s)p_{jk}(s)\quad (k,i=1,\dots,n-1) \tag{3.7}$$

where p_{jk} $(j,k=1,\dots,n-1)$ are given by

$$p_{jk}(s):=g\Big(R(e_j(s),\gamma'(s))\gamma'(s),e_k(s)\Big). \tag{3.8}$$

Notice that $p_{jk}=p_{kj}$. Now, we return to (3.6) and we set $B(s):=A'(s)A^{-1}(s)$. We calculate [1]

$$\frac{\partial}{\partial s}J(v,s)=J(v,s)trB(s)\quad,\quad\frac{\partial^2 J}{\partial s^2}=\frac{\partial J}{\partial s}trB+J\frac{\partial trB}{\partial s}. \tag{3.9}$$

We will now show that

$$\frac{\partial trB}{\partial s}=-tr(B^2)-Ric_{(M,g)}(\gamma',\gamma'). \tag{3.10}$$

Let $A^{-1}(s):=(\eta_{ij}(s))$. Then, $B(s):=(b_{ij}(s))=\Big(\sum_{\lambda=1}^{n-1}\alpha'_{i\lambda}(s)\eta_{\lambda j}(s)\Big)$. Differentiate the equation $\sum_{j=1}^{n-1}\eta_{\lambda j}\alpha_{jm}=\delta_{\lambda m}$ with respect to s to get

$$\sum_{k,\lambda=1}^{n-1}\alpha'_{k\lambda}\eta'_{\lambda k}=-tr(B^2). \tag{3.11}$$

Using (3.7), (3.11) and the symmetry of p_{kj} we get (3.10) as follows:

$$\begin{aligned}\frac{\partial}{\partial s}trB&=\sum_{k=1}^{n-1}\frac{\partial}{\partial s}b_{kk}=\sum_{k=1}^{n-1}\frac{\partial}{\partial s}\Big(\sum_{\lambda=1}^{n-1}\alpha'_{k\lambda}\eta_{\lambda k}\Big)=\sum_{k,\lambda=1}^{n-1}\Big(\alpha''_{k\lambda}\eta_{\lambda k}+\alpha'_{k\lambda}\eta'_{\lambda k}\Big)\\
&=\sum_{k,\lambda=1}^{n-1}\Big(-\eta_{\lambda k}\sum_{j=1}^{n-1}\alpha_{j\lambda}p_{jk}+\alpha'_{k\lambda}\eta'_{\lambda k}\Big)=-\sum_{\lambda=1}^{n-1}\Big(\sum_{k,j=1}^{n-1}\eta_{\lambda k}p_{kj}\alpha_{j\lambda}\Big)+\sum_{k,\lambda=1}^{n-1}\alpha'_{k\lambda}\eta'_{\lambda k}\\
&=-\sum_{\lambda=1}^{n-1}\big(A^{-1}PA\big)_{\lambda,\lambda}+\sum_{k,\lambda=1}^{n-1}\alpha'_{k\lambda}\eta'_{\lambda k}=-tr(A^{-1}PA)+\sum_{k,\lambda=1}^{n-1}\alpha'_{k\lambda}\eta'_{\lambda k}\\
&=-tr(P)+\sum_{k,\lambda=1}^{n-1}\alpha'_{k\lambda}\eta'_{\lambda k}=-\sum_{j=1}^{n-1}p_{jj}+\sum_{k,\lambda=1}^{n-1}\alpha'_{k\lambda}\eta'_{\lambda k}\\
&=\sum_{k,\lambda=1}^{n-1}\alpha'_{k\lambda}\eta'_{\lambda k}-\sum_{j=1}^{n-1}g\big(e_j,R(e_j,\gamma')\gamma'\big)=\sum_{k,\lambda=1}^{n-1}\alpha'_{k\lambda}\eta'_{\lambda k}-Ric_{(M,g)}(\gamma',\gamma')\\
&=-tr(B^2)-Ric_{(M,g)}(\gamma',\gamma').\end{aligned}$$

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[1] In general, if $\Phi\equiv\Phi(t)$ is an n × n $C^1-matrix$ with $\frac{d}{ds}\Phi(t)=A(t)\Phi(t)$, then: $\frac{d}{ds}\big(det\Phi(t)\big)=\big(det\Phi(t)\big)\big(tr\Phi(t)\big)$. Here, apply this result with $A(t):=(\frac{d}{dt}\Phi(t))(\Phi(t))^{-1}$ and $\Phi(t):=J(s,t)$.

Now, (3.9) yields

$$\frac{\partial^2 J}{\partial s^2} = J(trB)^2 - J\Big(tr(B^2) + Ric_{(M,g)}(\gamma',\gamma')\Big). \tag{3.12}$$

Setting $\Phi(s) := \big(J(s,v)\big)^{\frac{1}{n-1}}$, this equation becomes

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial s^2} &= \frac{1}{n-1}\Big\{\frac{1}{n-1}(trB)^2 - tr(B^2)\Big\}\Phi - \frac{1}{n-1}\Phi Ric_{(M,g)}(\gamma',\gamma') \\ &\geq \frac{1}{n-1}\Big\{\frac{1}{n-1}(trB)^2 - tr(B^2)\Big\}\Phi - \frac{1}{n-1}\Phi(n-1)kg(\gamma',\gamma') \\ &= \frac{1}{n-1}\Big\{\frac{1}{n-1}(trB)^2 - tr(B^2)\Big\}\Phi - \frac{1}{n-1}\Phi(n-1)k. \end{aligned} \tag{3.13}$$

Now, for the non-linear term in (3.13) we have

$$\frac{1}{n-1}(trB)^2 - tr(B^2) \leq 0 \tag{3.14}$$

Indeed; If $\lambda_1,\dots,\lambda_{n-1}$ are the eigenvalues of $B(s)$, Cauchy-Schwartz inequality implies

$$\big(trB\big)^2 = \Big(\sum_{j=1}^{n-1}\lambda_j\Big)^2 \leq (n-1)\sum_{j=1}^{n-1}\lambda_j^2 = (n-1)tr(B^2),$$

Finally, we get that Φ satisfies the differential inequality

$$\frac{\partial^2 \Phi}{\partial s^2} + k\Phi \leq 0. \tag{3.15}$$

Consider the function

$$\Phi_0(s) := \big(J_k(s)\big)^{\frac{1}{n-1}} = \begin{cases} sinh(s\sqrt{-k}) & ,k<0 \\ s & ,k=0 \\ sin(s\sqrt{k}) & ,k>0 \end{cases} \tag{3.16}$$

which is the (unique) solution of the initial value problem

$$\begin{cases} \Phi_0''(s) + k\Phi_0(s) = 0 \\ \Phi_0(0) = 0\ ,\ \Phi_0'(0) = 1 \end{cases}$$

Then, for all $s \in (0,\alpha]$, we have

$$\begin{aligned} \big(\Phi'(s)\Phi_0(s) - \Phi(s)\Phi_0'(s)\big)' &= \Phi''(s)\Phi_0(s) - \Phi(s)\Phi_0''(s) \\ &\leq -k\Phi(s)\Phi_0(s) + k\Phi(s)\Phi_0(s) = 0. \end{aligned}$$

Integrate this inequality to find that, for all $s \in (0,\alpha]$,

$$\Phi'(s)\Phi_0(s) - \Phi(s)\Phi_0'(s) \leq \Phi'(0)\Phi_0(0) - \Phi(0)\Phi_0'(0) = 0.$$

Thus, the function $\Phi(s)/\Phi_0(s)$ is decreasing with respect to s. This proves (ii). Now, the local geometry indicates that

$$\lim_{s\to 0}\frac{J(v,s)}{J_k(s)} = 1,$$

and according to (ii), we have

$$J(v,s) \leq J_k(s), \quad \forall s \in (0,\alpha].$$

Integrate this inequality and use theorem 3.2.2 to find (i). In order to prove (iii) we need the following lemma.

Lemma 3.3.2. *Suppose f, g are positive integrable functions of a real variable r for which f/g is decreasing with respect to r. Then, the function*

$$\int_0^r f \Big/ \int_0^r g$$

is also decreasing with respect to r.

Proof. Let $0 \leq r \leq R$. Then

$$\int_0^r f \int_0^R g = \int_0^r f \int_0^r g + \int_0^r f \int_r^R g$$

and

$$\int_0^R f \int_0^r g = \int_0^r f \int_0^r g + \int_r^R f \int_0^r g.$$

Set $f = gh$. Then, by hypothesis, h is decreasing. This implies

$$\begin{aligned}\int_0^r f \int_r^R g &= \int_0^r gh \int_r^R g \geq \int_0^r gh(r) \int_r^R g = h(r) \int_0^r g \int_r^R g \\ &= \int_0^r g \int_r^R gh(r) \geq \int_0^r g \int_r^R gh = \int_0^r g \int_r^R f.\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^r f \int_0^R g &= \int_0^r f \int_0^r g + \int_0^r f \int_r^R g \\ &\geq \int_0^r f \int_0^r g + \int_r^R f \int_0^r g = \int_0^R f \int_0^r g,\end{aligned}$$

which proves that the function $\int_0^r f / \int_0^r g$ is decreasing with respect to r. □

Now, (iii) follows easily. From (ii), this lemma and theorem 3.2.2, we have that, for any $0 < r \leq R$,

$$\frac{\int_0^r J(v,s)ds}{\int_0^r J_k(s)ds} \geq \frac{\int_0^R J(v,s)ds}{\int_0^R J_k(s)ds} \Longrightarrow \frac{\int_0^r J(v,s)ds}{B_n \int_0^r J_k(s)ds} \geq \frac{\int_0^R J(v,s)ds}{B_n \int_0^R J_k(s)ds} \Longrightarrow$$
$$\frac{\int_0^r J(v,s)ds}{\int_{\|v\|=1} \int_0^r J_k(s)dsdv} \geq \frac{\int_0^R J(v,s)ds}{\int_{\|v\|=1} \int_0^R J_k(s)dsdv} \Longrightarrow$$
$$\frac{\int_{\|v\|=1} \int_0^r J(v,s)dsdv}{\int_{\|v\|=1} \int_0^r J_k(s)dsdv} \geq \frac{\int_{\|v\|=1} \int_0^R J(v,s)dsdv}{\int_{\|v\|=1} \int_0^R J_k(s)dsdv} \Longrightarrow \frac{Vol_g(B(p,R))}{V_k(R)} \leq \frac{Vol_g(B(p,r))}{V_k(r)},$$

where B_n stands for the volume of the unit ball in T_pM. □

As we have just seen, in complete Riemannian manifolds with Ricci curvature bounded from below, Bishop-Gromov's volume comparison theorem allows us to estimate from above, for each $x \in M$, $0 < r < R$, the ratio $Vol_g(B(x,r))/Vol_g(B(x,r))$ from the corresponding ratio in the simply connected space form $M_{0,k}$. Notice that in the inequality $Ric_{(M,g)} \geq (n-1)kg$ we may take, without loss of generality, $k < 0$. Thus, let us focus for the moment in the hyperbolic space (H^n, g) with constant sectional curvature -1, that is the set

$$H^n := \{x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : g(x,x) = -1 \ , \ x_0 > 0\},$$

equipped with the Lorentz metric

$$g = -dx_0 \otimes dx_0 + \sum_{j=1}^{n} dx_j \otimes dx_j.$$

For this case, we have the following result [7].

Proposition 3.3.3. *Let b_n be the volume of the Euclidean ball of radius one. Maintaining the notation of the theorem above, one has that for each $t > 0$, $\lambda > 0$*

$$(i)\ V_{-1}(t) = nb_n \int_0^t (sinhs)^{n-1} ds.$$
$$(ii)\ b_n t^n \leq V_{-\lambda}(t) \leq b_n t^n e^{(n-1)t\sqrt{\lambda}}.$$

Proof. Pick any $x \in H^n$, $v \in T_xH^n \simeq (0,\infty) \times \mathbb{R}^n$ with $\|v\| = 1$ and let $\{e_1, \ldots, e_{n-1}, v\}$ be an orthonormal base for T_xH^n. Clearly, the geodesic curve

$$\gamma : \mathbb{R} \longrightarrow H^n, \quad \gamma(s) := coshs.x + sinhs.v$$

satisfies $\gamma(0) = x$ and $\gamma'(0) = v$. Consider the geodesic variation of γ,

$$f : \mathbb{R} \times (-\epsilon, \epsilon) \longrightarrow H^n, \quad f(s,t) := coshs.x + sinhs(cost + sint).v$$

The Jacobi field along γ which corresponds to this variation is given by

$$Y(s) := \frac{\partial f}{\partial t}(s,0) = sinhs.v, \;\; s \in \mathbb{R} \tag{3.17}$$

and clearly satisfies the initial conditions $Y(0) = 0, \;\; Y'(0) = v$. Now, let Y_i be the Jacobi field along γ with $Y_i(0) = 0, \;\; Y_i'(0) = e_i \; (i = 1, \dots, n-1)$. Then, theorem 1.1.2 implies

$$\begin{aligned} V_{-1}(t) &= \int_{\|v\|=1} \int_0^t \Big(det\big(g(Y_i(s), Y_j(s))\big)\Big)^{1/2} dsdv \\ &= \int_{\|v\|=1} \int_0^t \Big(det\big((sinhs)^2 g(e_i(s), e_j(s))\big)\Big)^{1/2} dsdv \\ &= \int_{\|v\|=1} \int_0^t (sinhs)^{n-1} dvds \\ &= Vol_{can.}(S^{n-1}) \int_0^t (sinhs)^{n-1} ds \\ &= nb_n \int_0^t (sinhs)^{n-1} ds. \end{aligned}$$

For (ii), the obvious inequality $s \leq sinhs \leq se^s, \; \forall \; s \geq 0$ and (i) imply

$$\begin{aligned} & nb_n \int_0^\infty s^{n-1} ds \leq nb_n \int_0^\infty (sinhs)^{n-1} ds \leq nb_n \int_0^\infty s^{n-1} e^{(n-1)s} ds \Longrightarrow \\ & b_n t^n \leq V_{-1}(t) \leq nb_n \int_0^\infty e^{(n-1)s} d(\frac{s^n}{n}) = nb_n \Big(t^n e^{(n-1)t} - \int_0^\infty \frac{s^n}{n} de^{(n-1)s} \Big) \Longrightarrow \\ & b_n t^n \leq V_{-1}(t) \leq nb_n \Big(t^n e^{(n-1)t} - \int_0^\infty \frac{s^n}{n} de^{(n-1)s} \Big) \leq b_n t^n e^{(n-1)t}. \end{aligned} \tag{3.18}$$

Let λ be a positive fixed number. We consider the Riemannian manifold (H^n, g') where $g' = \frac{1}{\lambda} g$. One can easily verify that the sectional curvature of (H^n, g') is given by $k' \equiv K_{g'}(\sigma) = -\lambda$, for all 2-dimenional subspaces $\sigma \subseteq T_x H^n$. As a consequence, for all $\lambda > 0$ and $t > 0$, we have

$$\begin{aligned} V_{-\lambda}(t) \equiv Vol_{g'}\big(B(x,t)\big) &= \int_{\|v\|=1} \int_0^t \Big(det(g'(Y_i(s), Y_j(s)))\Big)^{1/2} dsdv \\ &= \frac{1}{\sqrt{\lambda^{n-1}}} \int_{\|v\|=1} \int_0^t \Big(det(g(Y_i(s), Y_j(s)))\Big)^{1/2} dsdv \\ &= \frac{1}{\sqrt{\lambda}\sqrt{\lambda^{n-1}}} \int_{\|v\|=1} \int_0^{t\sqrt{\lambda}} \Big(det(g(Y_i(\frac{s}{\sqrt{\lambda}}), Y_j(\frac{s}{\sqrt{\lambda}})))\Big)^{1/2} dsdv \\ &= \frac{1}{\sqrt{\lambda^n}} V_{-1}(t\sqrt{\lambda}). \end{aligned} \tag{3.19}$$

Rescaling (3.18) and inserting (3.19) into it, the result follows. □

We will close this section with the following explicit inequality that will be used many times in the following pages.

Proposition 3.3.4. *Let (M,g) be a complete Riemannian manifold with $Ric_{(M,g)} \geq kg$ for some $k \in \mathbb{R}$. Furthermore, let $R > 0$ be given. Then, for all $x \in M$ and $0 < r < R$,*

$$Vol_g(B(x,R)) \leq \left(\frac{R}{r}\right)^n e^{(n-1)R\sqrt{|k|}} Vol_g\big(B(x,r)\big).$$

Proof. *Let $x \in M$, $R > 0$ and $0 < r < R$ and assume that $k < 0$. Then, according to what we said above, we have*

$$\begin{aligned} Vol_g\big(B(x,R)\big) &\leq \frac{V_k(R)}{V_k(r)} Vol_g\big(B(x,r)\big) \leq \frac{b_n R^n e^{(n-1)\sqrt{-k}R}}{b_n r^n} Vol_g\big(B(x,r)\big) \\ &= \left(\frac{R}{r}\right)^n e^{(n-1)R\sqrt{-k}} Vol_g\big(B(x,r)\big). \end{aligned}$$

□

3.4. A covering lemma

We will now present a covering lemma the proof of which is based on Bishop-Gromov's theorem. This lemma will allow us to pass from local to global analysis.

Lemma 3.4.1. *Let (M,g) a complete Riemannian manifold with $Ric_{(M,g)} \geq kg$ for some $k \in \mathbb{R}$ and let $\varrho > 0$ be given. Then, there exists a sequence $\{x_i\}_{i=1}^{\infty} \subseteq M$ such that*

(i) $\forall i \neq j : B(x_i, \frac{\varrho}{2}) \cap B(x_j, \frac{\varrho}{2}) = \emptyset$.

(ii) For any $r \geq \varrho$, the family $\{B(x_i,r)\}_{i=1}^{\infty}$ is a uniformly locally finite covering of M, that is

- *(a) $\bigcup_{i=1}^{\infty} B(x_i,r) = M$.*
- *(b) There exists a positive constant $N = N(n,\varrho,r,k)$ and an upper bound of N in terms of n,ϱ,r and k such that each point of M has a neighbourhood which intersects at most N of the $B(x_i,r)$'s.*

Proof. Let $\varrho > 0$ be given. We define

$$X_\varrho := \Big\{ \{x_i\}_{i\in I} : d(x_i,x_j) \geq \varrho, \forall i,j \in I \text{ with } i \neq j \text{ and } I \text{ countable} \Big\}.$$

It is not difficult to verify that X_ϱ is partially ordered by inclusion and every chain in X_ϱ has upper bound. Hence, by Zorn's lemma, there exists a maximal element $\{x_i\}_{i\in I} \subseteq X_\varrho$ for some countable set I, that is

$$\{z_j\}_{j\in J} \subseteq \{x_i\}_{i\in I}, \text{ for any sequence } \{z_j\}_{j\in J} \subseteq X_\varrho.$$

We will now show that this maximal element is the desired sequence. In order to prove (iia) we assume that there exists a point $x \in M$ such that $x \notin B(x_i,\varrho)$ for all $i \in I$. This leads to contradiction since the sequence $\{y_i\}_{i\in I} := \{x, x_i\}_{i\in I} \subseteq X_\varrho$ is such that

$\{y_i\}_{i\in I} \nsubseteq \{x_i\}_{i\in I}$ and so $\{x_i\}_{i\in I}$ is no longer a maximal element of X_ϱ. Now, for (i), we assume that there exists $i, j \in I$ with $i \neq j$ such that

$$B(x_i, \frac{\varrho}{2}) \cap B(x_j, \frac{\varrho}{2}) \neq \emptyset \Longleftrightarrow \exists z \in M : d(x_i, z) < \frac{\varrho}{2} \text{ and } d(x_j, z) < \frac{\varrho}{2}.$$

But then, the triangular inequality implies $d(x_i, x_j) < \varrho$ and so $\{x_i\}_{i\in I}$ is no longer an element of X_ϱ. We will now show the validity of (iib) in terms of n, ϱ, r and k. For fixed $x \in M$ and $r > 0$, we define

$$I_r(x) := \{i \in I : x \in B(x_i, r)\}.$$

Observe that, since $r \geq \varrho$, the triangular inequality implies

$$\bigcup_{i\in I_r(x)} B(x_i, \frac{\varrho}{2}) \subseteq B(x, 2r) \quad \text{and} \quad \forall i \in I_r(x),\ B(x, r) \subseteq B(x_i, 2r). \tag{3.20}$$

Now, using the proposition 3.3.4, (3.20) and (i) we get

$$\begin{aligned}
Vol_g(B(x,r)) &\geq e^{-2r(n-1)\sqrt{|k|}}\Big(\frac{r}{2r}\Big)^n Vol_g(B(x,2r)) \\
&\geq \frac{1}{2^n} e^{-2r(n-1)\sqrt{|k|}} Vol_g\Big(\bigcup_{i\in I_r(x)} B(x_i, \frac{\varrho}{2})\Big) \\
&= \frac{1}{2^n} e^{-2r(n-1)\sqrt{|k|}} \sum_{i\in I_r(x)} Vol_g(B(x_i, \frac{\varrho}{2})) \\
&\geq \frac{1}{2^n} e^{-2r(n-1)\sqrt{|k|}} \sum_{i\in I_r(x)} e^{-2r(n-1)\sqrt{|k|}}\Big(\frac{\varrho}{4r}\Big)^n Vol_g(B(x_i, 2r)) \\
&= \Big(\frac{\varrho}{8r}\Big)^n e^{-4r(n-1)\sqrt{|k|}} \sum_{i\in I_r(x)} Vol_g(B(x_i, 2r)) \\
&\geq \Big(\frac{\varrho}{8r}\Big)^n e^{-4r(n-1)\sqrt{|k|}} \sum_{i\in I_r(x)} Vol_g(B(x, r)) \\
&= \Big(\frac{\varrho}{8r}\Big)^n e^{-4r(n-1)\sqrt{|k|}} Vol_g(B(x,r))\Big(\sum_{i\in I_r(x)} 1\Big) \\
&= \Big(\frac{\varrho}{8r}\Big)^n e^{-4r(n-1)\sqrt{|k|}} Vol_g(B(x,r))\mathrm{Card} I_r(x)
\end{aligned}$$

Hence, for any $r \geq \varrho$ and $x \in M$,

$$Card I_r(x) \leq \Big(\frac{8r}{\varrho}\Big)^n e^{4r(n-1)\sqrt{|k|}} := C(n, \varrho, r, k). \tag{3.21}$$

Finally, given $r \geq \varrho$ and $i \in I$, assume that

$$Card\{j \in I : B(x_i, r) \cap B(x_j, r) \neq \emptyset\} = N.$$

Then,

$$C(n, \varrho, 2r, k) \geq \mathrm{Card} I_{2r}(x_i) \geq N + 1 \Longrightarrow N \leq C(n, \varrho, 2r, k) - 1,$$

which concludes the proof of the lemma. □

Chapter 4

Buser's Isoperimetric Inequality and Consequences

Let (M, g) be a complete Riemannian manifold of dimension n satisfying $Ric_{(M,g)} \geq kg$ for some $k \in \mathbb{R}$ and pick any $\varrho > 0$. In this chapter we will be concerned with the validity of the L^1-Poincare inequality on the manifold, that is, the existence of a positive constant $C \equiv C(n, k, \varrho)$ such that for any $0 < r < \varrho$ and any $u \in C^\infty(M)$,

$$\int_M |u(x) - \overline{u}_r(x)| d\mu_g(x) \leq Cr \int_M |\nabla u(x)| d\mu_g(x). \tag{4.1}$$

Here, $\overline{u}_r(x)$ denotes the average of u over the ball $B(x, r)$, namely

$$\overline{u}_r(x) := \frac{1}{Vol_g\big(B(x,r)\big)} \int_{B(x,r)} u(y) d\mu_g(y) \ , \ x \in M. \tag{4.2}$$

Such an inequality will play a critical role in our study. First, we will introduce Buser's isoperimetric inequality and we will see that such an inequality gives a local version of (4.1) over the ball $B(x, r)$, namely, the existence of a positive constant $C \equiv C(n, k, \varrho)$ such that for any $x \in M$, any $0 < r < 2\varrho$ and any $u \in C^\infty\big(B(x, r)\big)$,

$$\int_{B(x,r)} |u(y) - \overline{u}_r(x)| d\mu_g(y) \leq Cr \int_{B(x,r)} |\nabla u(y)| d\mu_g(y). \tag{4.3}$$

Inequalities of this type are called pseudo-Poincare inequalities. Later, using the covering lemma 3.4.1, we will expand (4.3) throughout the whole manifold proving (4.1).

4.1. Buser's isoperimetric inequality

For any given $x \in M$ and $r > 0$ let $\mathcal{G}$ be the set of all smooth hypersurfaces Γ in $B(x, r)$ with $\overline{\Gamma}$ embedded in $\overline{B(x, r)}$ which divides $B(x, r)$, $B(x, r) \setminus \Gamma = D_1 \cup D_2$, into two disjoint, open in $B(x, r)$ sets D_1, D_2.

Theorem 4.1.1. (**Buser**) *Let (M, g) be a complete Riemannian manifold of dimension n with $Ric_{(M,g)} \geq kg$, for some $k < 0$, on all of M. For any given $x \in M$, $r > 0$ and $\Gamma \in \mathcal{G}$ there exists a positive constant $C_B \equiv C_B(n, k, r)$ such that*

$$\min\{Vol_g(D_1), Vol_g(D_2)\} \leq C_B Area_g(\Gamma). \tag{4.4}$$

Proof. We shall actually prove a more general result [2], [3]. We shall consider, instead of a ball $B(x,r) \subseteq M$, a domain $D \subseteq M$ and we assume that there exists a fixed origin $0 \in D$ and two positive numbers r, R with $0 < r \leq R$ such that

(i) $\forall q \in D,\ Ric_{(M,g)}(q) \geq kg(q)$, for some constant $k < 0$,

(ii) $\overline{B(0,R)}$ is complete, that is the exponential map exp_0 is defined for every tangent vector v with $\|v\| = R$,

(iii) $B(0,r) \subseteq D \subseteq B(0,R)$,

(iv) D is starlike, that is, for any point $q \in D$ any minimizing geodesic joining 0 to p is contained in D.

We also define $\mathcal{G}$ in a similar way replacing $B(x,r)$ by D. In order to find the best constant in (4.4) we will consider a parameter $0 < t < r/2$ and we will prove that for every $\Gamma \in \mathcal{G}$,

$$\frac{Area_g(\Gamma)}{\min\{Vol_g(D_1), Vol_g(D_1)\}} \geq C(n,k,r,R,t). \tag{4.5}$$

Hence, maximizing the right hand side of this inequality with respect to t we obtain the best lower bound. Before we proceed to the proof let us give some notation and preliminaries. For any $\xi \in T_0M$ with $\|\xi\| = 1$, γ_ξ will denote the geodesic curve having $\gamma_\xi(0) = 0,\ \gamma'_\xi(0) = \xi$. From (3.3), recall that $c(\xi)$ denotes the distance along the geodesic γ_ξ from 0 to its cut point along γ_ξ. Furthermore, we supply the unit tangent sphere at 0,

$$\mathbb{S}_0^{n-1}(1) := \{\xi \in T_0M : \|\xi\| = 1\},$$

with the endowed $(n-1)-$measure $d\xi$. According to the theorem 3.2.2, for each $\xi \in \mathbb{S}_0^{n-1}(1)$ and $0 < s < c(\xi)$, the n-dimensional measure on M and the $(n-1)$-dimensional measure on the smooth points of the sphere $\partial B(0,s)$, can be written respectively as

$$d\mu_g(exp_0(s\xi)) = \sqrt{g(s,\xi)}dsd\xi \quad , \quad dA_g\Big|_{\partial B(0,r)}(exp_0(s\xi)) = \sqrt{g(s,\xi)}d\xi. \tag{4.6}$$

In addition, when we refer to the simply connected space form $M_{0,k}$ of constant sectional curvature k $(k < 0)$, the area of the $(n-1)-$dimensional sphere $\partial B(0,r)$ and the volume of the $n-$dimensional disk $B(0,s)$ will be $A_k(s), V_k(s)$ respectively. From Bishop-Gromov's theorem, (see theorem 3.3.1), we have

$$\frac{\sqrt{g(s_1,\xi)}}{A_k(s_1)} \geq \frac{\sqrt{g(s_2,\xi)}}{A_k(s_2)} \ , \text{ for all } 0 < s_1 \leq s_2, \tag{4.7}$$

from which, for every $0 \leq r < R$, we obtain

$$\begin{aligned}
&\forall \tau : r \leq \tau \leq R, \ \frac{\sqrt{g(r,\xi)}}{A_k(r)} \geq \frac{\sqrt{g(\tau,\xi)}}{A_k(\tau)} \Longrightarrow \\
&\int_r^R \sqrt{g(\tau,\xi)}d\tau \leq \frac{\sqrt{g(r,\xi)}}{A_k(r)} \int_r^R A_k(\tau)d\tau \Longrightarrow \\
&\int_r^R \sqrt{g(\tau,\xi)}d\tau \leq \frac{\sqrt{g(r,\xi)}}{A_k(r)} (V_k(R) - V_k(r)) \Longrightarrow \\
&\frac{\sqrt{g(r,\xi)}}{A_k(r)} \geq \frac{1}{V_k(R) - V_k(r)} \int_r^R \sqrt{g(\tau,\xi)}d\tau.
\end{aligned} \tag{4.8}$$

Furthermore, from (4.7) and lemma 3.3.2, the function

$$\frac{\int_0^r \sqrt{g(s,\xi)}ds}{\int_0^r A_k(s)ds}$$

is decreasing with respect to r and so, for every $0 \leq r_0 < r_1 \leq r_2$,

$$\begin{aligned}
&\frac{\int_0^{r_1} \sqrt{g(s,\xi)}ds}{\int_0^{r_1} A_k(s)ds} \geq \frac{\int_0^{r_2} \sqrt{g(s,\xi)}ds}{\int_0^{r_2} A_k(s)ds} \Longrightarrow \\
&\frac{\int_{r_0}^{r_1} \sqrt{g(s,\xi)}ds}{\int_{r_0}^{r_1} A_k(s)ds} \geq \frac{\int_{r_1}^{r_2} \sqrt{g(s,\xi)}ds}{\int_{r_1}^{r_2} A_k(s)ds} \Longrightarrow \\
&\frac{1}{V_k(r_1) - V_k(r_0)} \int_{r_0}^{r_1} \sqrt{g(s,\xi)}ds \geq \frac{1}{V_k(r_2) - V_k(r_1)} \int_{r_1}^{r_2} \sqrt{g(s,\xi)}ds.
\end{aligned} \tag{4.9}$$

In particular, for every $0 < r < R$,

$$\frac{1}{V_k(r)} \int_0^r \sqrt{g(s,\xi)}ds \geq \frac{1}{V_k(R)} \int_0^R \sqrt{g(s,\xi)}ds. \tag{4.10}$$

Given any $\Gamma \in \mathcal{G}$ the sets D_1, D_2 are uniquely defined. By choosing appropriately the set D_1, we shall prove that

$$Area_g(\Gamma) \geq C(n,k,r,R,t)Vol_g(D_1) \tag{4.11}$$

independent of whether D_1 has the minimum volume of D_1, D_2. This inequality will give immediately (4.5). In the "ideal" case where $\Gamma = \partial B(0, r/2)$ we choose $D_1 :=$

$D \setminus B(0, r/2)$. Then, (4.11) easily follows from (4.8):

$$\frac{\sqrt{g(r/2,\xi)}}{A_k(r/2)} \geq \frac{1}{V_k(R) - V_k(r/2)} \int_{r/2}^{R} \sqrt{g(\tau,\xi)}d\tau, \quad for\ all\ \xi \in \mathbb{S}_0^{n-1}(1) \Longrightarrow$$
$$\sqrt{g(r/2,\xi)} \geq \frac{A_k(r/2)}{V_k(R) - V_k(r/2)} \int_{r/2}^{R} \sqrt{g(\tau,\xi)}d\tau, \quad for\ all\ \xi \in \mathbb{S}_0^{n-1}(1) \Longrightarrow$$
$$\int_{\mathcal{S}_0^{n-1}(1)} \sqrt{g(r/2,\xi)}d\xi \geq \frac{A_k(r/2)}{V_k(R) - V_k(r/2)} \int_{\mathcal{S}_0^{n-1}(1)} \int_{r/2}^{R} \sqrt{g(\tau,\xi)}d\tau d\xi \Longrightarrow$$
$$\begin{aligned} Area_g\big(\partial B(0,r/2)\big) &\geq \frac{A_k(r/2)}{V_k(R) - V_k(r/2)} Vol_g\big(B(0,R) \setminus B(0,r/2)\big) \\ &\geq \frac{A_k(r/2)}{V_k(R) - V_k(r/2)} Vol_g(D_1). \end{aligned}$$

Let $\Gamma \in \mathcal{G}$. Then, the inequality

$$Vol_g(D_i \cap B(0,r/2)) \leq \frac{1}{2} Vol_g(B(0,r/2)) \tag{4.12}$$

must be satisfied for at least one $i = 1, 2$ because otherwise

$$\begin{aligned} Vol_g(D) &= Vol_g(D_1 \cup D_2 \cup G) = Vol_g(D_1) + Vol_g(D_2) \\ &= Vol_g(D_1 \cap B(0,r/2)) + Vol_g(D_1 \cap B(0,r/2)^c) \\ &\quad + Vol_g(D_2 \cap B(0,r/2)) + Vol_g(D_2 \cap B(0,r/2)^c) \\ &> \frac{1}{2} Vol_g(B(0,r/2)) + \frac{1}{2} Vol_g(D_1 \setminus B(0,r/2)) \\ &\quad + \frac{1}{2} Vol_g(B(0,r/2)) + \frac{1}{2} Vol_g(D_2 \setminus B(0,r/2)) \\ &= Vol_g(B(0,r/2)) + Vol_g(D_1 \setminus B(0,r/2)) + Vol_g(D_2 \setminus B(0,r/2)) \\ &= Vol_g\big(D_1 \cup D_2\big) = Vol_g(D). \end{aligned}$$

Once and for all in this proof we choose D_1 the set which satisfies (4.12), namely

$$Vol_g(D_1 \cap B(0,r/2)) \leq \frac{1}{2} Vol_g(B(0,r/2)). \tag{4.13}$$

This means that we call D_1 the set which inside $B(0, r/2)$ its volume is smaller than the volume of D_2. We will now consider two cases. Case 1 will be the case where $D \setminus B(0, r/2)$ contains a "significant" portion of D_1. In this case we will adjust the argument of the above "ideal" case while in case 2 we will not have this "significant" portion of D_1 and one gives the argument relative to a new origin $0' \in D$, replacing $0 \in D$.

Case 1 : In this case we assume that there exists a significant portion of D_1 outside the ball $B(0, r/2)$, that is

$$Vol_g(D_1 \cap B(0,r/2)) \leq \alpha Vol_g(D_1). \tag{4.14}$$

for some fixed constant $0 < \alpha < 1$. Let γ_{p0} be the minimized geodesic segment from p to 0. Recall that $Cut(0)$ denotes the cut locus of $0 \in M$, i.e the set of all points in the manifold where the geodesics starting at 0 stop being minimizing. For any given point $p \in D_1 \setminus Cut(0)$, we define $p^\star$ to be the first point where γ_{p0} meets Γ. If γ_{p0} is completely contained in D_1, then we set $p^\star = 0$. Since $p \notin Cut(0)$, this point $p^\star$ is uniquely defined. Notice that each point $p \in D_1 \setminus (Cut(0) \cup B(0, r/2))$ can be written as

$$p = exp_0(s\xi) \text{ ,for some } r/2 < s < c(\xi) \text{ , } \|\xi\| = 1.$$

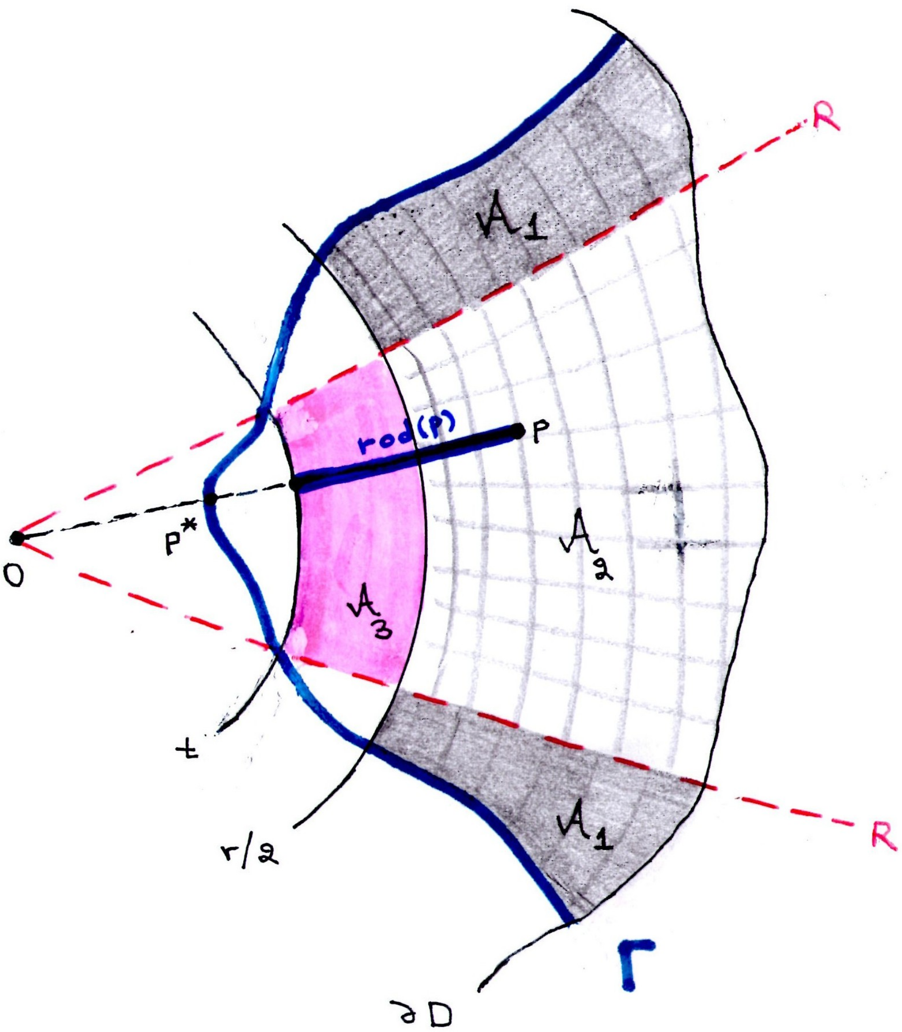
In order to prove (4.11), we consider

$$rodp := \{exp_0(\tau\xi) : t \leq \tau \leq s\} \tag{4.15}$$

$$\mathcal{A}_1 := \{p \in D_1 \setminus (Cut(0) \cup \overline{B(0, r/2)}) : p^\star \notin \overline{B(0,t)}\}, \tag{4.16}$$

$$\mathcal{A}_2 := \{p \in D_1 \setminus (Cut(0) \cup \overline{B(0, r/2)}) : p^\star \in \overline{B(0,t)}\}, \tag{4.17}$$

$$\mathcal{A}_3 := \big(B(0, r/2) \setminus B(0,t)\big) \cap \big(\bigcup_{p \in \mathcal{A}_2} rodp\big). \tag{4.18}$$



Furthermore, let

$$\mathbf{v} : B(0,R) \setminus (Cut(0) \cup \{0\}) \longrightarrow \mathbb{S}_0^{n-1}(1) \ , \ \mathbf{v}(exp_0(s\xi)) := \xi,$$

and observe that $\mathbf{v}(\mathcal{A}_2) = \mathbf{v}(\mathcal{A}_3)$. According to (4.9), we have

$$\frac{1}{V_k(r/2) - V_k(t)} \int_t^{r/2} \sqrt{g(s,\xi)}ds \geq \frac{1}{V_k(R) - V_k(r/2)} \int_{r/2}^{R} \sqrt{g(s,\xi)}ds \Longrightarrow$$

$$\frac{V_k(R) - V_k(r/2)}{V_k(r/2) - V_k(t)} \int_t^{r/2} \sqrt{g(s,\xi)}ds \geq \int_{r/2}^{R} \sqrt{g(s,\xi)}ds \Longrightarrow$$

$$\gamma(r,k,R,t) \int_{\mathbf{v}(\mathcal{A}_3)} \int_t^{r/2} \sqrt{g(s,\xi)}dsd\xi \geq \int_{\mathbf{v}(\mathcal{A}_3)} \int_{r/2}^{R} \sqrt{g(s,\xi)}dsd\xi = \int_{\mathbf{v}(\mathcal{A}_2)} \int_{r/2}^{R} \sqrt{g(s,\xi)}dsd\xi \Longrightarrow$$

$$\gamma(r,k,R,t)\ Vol_g(\mathcal{A}_3) \geq Vol_g(\text{something greater than } \mathcal{A}_2) \geq Vol_g(\mathcal{A}_2).$$

Hence,

$$\frac{Vol_g(\mathcal{A}_2)}{Vol_g(\mathcal{A}_3)} \leq \gamma(r,k,R,t) := \frac{V_k(R) - V_k(r/2)}{V_k(r/2) - V_k(t)}. \tag{4.19}$$

But since we are in case 1, inequality (4.14), implies

$$\begin{aligned}
Vol_g(D_1) &= Vol_g(D_1 \cap B(0,r/2)) + Vol_g(D_1 \setminus B(0,r/2)) \\
&\leq \alpha Vol_g(D_1) + Vol_g(D_1 \setminus B(0,r/2)) \\
&= \alpha Vol_g(D_1) + Vol_g(\mathcal{A}_1 \cup \mathcal{A}_2) \Longrightarrow \\
(1-\alpha)Vol_g(D_1) &= Vol_g(\mathcal{A}_1) + Vol_g(\mathcal{A}_2) \\
&\leq Vol_g(\mathcal{A}_1) + \gamma(r,k,R,t)Vol_g(\mathcal{A}_3) \\
&\leq Vol_g(\mathcal{A}_1) + \gamma(r,k,R,t)Vol_g(D_1 \cap B(0,r/2)) \\
&\leq Vol_g(\mathcal{A}_1) + \gamma(r,k,R,t)\alpha Vol_g(D_1),
\end{aligned}$$

and so,

$$\frac{Vol_g(\mathcal{A}_1)}{Vol_g(D_1)} \geq 1 - \alpha(1 + \gamma(r,k,R,t)). \tag{4.20}$$

Clearly, according to this inequality above, in order to prove (4.11), we need to find a lower bound for the quantity $Area_g(\Gamma)/Vol_g(\mathcal{A}_1)$, since

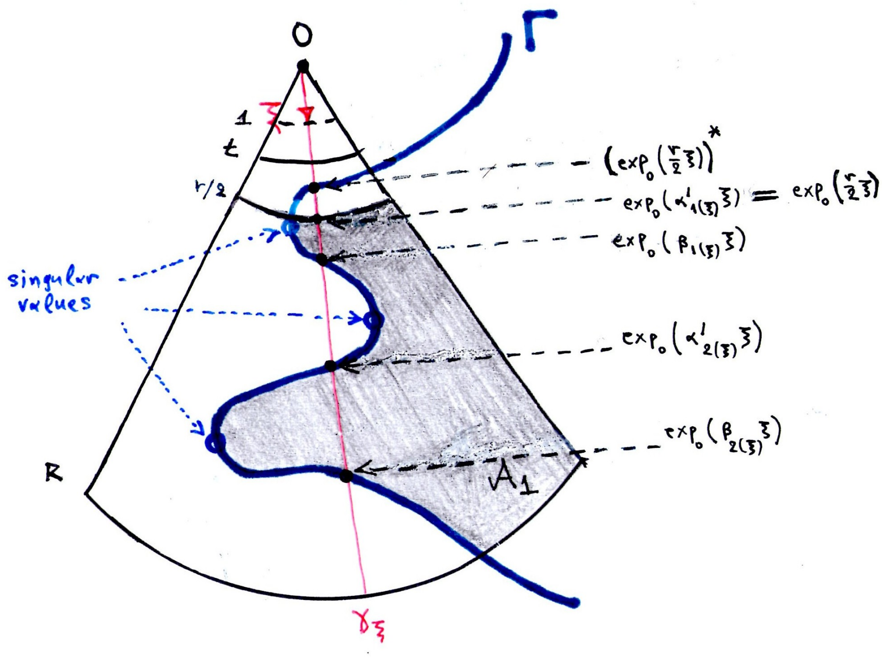
$$\frac{Area_g(\Gamma)}{Vol_g(D_1)} = \frac{Area_g(\Gamma)}{Vol_g(\mathcal{A}_1)} \frac{Vol_g(\mathcal{A}_1)}{Vol_g(D_1)} \geq \frac{Area_g(\Gamma)}{Vol_g(\mathcal{A}_1)} \Big(1 - \alpha(1 + \gamma(r,k,R,t))\Big).$$

Let $\mathbb{1}_{\mathcal{A}_1}$ be the indicator function of $\mathcal{A}_1$. Using (4.6), we may write

$$\begin{aligned}
Vol_g(\mathcal{A}_1) &= \int_{\mathcal{A}_1} \mathbb{1}_{\mathcal{A}_1}(x)d\mu_g(x) \Longrightarrow \\
Vol_g(\mathcal{A}_1) &= \int_{\|\xi\|=1} \int_0^{\min\{R,c(\xi)\}} \mathbb{1}_{\mathcal{A}_1}(exp_0(s\xi))\sqrt{g(s,\xi)}dsd\xi \Longrightarrow \\
Vol_g(\mathcal{A}_1) &= \int_{\mathbf{v}(\mathcal{A}_1)} \int_{r/2}^{\min\{R,c(\xi)\}} \mathbb{1}_{\mathcal{A}_1}(exp_0(s\xi))\sqrt{g(s,\xi)}dsd\xi.
\end{aligned} \tag{4.21}$$

For any $\xi \in \mathbf{v}(\mathcal{A}_1)$, we have

$$\int_{r/2}^{\min\{R,c(\xi)\}} \mathbb{1}_{\mathcal{A}_1}(exp_0(s\xi))\sqrt{g(s,\xi)}ds = \sum_{j(\xi)} \left(\int_{\alpha'_{j(\xi)}}^{\beta_{j(\xi)}} 1 \cdot \sqrt{g(s,\xi)}ds + \int_{\beta_{j(\xi)}}^{\alpha'_{j(\xi)+1}} 0 \cdot \sqrt{g(s,\xi)}ds \right).$$



Let

$$\alpha_{j(\xi)} := \begin{cases} \alpha'_{j(\xi)} & , r/2 < \alpha'_{j(\xi)} \\ \left|exp_0^{-1}\left(exp_0(\frac{r}{2}\xi)^\star\right)\right| & , r/2 = \alpha'_{j(\xi)} \end{cases}$$

where $\left|exp_0^{-1}\left(exp_0(\frac{r}{2}\xi)^\star\right)\right|$ is simply the length of the geodesic segment $\gamma_{0,exp_0(\frac{r}{2}\xi)^\star}$. It is not difficult to verify that

(1) $\alpha_{j(\xi)} \leq \alpha'_{j(\xi)}$,

(2) $exp_0(\alpha_{j(\xi)}\xi) \in \Gamma$,

(3) $\mathbf{v}(\mathcal{A}_1) = \mathbf{v}(\Gamma_0)$, where $\Gamma_0 := \{exp_0(\alpha_{j(\xi)}\xi) : \xi \in \mathbf{v}(\mathcal{A}_1),\ j(\xi) = 1, 2, 3, \ldots\}$.

Now, applying (4.8), we get

$$\int_{r/2}^{\min\{R,c(\xi)\}} \mathbb{1}_{\mathcal{A}_1}(exp_0(s\xi))\sqrt{g(s,\xi)}ds = \sum_{j(\xi)} \int_{\alpha'_{j(\xi)}}^{\beta_{j(\xi)}} \sqrt{g(s,\xi)}ds \leq \sum_{j_\xi} \int_{\alpha_{j(\xi)}}^{\beta_{j(\xi)}} \sqrt{g(s,\xi)}ds$$
$$\leq \sum_{j(\xi)} \frac{V_k\big(\beta_{j(\xi)}(\xi)\big) - V_k\big(\alpha_{j(\xi)}(\xi)\big)}{A_k\big(\alpha_{j(\xi)}(\xi)\big)} \sqrt{g(\alpha_{j(\xi)}(\xi),\xi)}$$
$$\leq \frac{V_k(R) - V_k(t)}{A_k(t)} \sum_{j(\xi)} \sqrt{g(\alpha_{j(\xi)}(\xi),\xi)}.$$

Integrate this inequality for all $\xi \in \mathbf{v}(\mathcal{A}_1) = \mathbf{v}(\Gamma_0)$ to find

$$\begin{aligned}
Vol_g(\mathcal{A}_1) &= \int_{\mathbf{v}(\mathcal{A}_1)} \int_{r/2}^{\min\{R,c(\xi)\}} \mathbb{1}_{\mathcal{A}_1}(exp_0(s\xi))\sqrt{g(s,\xi)}dsd\xi \\
&\leq \frac{V_k(R)-V_k(t)}{A_k(t)} \int_{\mathbf{v}(\Gamma_0)} \sum_{j(\xi)} \sqrt{g(\alpha_{j(\xi)}(\xi),\xi)}d\xi \\
&= \frac{V_k(R)-V_k(t)}{A_k(t)} \Big\{ \underbrace{\int_{\mathbf{v}(\Gamma_0)\cap R_{\mathbf{v}|\Gamma_0}} \sum_{j(\xi)} \sqrt{g(\alpha_{j(\xi)}(\xi),\xi)}d\xi}_{\text{Regular Values}} + \underbrace{\int_{\mathbf{v}(\Gamma_0)\cap R^c_{\mathbf{v}|\Gamma_0}} \sum_{j(\xi)} \sqrt{g(\alpha_{j(\xi)}(\xi),\xi)}d\xi}_{\text{Singular Values}} \Big\} \\
&= \frac{V_k(R)-V_k(t)}{A_k(t)} \int_{\mathbf{v}(\Gamma_0)\cap R_{\mathbf{v}|\Gamma_0}} \sum_{j(\xi)} \sqrt{g(\alpha_{j(\xi)}(\xi),\xi)}d\xi,
\end{aligned}$$

since, by Sard's theorem the set of the singular values has measure zero. Finally, observe that the integral

$$\int_{\mathbf{v}(\Gamma_0)\cap R_{\mathbf{v}|\Gamma_0}} \sum_{j(\xi)} \sqrt{g(\alpha_{j(\xi)}(\xi),\xi)}d\xi$$

gives only a portion of Γ's area, and so,

$$\int_{\mathbf{v}(\Gamma_0)\cap R_{\mathbf{v}|\Gamma_0}} \sum_{j(\xi)} \sqrt{g(\alpha_{j(\xi)}(\xi),\xi)}d\xi \leq Area_g(\Gamma),$$

which leads to

$$Vol_g(\mathcal{A}_1) \leq \frac{V_k(R)-V_k(t)}{A_k(t)} Area_g(\Gamma). \tag{4.22}$$

In summary,

$$\begin{aligned}
&\frac{Area_g(\Gamma)}{Vol_g(D_1)} \geq \frac{Area_g(\Gamma)}{Vol_g(\mathcal{A}_1)}\Big(1-\alpha(1+\gamma(r,k,R,t))\Big) \Longrightarrow \\
&\frac{Area_g(\Gamma)}{Vol_g(D_1)} \geq \frac{A_k(t)}{V_k(R)-V_k(t)}\Big(1-\alpha\big(1+\frac{V_k(R)-V_k(r/2)}{V_{r/2}-V_k(t)}\big)\Big).
\end{aligned}$$

Case 2: In this case we assume that

$$Vol_g(D_1 \cap B(0,r/2)) \geq \alpha Vol_g(D_1), \tag{4.23}$$

for some fixed constant $0<\alpha<1$. As we said above, in this case, we need the following lemma.

Lemma 4.1.2. *Set $W_0 := D_1 \cap B(0,r/2)$, $W_1 := D_2 \cap B(0,r/2)$ (or $W_0 := D_2 \cap B(0,r/2)$, $W_1 := D_1 \cap B(0,r/2)$). Then, for at least one of the two choices above, there exists a point $w_0 \in W_0$ and a measurable set $\mathcal{W}_1 \subseteq W_1$ such that* *(i)* $Vol_g(\mathcal{W}_1) \geq \frac{1}{2} Vol_g(W_1)$,

(ii) *For each* $q \in \mathcal{W}_1$*, each minimized geodesic segment* γ_{qw_0} *intersects* Γ *in a first point* $q^\star$ *such that* $d(q, q^\star) \leq d(q^\star, w_0)$*.*

Proof. Let $\mu^2 := \mu \otimes \mu$ be the product measure on $W_1 \times W_0$. Since the cut locus of any point has $n-$dimensional measure zero, there exist a null set N such that for all $(q, w) \in (W_1 \times W_0) \setminus N$, there exists a unique minimized geodesic segment γ_{qw}. This can be easily proved as follows: Let

$$A := \{(q, w) \in W_1 \times W_0 : \exists \text{ unique minimized geodesic segment } \gamma_{qw}\}$$

and set $N := A^c$. By the definition of the cut locus, we have

$$\begin{aligned}
&\text{For all } q \in W_1 \text{ and for all } w \notin Cut(q),\ (q, w) \in A \Longrightarrow \\
&A^c \subseteq \bigcup_{q \in W_1} \big(\{q\} \times (Cut(q) \cap W_0)\big) \Longrightarrow \\
&\mu^2(A^c) \leq \mu^2\Big(\bigcup_{q \in W_1} \big(\{q\} \times (Cut(q) \cap W_0)\big)\Big) = 0.
\end{aligned}$$

Furthermore, the ball $B(0, r/2)$ is not necessarily convex, but since γ_{qw} is minimizing, we must have

$$\gamma_{qw} \subseteq B(0, r) \subseteq D. \tag{4.24}$$

Indeed; if this is not the case, there exists a point $x \in \gamma_{qw} \setminus B(0, r)$. Then, the triangular inequality yields

$$\begin{aligned}
r = (r - \frac{r}{2}) + (r - \frac{r}{2}) &< (d(0, x) - d(0, q)) + (d(0, x) - d(0, w)) \\
&= |d(0, x) - d(0, q)| + |d(0, x) - d(0, w)| \leq d(q, x) + d(w, x) \\
&= d(w, q) \leq d(0, q) + d(0, w) < \frac{r}{2} + \frac{r}{2} = r, \text{ a contradiction.}
\end{aligned}$$

Clearly, (4.24) implies that γ_{qw} must intersect Γ. Let $q^\ddagger$ (respectively $w^\ddagger$) be the intersection of γ_{qw} (respectively γ_{wq}) with the hypersurface Γ. We define

$$\begin{aligned}
V_0 :=& \{(q, w) \in (W_1 \times W_0) \setminus N : d(q, q^\ddagger) \leq d(q^\ddagger, w)\}, \\
V_1 :=& \{(q, w) \in (W_1 \times W_0) \setminus N : d(q, w^\ddagger) \geq d(w^\ddagger, w)\}.
\end{aligned}$$

Notice that V_0, V_1 are not necessarily disjoint. If we assume that

$$\exists (q, w) \in (W_1 \times W_0) \setminus N : (q, w) \notin V_0,$$

then

$$d(q, w^\ddagger) \geq d(q, q^\ddagger) > d(w, q^\ddagger) \geq d(w, w^\ddagger) \Longrightarrow (q, w) \in V_1$$

Similarly,

$$\exists (q,w) \in (W_1 \times W_0) \setminus N : (q,w) \notin V_1 \Longrightarrow (q,w) \in V_0.$$

As a consequence,

$$V_0 \cup V_1 = (W_1 \times W_0) \setminus N, \tag{4.25}$$

from which we either have

$$Vol_{g\otimes g}(V_0) \geq \frac{1}{2} Vol_{g\otimes g}(W_1 \times W_0) \quad \text{or} \quad Vol_{g\otimes g}(V_1) \geq \frac{1}{2} Vol_{g\otimes g}(W_1 \times W_0). \tag{4.26}$$

Let's say that $Vol_{g\otimes g}(V_0) \geq \frac{1}{2} Vol_{g\otimes g}(W_1 \times W_0)$ holds. We define the map

$$V_0 \longrightarrow \mathcal{P}(W_1) = \text{the power set of } W_1, \quad x \longmapsto V_{0,x} := \{y \in W_1 : (y,x) \in V_0\}.$$

and we claim that

$$\exists w_0 \in W_0 : Vol_g(V_{0,w_0}) \geq \frac{1}{2} Vol_g(W_1). \tag{4.27}$$

In contrary, if we assume that

$$\forall x \in W_0 : Vol_g(V_{0,x}) < \frac{1}{2} Vol_g(W_1), \tag{4.28}$$

then, Fubini's theorem implies

$$\begin{aligned}
\frac{1}{2} Vol_{g\otimes g}(W_1 \times W_0) \leq Vol_{g\otimes g}(V_0) &= \int_{W_1 \times W_0} \mathbb{1}_{V_0}(x,y) d\mu(x) d\mu(y) \\
&= \int_{W_0} \Big(\int_{W_1} \mathbb{1}_{V_{0,x}}(y) d\mu(y) \Big) d\mu(x) = \int_{W_0} Vol_g(V_{0,x}) d\mu(x) \\
&< \int_{W_0} \frac{1}{2} Vol_g(W_1) d\mu(x) \quad \big(\text{since } \mu_g(W_0) > 0\big) \\
&= \frac{1}{2} Vol_g(W_1) Vol_g(W_2) = \frac{1}{2} Vol_{g\otimes g}(W_1 \times W_0),
\end{aligned}$$

which proves (4.27). Setting $\mathcal{W}_1 := V_{0,w_0}$ the result of the lemma follows. □

Now, we return to the main proof choosing $\mathcal{W}_1$ as above. Recall that, since we are in case 2, we have

$$\alpha Vol_g(D_1) \leq Vol_g(D_1 \cap B(0,r/2)),$$

for some fixed constant $0 < \alpha < 1$. From this we obtain

$$\alpha Vol_g(D_1) \leq Vol_g(D_2 \cap B(0,r/2)),$$

because

$$\alpha Vol_g(D_1) \leq Vol_g(D_1 \cap B(0,r/2)) \leq \frac{1}{2} Vol_g(B(0,r/2)).$$

Hence, however W_1 is picked, according to the conclusion of the lemma, we have

$$\begin{aligned} &\alpha Vol_g(D_1) \leq Vol_g(D_1 \cap B(0,r/2)) \leq Vol_g(W_1) \leq 2Vol_g(\mathcal{W}_1) \Longrightarrow \\ &\frac{Area_g(\Gamma)}{Vol_g(D_1)} = \frac{Area_g(\Gamma)}{Vol_g(\mathcal{W}_1)}\frac{Vol_g(\mathcal{W}_1)}{Vol_g(D_1)} \geq \frac{\alpha}{2}\frac{Area_g(\Gamma)}{Vol_g(\mathcal{W}_1)}. \end{aligned} \tag{4.29}$$

Thus, it is sufficient to find a lower bound for $\frac{Area_g(\Gamma)}{Vol_g(\mathcal{W}_1)}$. Center geodesic coordinates at w_0 and define the map

$$\mathbf{v} : B(0,r/2) \setminus Cut(w_0) \longrightarrow \mathbb{S}^{n-1}_{w_0}(1), \quad \mathbf{v}(exp(s\xi)) = \xi.$$

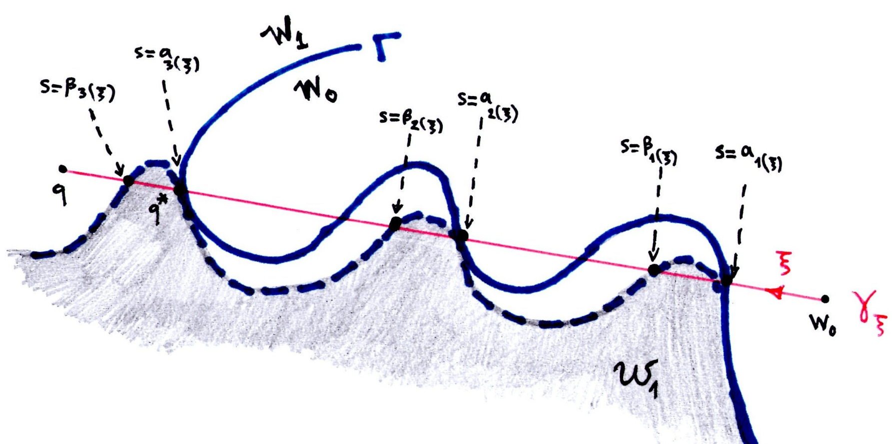
Although this projection is different from $\mathbf{v}$ in case 1, we keep the same notation for simplicity. To each $\xi \in \mathbf{v}(\mathcal{W}_1 \setminus Cut(w_0))$ we determine a collection of disjoint intervals

$$\{(\alpha_{\xi,j}, \beta_{\xi,j}) : j \in \mathcal{J}_\xi\}\ ,\ \mathcal{J}_\xi \text{ is a finite or countably infinite index set}$$

as follows: For $q = exp_{w_0}(t_0\xi) \in \mathcal{W}_1$,

$$\begin{aligned} &\alpha_q := d(w_0, q^\star), \\ &\beta_q := \sup\{t > \alpha_q : \gamma_\xi(t) \in \mathcal{W}_1\ ,\ \gamma_\xi\big((\alpha_q, t)\big) \subseteq D \setminus \Gamma\}. \end{aligned}$$

where $q^\star$ is described in the lemma above. Notice that β_q is either $d(w_0, \partial\mathcal{W}_1)$, $d(w_0, Cut(w_0))$ or $d(w_0, D)$.

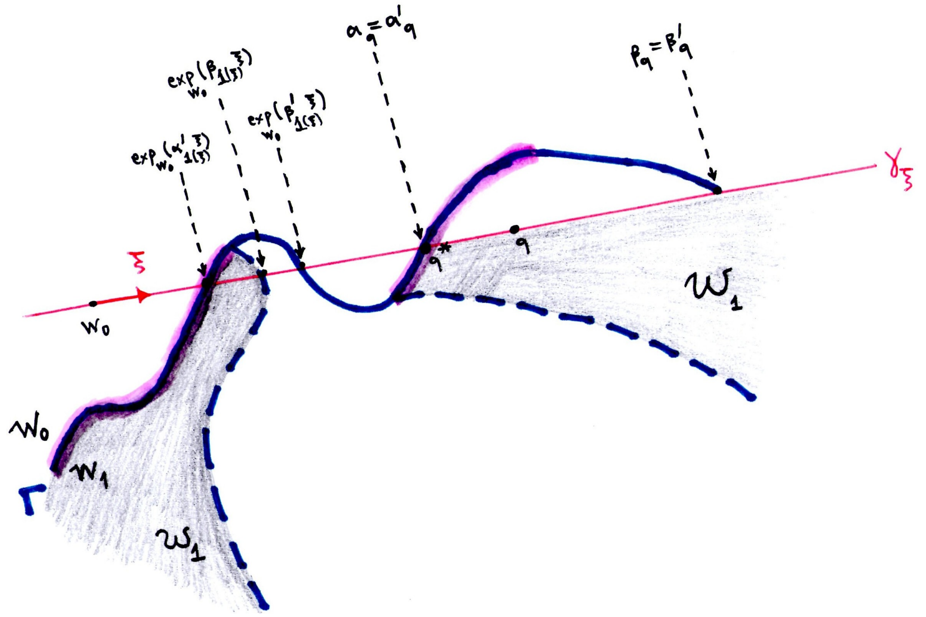


By the definition of $\alpha_{\xi,j}$ and $\beta_{\xi,j}$, using elementary topology, we can prove that

$$\mathcal{W}_1 \cap \gamma_\xi\big((0,+\infty)\big) = \bigcup_{j \in \mathcal{J}_\xi} \gamma_\xi\big((\alpha_{\xi,j}, \beta_{\xi,j})\big).$$

Furthermore, from the definition of $q^\star$ and the fact that the geodesic segment $\gamma_\xi = w_0q$ is contained inside $B(0,r/2)$, we have $\beta_{\xi,j} \leq 2\alpha_{\xi,j} < r$. Therefore, using these estimates about $\alpha_{j,\xi}, \beta_{j,\xi}$ and (4.8), we have

$$\begin{aligned} Vol_g(\mathcal{W}_1) &= \int_{\|\xi\|=1} \int_{W_1} \mathbb{1}_{\mathcal{W}_1}\big(exp_{w_0}(s\xi)\big)\sqrt{g(s,\xi)}ds d\xi \\ &= \int_{\mathbf{v}\left(\mathcal{W}_1 \setminus Cut(w_0)\right)} \Big(\sum_{j\in\mathcal{J}_\xi} \int_{\alpha_{j,\xi}}^{\beta_{j,\xi}} \sqrt{g(s,\xi)}ds\Big)d\xi \\ &\leq \int_{\mathbf{v}\left(\mathcal{W}_1 \setminus Cut(w_0)\right)} \sum_{j\in\mathcal{J}_\xi} \frac{V_k(\beta_{j,\xi}) - V_k(\alpha_{j,\xi})}{A_k(\alpha_{j,\xi})}\sqrt{g(\alpha_{j,\xi},\xi)}d\xi \\ &\leq \int_{\mathbf{v}\left(\mathcal{W}_1 \setminus Cut(w_0)\right)} \sum_{j\in\mathcal{J}_\xi} \frac{V_k(r) - V_k(r/2)}{A_k(r/2)}\sqrt{g(\alpha_{j,\xi},\xi)}d\xi \\ &\leq \frac{V_k(r) - V_k(r/2)}{A_k(r/2)} Area_g(\Gamma). \end{aligned}$$



Finally, inserting this inequality into (4.29), we get

$$\frac{Area_g(\Gamma)}{Vol_g(D_1)} \geq \frac{\alpha}{2}\frac{A_k(r/2)}{V_k(r) - V_k(r/2)}.$$

Conclusion: For any $n-$dimensional starlike domain $D \subseteq M$ with Ricci curvature bounded from below for some $k < 0$, any fixed origin $0 \in D$, any $0 < r \leq R$ with complete $\overline{B(0,R)}$ and $B(0,r) \subseteq D \subseteq B(0,R)$ and any $\Gamma \in \mathcal{G}$,

$$\frac{Area_g(\Gamma)}{\min\{Vol_g(D_1), Vol_g(D_2)\}} \geq C_B(k,n,r,R) := \tag{4.30}$$

$$\max_{0<t<r/2} \max_{0<\alpha<1} \left\{ \min \left\{ \frac{\alpha}{2}\frac{A_k(r/2)}{V_k(r) - V_k(r/2)}, \Big(1 - \alpha(1 + \gamma(r,k,R,t))\Big)\frac{A_k(t)}{V_k(R) - V_k(t)} \right\} \right\},$$

where

$$\gamma(r, k, R, t) := \frac{V_k(R) - V_k(r/2)}{V_k(r/2) - V_k(t)}. \tag{4.31}$$

This concludes the proof of Buser's inequality (4.4). □

4.2. A lower bound on Buser's isoperimetric constant

In order to make the factor r come up the the right side side of (4.1) we need the following result [2].

Proposition 4.2.1. *For any* $0 < r \leq R$ *with complete* $\overline{B(0,R)}$ *and* $B(0,r) \subseteq D \subseteq B(0,R)$,

$$C_B(k, n, r, R) \geq \frac{r^{n-1}}{R^n} \frac{n}{2^{4n-1}} \big(c(n,k)\big)^R, \tag{4.32}$$

where $c(n,k) := e^{-2(n-1)\sqrt{-k}} \in (0,1)$.

Proof. Fix any $0 < r \leq R$ with complete $\overline{B(0,R)}$ and $B(0,r) \subseteq D \subseteq B(0,R)$. In (4.30), for any $t \in (0, \frac{r}{2})$, we may choose the constant α such that

$$\begin{aligned}
&\frac{\alpha}{2} \frac{A_k(r/2)}{V_k(r) - V_k(r/2)} = \big(1 - \alpha(1 + \gamma(r,k,R,t))\big) \frac{A_k(t)}{V_k(R) - V_k(t)} \iff \\
&\frac{\alpha}{2} = \frac{A_k(t)\big(V_k(r) - V_k(r/2)\big)\big(V_k(r/2) - V_k(t)\big)}{\big(V_k(R) - V_k(t)\big)\Big\{A_k(r/2)\big(V_k(r/2) - V_k(t)\big) + 2A_k(t)\big(V_k(r) - V_k(r/2)\big)\Big\}}.
\end{aligned}$$

It is easy to verify that $0 < \alpha < 1$. Note that, for $k < 0$, the function $A_k(t)$ is decreasing with respect to t, since from proposition 3.3.3,

$$\begin{aligned}
V_k(t) &= \frac{1}{\sqrt{(-k)^n}} V_{-1}(\sqrt{-k}t) = \frac{nb_n}{\sqrt{(-k)^n}} \int_0^{t\sqrt{-k}} (\sinh s)^{n-1} ds \\
&= \frac{nb_n}{\sqrt{(-k)^{n-1}}} \int_0^t \big(\sinh(s\sqrt{-k})\big)^{n-1} ds, \quad t \geq 0 \Longrightarrow \\
A_k(t) &= \frac{nb_n}{\sqrt{(-k)^{n-1}}} \big(\sinh(t\sqrt{-k})\big)^{n-1}, \quad t \geq 0 \Longrightarrow \\
\frac{d}{dt} A_k(t) &= \frac{n(n-1)b_n}{\sqrt{(-k)^{n-2}}} \big(\sinh(t\sqrt{-k})\big)^{n-2} \big(\cosh(t\sqrt{-k})\big) \geq 0, \quad t \geq 0.
\end{aligned}$$

By (4.30), with this particular choice of α, we have

$$
\begin{aligned}
C_B(k,n,r,R) &\geq \frac{A_k(t)A_k(r/2)\big(V_k(r/2)-V_k(t)\big)}{\big(V_k(R)-V_k(t)\big)\Big\{1\cdot A_k(r/2)\big(V_k(r/2)-V_k(t)\big)+2\cdot A_k(t)\big(V_k(r)-V_k(r/2)\big)\Big\}} \\
&\geq \frac{A_k(t)A_k(r/2)\big(V_k(r/2)-V_k(t)\big)}{\big(V_k(R)-V_k(t)\big)\Big\{2\cdot A_k(r/2)\big(V_k(r/2)-V_k(t)\big)+2\cdot A_k(r/2)\big(V_k(r)-V_k(r/2)\big)\Big\}} \\
&\geq \frac{A_k(t)\big(V_k(r/2)-V_k(t)\big)}{\big(V_k(R)-0\big)\cdot 2\cdot\Big\{V_k(r/2)-V_k(t)+V_k(r)-V_k(r/2)\Big\}} \\
&= \frac{A_k(t)\big(V_k(r/2)-V_k(t)\big)}{2V_k(R)\big(V_k(r)-V_k(t)\big)} \geq \frac{A_k(t)\big(V_k(r/2)-V_k(t)\big)}{2V_k(R)V_k(r)\big)}.
\end{aligned}
$$

Choosing $t = r/4$ we obtain

$$
C_B(k,n,r,R) \geq \frac{A_k(r/4)\big(V_k(r/2)-V_k(r/4)\big)}{2V_k(R)V_k(r)}.
$$

Using once more proposition 3.3.3, for all $r > 0$, we get

$$
V_k(r/2) - V_k(r/4) \geq V_k(r/4)
$$

since

$$
\begin{aligned}
V_k(r/2) - 2V_k(r/4) &= \frac{1}{\sqrt{(-k)^{n-1}}}\Big(V_{-1}(r\sqrt{-k}/2) - 2V_{-1}(r\sqrt{-k}/4)\Big) \\
&= \frac{nb_n}{\sqrt{(-k)^{n-1}}}\Big(\int_0^{r\sqrt{-k}/2}(\sinh s)^{n-1}ds - 2\int_0^{r\sqrt{-k}/4}(\sinh s)^{n-1}ds\Big) \\
&= \frac{nb_n}{\sqrt{(-k)^{n-1}}}\int_0^{r\sqrt{-k}/2}\Big((\sinh s)^{n-1} - (\sinh(s/2))^{n-1}\Big)ds \geq 0
\end{aligned}
$$

and thus,

$$
C_B(k,n,r,R) \geq \frac{A_k(r/4)V_k(r/4)}{2V_k(R)V_k(r)}.
$$

Now, proposition 3.3.4 implies

$$
\begin{aligned}
C_B(k,n,r,R) &\geq \frac{1}{2}A_k(r/4)\frac{V_k(r/4)}{2V_k(R)V_k(r)} \\
&\geq \frac{1}{2}A_k(r/4)\frac{b_n(r/4)^n}{b_nR^ne^{(n-1)R\sqrt{-k}}b_nr^ne^{(n-1)r\sqrt{-k}}} \\
&\geq \frac{1}{2^{2n+1}}A_k(r/4)\frac{1}{b_nR^ne^{2(n-1)R\sqrt{-k}}}.
\end{aligned}
$$

Observe that the function $A_k(t)$ is decreasing with respect to k, since for any fixed $t > 0$,

$$A_k(t) = nb_n\Big(\frac{\sinh(t\sqrt{-k})}{\sqrt{-k}}\Big)^{n-1} \geq nb_n t^{n-1} = A_0(t), \text{ for all } k < 0.$$

Finally,

$$\begin{aligned} C_B(k,n,r,R) &\geq \frac{1}{2^{2n+1}} A_k(r/4) \frac{1}{b_n R^n e^{2(n-1)R\sqrt{-k}}} \\ &\geq \frac{1}{2^{2n+1}} A_0(r/4) \frac{1}{b_n R^n e^{2(n-1)R\sqrt{-k}}} \\ &= \frac{1}{2^{2n+1}} nb_n (r/4)^{n-1} \frac{1}{b_n R^n e^{2(n-1)R\sqrt{-k}}} \\ &= \frac{r^{n-1}}{R^n} \frac{n}{2^{4n-1}} \Big(e^{-2(n-1)\sqrt{-k}}\Big)^R. \end{aligned}$$

□

In the case of the ball, the above estimate is simplified.

Remark 4.2.2. *Let any $x \in M$ and $r > 0$. If $D = B(x,r)$, then we may take $R = r$ in (4.32) which leads to*

$$C_B(k,n,r) \geq \frac{1}{r} \frac{n}{2^{4n-1}} \big(c(n,k)\big)^r, \tag{4.33}$$

where $0 < c(n,k) < 1$.

4.3. Local L^1−Poincare inequality on balls.

Let (M,g) be a complete Riemannian manifold of dimension n and fix $x \in M$ and $r > 0$. We define the isoperimetric and the Sobolev constant, respectively, by

$$i_\infty \equiv i_\infty\big(B(x,r)\big) := \inf_{\Gamma \in \mathcal{G}} \frac{Area_g(\Gamma)}{\min\{Vol_g(D_1), Vol_g(D_2)\}} \tag{4.34}$$

$$s'_\infty \equiv s'_\infty\big(B(x,r)\big) := \inf_{u \in C^\infty(B(x,r))} \frac{\|\nabla u\|_1}{\|u - \overline{u}_r(x)\|_1} \tag{4.35}$$

where Γ varies over all compact $(n-1)$−dimensional C^∞ submanifolds of $B(x,r)$ that divide $B(x,r)$ into two disjoint and open submanifolds D_1, D_2. Here, $\|\cdot\|_1$ stands for the L^1−norm on the ball $B(x,r)$ and $\overline{u}_r(x)$ for the average of u over the ball $B(x,r)$, that is

$$\overline{u}_r(x) := \frac{1}{Vol_g(B(x,r))} \int_{B(x,r)} u(y) d\mu_g(y).$$

Let us also define

$$s_\infty \equiv s_\infty\big(B(x,r)\big) := \inf_{u \in C^\infty(B(x,r))} \frac{\|\nabla u\|}{\inf_{\alpha \in \mathbb{R}} \|u - \alpha\|_1}. \tag{4.36}$$

In this section, we will first prove the equivalence between these constants although, in fact, only inequality $i_\infty/2 \leq s'_\infty$ will be needed [4].

Theorem 4.3.1. *Let i_∞, s'_∞ and s_∞ be the constants defined above. Then,*

$$\frac{i_\infty}{2} \leq s'_\infty \leq s_\infty \leq i_\infty. \tag{4.37}$$

Proof. **Step 1**: Inequality $s'_\infty \leq s_\infty$ is obvious since $\inf_{\alpha\in\mathbb{R}} \|u-\alpha\|_1 \leq \|u - \overline{u}_r(x)\|$.
Step 2: In order to prove that $s_\infty \leq i_\infty$ we consider two open and disjoint subsets D_1, D_2 of $B(x,r)$ and a hypersurface $\Gamma \in \mathcal{G}$ with $B(x,r) = D_1 \cup D_2 \cup \Gamma$. Without loss of generality, we assume that $Vol_g(D_1) \leq Vol_g(D_2)$. Also, let $\epsilon > 0$ be given. We define

$$\Gamma_\epsilon := \{y \in B(x,r) : d(y,\Gamma) < \epsilon\}$$

and a function f_ϵ on $B(x,r)$ by

$$f_\epsilon(y) = \begin{cases} 1, & y \in D_1 \setminus \Gamma_\epsilon \\ -1, & y \in D_2 \setminus \Gamma_\epsilon \\ \text{linear}, & \text{across } \Gamma \text{ on } \Gamma_\epsilon \end{cases}$$

Notice that f_ϵ may not be smooth but it can be properly approximated by smooth functions since it is Lipschitz. In addition, for almost every y across Γ on Γ_ϵ we assume that $|\nabla f_\epsilon(y)| = \frac{1}{\epsilon}$. Coarea formula implies

$$\begin{aligned} \lim_{\epsilon\to 0} \int_{B(x,r)} |\nabla f_\epsilon| d\mu_g &= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} \int_{\Gamma_\epsilon} dA(s) ds = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} Area(\Gamma_\epsilon) ds \\ &= 2Area(\Gamma_0) = 2Area(\Gamma). \end{aligned} \tag{4.38}$$

Furthermore, for any $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} \int_{B(x,r)} |f_\epsilon - \alpha| d\mu_g &\geq \int_{D_1\setminus\Gamma_\epsilon} |f_\epsilon - \alpha| d\mu_g + \int_{D_2\setminus\Gamma_\epsilon} |f_\epsilon - \alpha| d\mu_g \\ &= \int_{D_1\setminus\Gamma_\epsilon} |1-\alpha| d\mu_g + \int_{D_2\setminus\Gamma_\epsilon} |1+\alpha| d\mu_g \\ &= |1-\alpha| Vol_g\big(B(D_1 \setminus \Gamma_\epsilon)\big) + |1+\alpha| Vol_g\big(B(D_2 \setminus \Gamma_\epsilon)\big) \\ &= |1-\alpha|\big(Vol_g(D_1) - Vol_g(\Gamma_\epsilon)\big) + |1+\alpha|\big(Vol_g(D_2) - Vol_g(\Gamma_\epsilon)\big) \\ &\geq |1-\alpha|\big(Vol_g(D_1) - Vol_g(\Gamma_\epsilon)\big) + |1+\alpha|\big(Vol_g(D_1) - Vol_g(\Gamma_\epsilon)\big) \\ &= \big(|1-\alpha| + |1+\alpha|\big)\big(Vol_g(D_1) - Vol_g(\Gamma_\epsilon)\big). \end{aligned}$$

Notice that for small values of ϵ the quantity $Vol_g(D_1) - Vol_g(\Gamma_\epsilon)$ is positive. Taking the infimum over α we get

$$\begin{aligned} \inf_{\alpha\in\mathbb{R}} \|f_\epsilon - \alpha\|_1 &\geq \Big(\inf_{\alpha\in\mathbb{R}} \big(|1-\alpha| + |1+\alpha|\big)\Big)\big(Vol_g(D_1) - Vol_g(\Gamma_\epsilon)\big) \\ &= 2\big(Vol_g(D_1) - Vol_g(\Gamma_\epsilon)\big) \end{aligned}$$

Thus,

$$s_\infty \leq \frac{\|\nabla f_\epsilon\|_1}{\inf_{\alpha\in\mathbb{R}} \|f_\epsilon - \alpha\|_1} \leq \frac{\|\nabla f_\epsilon\|_1}{2\big(Vol_g(D_1) - Vol_g(\Gamma_\epsilon)\big)} \longrightarrow \frac{Area(\Gamma)}{Vol_g(D_1)}, \text{ as } \epsilon \longrightarrow 0.$$

Step 3: We will now prove that $s_\infty \geq i_\infty$. Let any $u \in C^\infty(B(x,r))$ and consider the sets

$$D_1 = \{y \in B(x,r) : u(y) > \beta\}\ ,\ D_2 = \{y \in B(x,r) : u(y) < \beta\}.$$

Pick the constant $\beta \in \mathbb{R}$ such that $Vol_g(D_1) = Vol_g(D_2)$. This will happen if we choose β to realize the $\inf_{\alpha\in\mathbb{R}} \|u-\alpha\|_1$. Indeed; if $\frac{\partial}{\partial\alpha}$ denotes the weak derivative with respect to α, then

$$\beta = \inf_{\alpha\in\mathbb{R}} \|u-\alpha\|_1 \Longrightarrow \frac{\partial}{\partial\alpha}\Big|_\beta \|u-\alpha\|_1 = 0 \Longrightarrow \int_{B(x,r)} \frac{\partial}{\partial\alpha}\Big|_\beta |u-\alpha| d\mu_g = 0 \Longrightarrow$$
$$\int_{B(x,r)} (+1)\cdot \mathbb{1}_{\{u>\beta\}} + (-1)\cdot \mathbb{1}_{\{u<\beta\}} d\mu_g = 0 \Longrightarrow Vol_g(D_1) = Vol_g(D_2).$$

Now, for any $t > 0$, set

$$D_t := \{q \in D_1 : u(q) > \beta + t\} \subseteq D_1$$

and note that, set D_t does not contain important portion of the $B(x,r)$'s volume, since

$$Vol_g(B(x,r)) = 2Vol_g(D_1) \geq 2Vol_g(D_t) \Longrightarrow Vol_g(D_t) \leq \frac{1}{2} Vol_g(B(x,r)).$$

Coarea formula, Buser's isoperimetric inequality and Cavalieri's principle imply

$$\int_{D_1} |\nabla(u-\beta)| d\mu_g = \int_0^\infty Area\big(\{x \in D_1 : u(x) - \beta = t\}\big) dt = \int_0^\infty Area(\partial D_t) dt$$
$$\geq \int_0^\infty i_\infty Vol_g(D_t) dt = i_\infty \int_{D_1} |u-\beta| d\mu_g.$$

Working with the same way, we can prove that

$$\int_{D_2} |\nabla(u-\beta)| d\mu_g \geq i_\infty \int_{D_2} |u-\beta| d\mu_g.$$

Thus,

$$\begin{aligned}\int_{B(x,r)} |\nabla u| d\mu_g &= \int_{B(x,r)} |\nabla(u-\beta)| d\mu_g \\ &= \int_{D_1} |\nabla(u-\beta)| d\mu_g + \int_{D_2} |\nabla(u-\beta)| d\mu_g \\ &\geq i_\infty \Big(\int_{D_1} |u-\beta| d\mu_g + \int_{D_2} |u-\beta| d\mu_g \Big) \\ &= i_\infty \int_{B(x,r)} |u-\beta| d\mu_g \geq i_\infty \inf_{\alpha\in\mathbb{R}} \int_{B(x,r)} |u-\alpha| d\mu_g \Longrightarrow\end{aligned}$$
$$\|\nabla u\|_1 \geq i_\infty \inf_{\alpha\in\mathbb{R}} \|u-\alpha\|_1 \Longrightarrow i_\infty \leq \frac{\|\nabla u\|_1}{\inf_{\alpha\in\mathbb{R}} \|u-\alpha\|_1}.$$

Step 4 : Finally, we will prove that $\frac{i_\infty}{2} \leq s'_\infty$. One can easily verify that at least one of $\{q \in D_1 : u(q) - \overline{u}_r(x) > 0\}$,$\{q \in D_1 : u(q) - \overline{u}_r(x) < 0\}$ has volume less than $\frac{1}{2}Vol_g(B(x,r))$. Thus, the same argument of step 3 with $\overline{u}_r(x)$ replaced by β and $\{q \in D_1 : u(q) - \overline{u}_r(x) > 0\}$ or $\{q \in D_1 : u(q) - \overline{u}_r(x) < 0\}$ replaced by D_t will give the desired result. □

We are now in position to prove the validity of the pseudo L^1-Poincare inequality on the ball $B(x,r) \subseteq M$.

Theorem 4.3.2. *Let (M,g) be a complete Riemannian manifold with $Ric_{(M,g)} \geq kg$ for some real constant $k \in \mathbb{R}$ and let $\varrho > 0$ be given. Then, there exists a positive constant $C \equiv C(n,k,\varrho)$ such that for any fixed point $x \in M$, any $0 < r < 2\varrho$ and any $u \in C^\infty\big(B(x,r)\big)$,*

$$\int_{B(x,r)} |u(y) - \overline{u}_r(x)| d\mu_g(y) \leq Cr \int_{B(x,r)} |\nabla u(y)| d\mu_g(y).$$

Proof. For any $x \in M$, any $0 < r < 2\varrho$ and any $u \in C^\infty\big(B(x,r)\big)$, according to that we said above, we have

$$\begin{aligned}\frac{\|\nabla u\|_1}{\|u - \overline{u}_r(x)\|_1} &\geq \frac{1}{2}\frac{Area_g(\Gamma)}{\min\{Vol_g(D_1), Vol_g(D_2)\}} \qquad \text{(from theorem 4.3.1)}\\ &\geq \frac{1}{2}\frac{n}{2^{4n-1}}\frac{1}{r}\big(c(n,k)\big)^r \qquad \text{(from (4.33))}\\ &\geq \frac{n}{2^{4n}}\frac{1}{r}\big(c(n,k)\big)^{2\varrho} \qquad \big(\text{since } 0 < c(n,k) < 1 \text{ and } 0 < r < 2\varrho\big)\\ &:= \frac{1}{\mathcal{C}(n,k,\varrho)}\frac{1}{r} \Longrightarrow\end{aligned}$$

$$\int_{B(x,r)} |u(y) - \overline{u}_r(x)| d\mu_g(y) \leq \mathcal{C}(n,k,\varrho) r \int_{B(x,r)} |\nabla u(y)| d\mu_g(y).$$

□

4.4. Global L^1-Poincare inequality on the whole manifold.

We will now present the L^1-Poincare inequality on (M,g) as it is proved in [17], [8].

Theorem 4.4.1. *Let (M,g) be a complete n-dimensional Riemannian manifold such that $Ric_{(M,g)} \geq kg$ for some constant $k \in \mathbb{R}$, and let $R > 0$ be given. There exists a positive constant $C \equiv C(n,k,R)$, such that for any $0 < r < R$ and any $u \in C_c^\infty(M)$,*

$$\int_M |u(x) - \overline{u}_r(x)| d\mu_g(x) \leq Cr \int_M |\nabla u(x)| d\mu_g(x). \tag{4.39}$$

Proof. Let $R > 0$ be given. From theorem 4.3.2, there exists a positive constant $C(n, k, R)$ such that for any $x \in M$, any $0 < \rho < 2R$ and any $u \in C^\infty(B(x, \rho))$,

$$\int_{B(x,\rho)} |u(y) - \overline{u}_\rho(x)| d\mu_g(y) \le C(n, k, R)\rho \int_{B(x,\rho)} |\nabla u(y)| d\mu_g(y). \tag{4.40}$$

Let $0 < r < R$ be given. Covering lemma 3.4.1 implies the existence of a sequence $\{x_i\}_{i=1}^\infty \subseteq M$ such that

(i) $M = \bigcup_{i=1}^\infty B(x_i, r)$,

(ii) $B(x_i, r/2) \cap B(x_j, r/2) = \emptyset$ for all $i, j = 1, \ldots, n$ with $i \neq j$,

(iii) Card $\{i \in \mathbb{N} : x \in B(x_i, 2r)\} \le N \equiv N(n, k, R)$.

where (iii) can be proved using same arguments as used in the proof of lemma 3.4.1. Let any $u \in C_c^\infty(M)$. We have

$$\begin{aligned}
&\int_M |u(y) - \overline{u}_r(y)| d\mu_g(y) \le \sum_{j=1}^\infty \int_{B(x_j,r)} |u(y) - \overline{u}_r(y)| d\mu_g(y) \\
&\le \sum_{j=1}^\infty \int_{B(x_j,r)} |u(y) - \overline{u}_r(x_j)| d\mu_g(y) + \sum_{j=1}^\infty \int_{B(x_j,r)} |\overline{u}_r(x_j) - \overline{u}_{2r}(x_j)| d\mu_g(y) \\
&\quad + \sum_{j=1}^\infty \int_{B(x_j,r)} |\overline{u}_{2r}(x_j) - \overline{u}_r(y)| d\mu_g(y) := I_1 + I_2 + I_3.
\end{aligned}$$

Since (4.40) holds for any $x \in M$ and since Card $\{j \in \mathbb{N} : y \in B(x_j, 2r)\} \le N \equiv N(n, k, R)$,

$$\begin{aligned}
I_1 &:= \sum_{j=1}^\infty \int_{B(x_j,r)} |u(y) - \overline{u}_r(x_j)| d\mu_g(y) \le \sum_{j=1}^\infty Cr \int_{B(x_j,r)} |\nabla u(y)| d\mu_g(y) \\
&\le \sum_{j=1}^N Cr \int_{B(x_j,r)} |\nabla u(y)| d\mu_g(y) \le NCr \int_M |\nabla u(y)| d\mu_g(y).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
I_2 :&= \sum_{j=1}^{\infty} \int_{B(x_j,r)} |\overline{u}_r(x_j) - \overline{u}_{2r}(x_j)| d\mu_g(y) = \sum_{j=1}^{\infty} |\overline{u}_r(x_j) - \overline{u}_{2r}(x_j)| Vol_g(B(x_j,r)) \\
&= \sum_{j=1}^{\infty} \Big| Vol_g(B(x_j,r))\overline{u}_r(x_j) - Vol_g(B(x_j,r))\overline{u}_{2r}(x_j) \Big| \\
&= \sum_{j=1}^{\infty} \Big| \int_{B(x_j,r)} u(y) d\mu_g(y) - \int_{B(x_j,r)} \overline{u}_{2r}(x_j) d\mu_g(y) \Big| \\
&\leq \sum_{j=1}^{\infty} \int_{B(x_j,r)} |u(y) - \overline{u}_{2r}(x_j)| d\mu_g(y) \leq \sum_{j=1}^{\infty} \int_{B(x_j,2r)} |u(y) - \overline{u}_{2r}(x_j)| d\mu_g(y) \\
&\leq \sum_{j=1}^{\infty} 2Cr \int_{B(x_j,2r)} |\nabla u(y)| d\mu_g(y) \leq \sum_{j=1}^{N} 2Cr \int_{B(x_j,2r)} |\nabla u(y)| d\mu_g(y) \\
&\leq 2NCr \int_M |\nabla u(y)| d\mu_g(y),
\end{aligned}$$

For I_3 we have

$$\begin{aligned}
I_3 :&= \sum_{j=1}^{\infty} \int_{B(x_j,r)} |\overline{u}_r(y) - \overline{u}_{2r}(x_j)| d\mu_g(y) \\
&= \sum_{j=1}^{\infty} \int_{B(x_j,r)} \Big| \frac{1}{Vol_g(B(y,r))} \int_{B(y,r)} u(x) d\mu_g(x) \\
&\qquad\qquad - \frac{1}{Vol_g(B(y,r))} \int_{B(y,r)} \overline{u}_{2r}(x_j) d\mu_g(x) \Big| d\mu_g(y) \\
&\leq \sum_{j=1}^{\infty} \int_{B(x_j,r)} \left(\frac{1}{Vol_g(B(y,r))} \int_{B(y,r)} |u(x) - \overline{u}_{2r}(x_j)| d\mu_g(x) \right) d\mu_g(y) \\
&\leq \sum_{j=1}^{\infty} \int_{B(x_j,r)} \left(\frac{1}{Vol_g(B(y,r))} \underbrace{\int_{B(x_j,2r)} |u(x) - \overline{u}_{2r}(x_j)| d\mu_g(x)}_{\text{independent of } y} \right) d\mu_g(y) \\
&= \sum_{j=1}^{\infty} \left(\int_{B(x_j,2r)} |u(x) - \overline{u}_{2r}(x_j)| d\mu_g(x) \right) \left(\int_{B(x_j,r)} \frac{1}{Vol_g(B(y,r))} d\mu_g(y) \right).
\end{aligned}$$

Now, inequality (4.40) gives

$$I_3 \leq \sum_{j=1}^{\infty} \left(2Cr \int_{B(x_j,2r)} |\nabla u(x)| d\mu_g(x) \right) \left(\int_{B(x_j,r)} \frac{1}{Vol_g(B(y,r))} d\mu_g(y) \right),$$

and from proposition 3.3.4,

$$\begin{aligned} I_3 &\leq \sum_{j=1}^{\infty}\left(2Cr\int_{B(x_j,2r)}|\nabla u(x)|d\mu_g(x)\right)\left(\int_{B(x_j,r)}\frac{1}{Vol_g(B(y,r))}d\mu_g(y)\right) \\ &\leq \sum_{j=1}^{\infty}\left(2Cr\int_{B(x_j,r)}|\nabla u(x)|d\mu_g(x)\right)\left(\int_{B(x_j,r)}\frac{e^{2r(n-1)\sqrt{|k|}}}{Vol_g(B(y,2r))}\left(\frac{2r}{r}\right)^n d\mu_g(y)\right) \\ &:= 2KCr\sum_{j=1}^{\infty}\left(\int_{B(x_j,2r)}|\nabla u(x)|d\mu_g(x)\right)\left(\int_{B(x_j,r)}\frac{1}{Vol_g(B(y,2r))}d\mu_g(y)\right), \end{aligned}$$

where $K \equiv K(n,k,r) := 2^n e^{2r(n-1)\sqrt{|k|}}$. Triangular inequality implies that for any $y \in B(x_j,r)$, $B(x_j,r) \subseteq B(y,2r)$ and so,

$$\begin{aligned} I_3 &\leq 2KCr\sum_{j=1}^{\infty}\left(\int_{B(x_j,2r)}|\nabla u(x)|d\mu_g(x)\right)\left(\int_{B(x_j,r)}\frac{1}{Vol_g(B(x_j,r))}d\mu_g(y)\right) \\ &= 2KCr\sum_{j=1}^{\infty}\int_{B(x_j,2r)}|\nabla u(x)|d\mu_g(x) \leq 2KCr\sum_{j=1}^{N}\int_{B(x_j,2r)}|\nabla u(x)|d\mu_g(x) \\ &\leq 2KCrN\int_M|\nabla u(x)|d\mu_g(x). \end{aligned}$$

Finally,

$$\int_M |u(y)-\overline{u}_r(y)|d\mu_g(y) \leq N\big(3+2K\big)Cr\int_M|\nabla u(y)|d\mu_g(y).$$

□

Chapter 5

Sobolev Inequalities on Complete Riemannian Manifolds

5.1. A necessary and sufficient condition

As already mentioned, in this section we will focus on the characterization of complete Riemannian manifolds with Ricci curvature bounded from below which support a Sobolev Embedding. Namely, will we prove the following result.

Theorem 5.1.1. *Let (M, g) be a complete Riemannian manifold of dimension n with Ricci curvature bounded from below. Then, all Sobolev embeddings are valid for (M, g) if and only if there is a uniform lower bound for the volume of balls which is independent of their center, namely if and only if*

$$\inf_{x \in M} Vol_g B(x, 1) > 0.$$

Remark 5.1.2. *Note that by Bishop-Gromov's result, the condition*

$$\inf_{x \in M} Vol_g B(x, 1) > 0$$

is equivalent to

$$\inf_{x \in M} Vol_g B(x, r) > 0, \quad \forall r > 0.$$

Indeed; If $\inf_{x \in M} Vol_g B(x, 1) > 0$, then $Vol_g B(x, 1) > v$ for all $x \in M$ and for some positive constant v. From proposition 3.3.4, for all $x \in M$ and for all $0 < r < 1$, we have

$$Vol_g(B(x, r)) \geq r^n e^{-(n-1)\sqrt{|k|}} Vol_g(B(x, 1)) \geq v r^n e^{-(n-1)\sqrt{|k|}}.$$

If $r \geq 1$, obviously,

$$B(x, r) \supseteq B(x, 1) \Longrightarrow Vol_g(B(x, r)) \geq Vol_g(B(x, 1)) > v.$$

Therefore, for all $r > 0$, there exists a positive constant

$$A_r := \begin{cases} v, & r \geq 1 \\ v r^n e^{-(n-1)\sqrt{|k|}}, & 0 < r < 1 \end{cases}$$

such that $Vol_g(B(x, r)) \geq A_r$ for all $x \in M$.

5.2. The embedding $H_1^1(M) \hookrightarrow L^{\frac{n}{n-1}}(M)$ implies all Sobolev embeddings

We will now show under a scaling argument, without any assumption on the Ricci curvature, that on complete $n-$dimensional Riemannian manifolds the embedding $H_1^1(M) \hookrightarrow L^{\frac{n}{n-1}}(M)$ is sufficient condition in order to have all others Sobolev embeddings.

Lemma 5.2.1. *Let (M, g) a $n-$dimensional complete Riemannian manifold. Suppose that the embedding $H_1^1(M) \hookrightarrow L^{\frac{n}{n-1}}(M)$ is valid. Then for any real numbers p, q with $1 \leq q < p$ and for any integers m, k with $0 \leq m < k$, satisfying $1/p = 1/q - (k-m)/n$, $H_k^q(M) \hookrightarrow H_m^p(M)$.*

Proof. **Step 1** . We will first prove that if $H_1^1(M) \hookrightarrow L^{\frac{n}{n-1}}(M)$ then, for any $1 \leq q < n$ and $1/p = 1/q - 1/n$, $H_1^q(M) \hookrightarrow L^p(M)$. Assume that $H_1^1(M) \hookrightarrow L^{\frac{n}{n-1}}(M)$ hold, namely there exists a constant $A \in \mathbb{R}$ such that for any $\varphi \in H_1^1(M)$,

$$\left(\int_M |\varphi(x)|^{\frac{n}{n-1}} d\mu_g(x) \right)^{\frac{n-1}{n}} \leq A \int_M |\varphi(x)| + |\nabla \varphi(x)| d\mu_g(x).$$

Let p, q be real numbers such that $1 \leq q < n$ and $1/p = 1/q - 1/n$. Set $p' := \frac{p(n-1)}{n} - 1$ and let q' the conjugate of q, that is $1/q' + 1/q = 1$. A direct calculation shows that $p'q' = p$. Also, for any given $u \in C_c^\infty(M)$, let $\varphi := |u|^{p'+1}$. We calculate

$$\nabla \varphi = (p'+1)|u|^{p'} \nabla |u|,$$
$$\left|\nabla \varphi\right| = (p'+1)|u|^{p'} \left|\nabla |u|\right| = (p'+1)|u|^{p'} |\nabla u|,$$

almost everywhere on M. Notice that $\varphi \in H_1^1(M)$. The assumption above and Holder's inequality imply

$$\begin{aligned}
\left(\int_M |u|^p d\mu_g\right)^{\frac{n-1}{n}} &= \left(\int_M |\varphi|^{\frac{n}{n-1}} d\mu_g\right)^{\frac{n-1}{n}} \leq A\left(\int_M |\nabla\varphi| + |\varphi| d\mu_g\right) \\
&= A\int_M (p'+1)|u|^{p'}|\nabla u| d\mu_g + A\int_M |u|^{p'+1} d\mu_g \\
&\leq A(p'+1)\left(\int_M |u|^{p'q'} d\mu_g\right)^{1/q'}\left(\int_M |\nabla u|^q d\mu_g\right)^{1/q} + A\int_M |u|^{p'+1} d\mu_g \\
&\leq A(p'+1)\left(\int_M |u|^{p'q'} d\mu_g\right)^{1/q'}\left(\int_M |\nabla u|^q d\mu_g\right)^{1/q} + \\
&\qquad\qquad + A\left(\int_M |u|^{p'q'} d\mu_g\right)^{1/q'}\left(\int_M |u|^q d\mu_g\right)^{1/q} \\
&= \left(\int_M |u|^p d\mu_g\right)^{1/q'}\left\{A(p'+1)\left(\int_M |\nabla u|^q d\mu_g\right)^{1/q} + A\left(\int_M |u|^q d\mu_g\right)^{1/q}\right\} \\
&\leq C(A,p,n)\left(\int_M |u|^p d\mu_g\right)^{1/q'}\left\{\left(\int_M |\nabla u|^q d\mu_g\right)^{1/q} + \left(\int_M |u|^q d\mu_g\right)^{1/q}\right\},
\end{aligned}$$

where $C(A,p,n) := \max\{A(p'+1), A\}$. Therefore,

$$\left(\int_M |u|^p d\mu_g\right)^{\frac{n-1}{n}-\frac{1}{q'}} \leq C(A,p,n)\left\{\left(\int_M |\nabla u|^q d\mu_g\right)^{1/q} + \left(\int_M |u|^q d\mu_g\right)^{1/q}\right\},$$

which proves that $H_1^1(M) \hookrightarrow L^p(M)$ for any $u \in C_c^\infty(M)$ and since $C_c^\infty(M)$ is dense in $H_1^1(M)$, this embedding holds for all $u \in H_1^1(M)$.

Step 2 . We will now prove that if $H_1^1(M) \hookrightarrow L^p(M)$ for any real numbers p, q with $1 \leq q < n$, then $H_k^q(M) \hookrightarrow H_m^p(M)$ for any integers m, k satisfying $0 \leq m < k$ and $1/p = 1/q - (k-m)/n$. Pick any real numbers p, q with $1 \leq q < n$ and suppose that $H_1^1(M) \hookrightarrow L^p(M)$. Also, pick two integers m, k with $0 \leq m < k$ satisfying $1/p = 1/q - (k-m)/n$ and let $r \in \mathbb{N}$ and $\psi \in C^{r+1}(M)$. Set $\varphi := |\nabla^r\psi|$ and suppose that $\varphi \in H_1^1(M)$. The assumption of this step yields

$$\begin{aligned}
&\|\varphi\|_{L^p(M)} \leq C\|\varphi\|_{H_1^1(M)} \Longrightarrow \\
&\|\nabla^r\psi\|_{L^p(M)} \leq C\Big(\|\nabla^r\psi\|_{L^q(M)} + \|\nabla\big|\nabla^r\psi\big|\|_{L^q(M)}\Big).
\end{aligned}$$

for some positive constant C. Applying Kato's inequality, $\big|\nabla\big|\nabla^r\psi\big|\big| \leq \big|\nabla^{r+1}\psi\big|$, we obtain

$$\|\nabla^r\psi\|_{L^p(M)} \leq C\Big(\|\nabla^r\psi\|_{L^q(M)} + \|\nabla^{r+1}\psi\|_{L^q(M)}\Big). \tag{5.1}$$

Now, let $\psi \in C^\infty(M) \cap H_k^q(M)$. Applying the last inequality for any $r = 1, 2, \dots, k-1$ we get

$$\sum_{j=1}^{k-1} \|\nabla^j \psi\|_{L^p(M)} \leq C\left(\sum_{j=1}^{k-1} \|\nabla^j \psi\|_{L^q(M)} + \sum_{j=2}^{k} \|\nabla^j \psi\|_{L^q(M)}\right). \tag{5.2}$$

Furthermore, since $H_1^1(M) \hookrightarrow L^p(M)$ and $\psi \in C^{r+1}(M)$, $r \in \mathbb{N}$ we have

$$\|\nabla^0 \psi\|_{L^p(M)} \equiv \|\psi\|_{L^p(M)} \leq C\Big(\|\psi\|_{L^q(M)} + \|\nabla\psi\|_{L^q(M)}\Big).$$

From (5.1) and (5.2) we finally get

$$\begin{aligned}
\|\psi\|_{H_{k-1}^p(M)} &:= \sum_{j=0}^{k-1} \|\nabla^j \psi\|_{L^p(M)} \\
&\leq C\Big(\sum_{j=0}^{k-1} \|\nabla^j \psi\|_{L^q(M)} + \sum_{j=1}^{k} \|\nabla^j \psi\|_{L^q(M)}\Big) \\
&\leq C\Big(\sum_{j=0}^{k} \|\nabla^j \psi\|_{L^q(M)} + \sum_{j=0}^{k} \|\nabla^j \psi\|_{L^q(M)}\Big) \\
&= 2C \sum_{j=0}^{k} \|\nabla^j \psi\|_{L^q(M)} := 2C\|\psi\|_{H_k^q(M)}.
\end{aligned}$$

Since the Sobolev space $H_k^q(M)$ is the completion of

$$C_k^q(M) := C^\infty(M) \cap \Big(\bigcap_{j=0}^{k} \{u : \nabla^j u \in L^q(M)\}\Big),$$

by a standard approximation argument we have that the above inequality holds for any $\psi \in H_k^q(M)$, proving $H_k^q(M) \hookrightarrow H_{k-1}^p(M)$. Proceeding as above, one can prove the other embeddings. □

5.3. The easy case: Compact Riemannian manifolds

Before considering the general case of complete Riemannian manifolds, we will now see how one can prove that the validity of Sobolev embeddings for compact manifolds.

Theorem 5.3.1. *Let (M, g) be a compact Riemannian manifold of dimension n. Then for any real numbers p, q and for any integers m, k with $0 \leq m < k$ and $1 \leq q < n$ satisfying $1/p = 1/q - (k-m)/n$, $H_k^q(M) \hookrightarrow H_m^p(M)$.*

Proof. According to lemma 5.2.1 we just have to prove the validity of the embedding $H_1^1(M) \hookrightarrow L^{\frac{n}{n-1}}(M)$. Recall that the Sobolev space $H_k^p(M)$ is the completion of

$$\mathcal{C}_k^p(M) := \Big\{u \in C^\infty(M) : \nabla^j u \in L^p(M)\ ,\ \forall j = 0, 1, \dots, k\Big\}$$

with respect to the norm

$$\|u\|_{H_k^p(M)} := \sum_{j=0}^{k} \|\nabla^j u\|_{L^p(M)} = \sum_{j=0}^{k} \Big(\int_M |\nabla^j u|^p d\mu(g) \Big)^{1/p}.$$

Since (M,g) is compact we have $C_k^p(M) = C^\infty(M)$ and thus it is sufficient to prove the above embedding for any $u \in C^\infty(M)$. Furthermore, since (M,g) is compact there exists a finite collection of charts $\{(\Omega_m, \varphi_m) : m = 1,2,\dots,N\}$ such that

(i) $M = \bigcup_{m=1}^N \Omega_m$

(ii) For any $m = 1,2,\dots,N$, the components g_{ij}^m of the metric g on the chart (Ω_m, φ_m) satisfy $\frac{1}{2}\delta_{ij} \leq g_{ij}^m \leq 2\delta_{ij}$ as bilinear forms.

Also, let $\{\eta_m\}$ be a smooth partition of unity subordinate to the covering $\{(\Omega_m, \varphi_m) : m = 1,2,\dots,N\}$, that is, for any $m = 1,2,\dots,N$,

(iii) $0 \leq \eta_m \leq 1$, (iv) $\eta_m \in C^\infty(\Omega_m)$, (v) $\sum_{m=1}^N \eta_m = 1$, (vi) $\operatorname{supp}(\eta_m) \subseteq \Omega_m$.

We have

$$\begin{aligned}
\int_M |\eta_m u|^{\frac{n}{n-1}} d\mu_g &= \int_{\Omega_m} |\eta_m u|^{\frac{n}{n-1}} d\mu_g \\
&= \int_{\Omega_m} |\eta_m u|^{\frac{n}{n-1}} \sqrt{det(g_{ij}^m)} dx_1 \wedge \cdots \wedge dx_n \\
&\leq \int_{\Omega_m} |\eta_m u|^{\frac{n}{n-1}} \sqrt{det(2\delta_{ij})} dx_1 \wedge \cdots \wedge dx_n \\
&= 2^{n/2} \int_{\Omega_m} |\eta_m u|^{\frac{n}{n-1}} dx_1 \wedge \cdots \wedge dx_n \\
&= 2^{n/2} \int_{\varphi_m(\Omega_m) \subseteq \mathbb{R}^n} |\eta_m u|^{\frac{n}{n-1}} \circ \varphi_m^{-1} dx \\
&\leq 2^{n/2} \int_{\mathbb{R}^n} |\eta_m u|^{\frac{n}{n-1}} \circ \varphi_m^{-1} dx \\
&= 2^{n/2} \int_{\mathbb{R}^n} |\eta_m u \circ \varphi_m^{-1}|^{\frac{n}{n-1}} dx,
\end{aligned}$$

where dx is the Lebesgue's volume element of $\mathbb{R}^n$. Now, Sobolev's inequality on $\mathbb{R}^n$ (see theorem 2.1.2) implies

$$\begin{aligned}
\left(\int_M |\eta_m u|^{\frac{n}{n-1}} d\mu_g \right)^{\frac{n-1}{n}} &= \left(2^{n/2} \int_{\mathbb{R}^n} |\eta_m u \circ \varphi_m^{-1}|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \\
&\leq C(n) \int_{\mathbb{R}^n} |\nabla_e \big((\eta_m u) \circ \varphi_m^{-1} \big)| dx,
\end{aligned}$$

where ∇_e is the gradient with respect to the Euclidean metric on $\mathbb{R}^n$. On each chart

(Ω_m, φ_m), we compute

$$\begin{aligned}
|\nabla(\eta_m u)|^2 := g(\nabla(\eta_m u), \nabla(\eta_m u)) &= \sum_{i,j=1}^{n} g^{m,ij}\big(\nabla(\eta_m u)\big)_i\big(\nabla(\eta_m u)\big)_j \\
&= \sum_{i,j=1}^{n} g^{m,ij}\frac{\partial\big((\eta_m u)\circ\varphi_m^{-1}\big)}{\partial x_i}\frac{\partial\big((\eta_m u)\circ\varphi_m^{-1}\big)}{\partial x_j} \\
&\geq \sum_{i,j=1}^{n} \frac{1}{2}\delta^{ij}\frac{\partial\big((\eta_m u)\circ\varphi_m^{-1}\big)}{\partial x_i}\frac{\partial\big((\eta_m u)\circ\varphi_m^{-1}\big)}{\partial x_j} \\
&= \frac{1}{2}\sum_{i=1}^{n}\Big(\frac{\partial\big((\eta_m u)\circ\varphi_m^{-1}\big)}{\partial x_i}\Big)^2 = \frac{1}{2}|\nabla_e\big((\eta_m u)\circ\varphi_m^{-1}\big)|^2.
\end{aligned}$$

and finally, since $\mathrm{supp}\eta_m \subseteq \Omega_m$, we get

$$\left(\int_M |\eta_m u|^{\frac{n}{n-1}} d\mu_g\right)^{\frac{n-1}{n}} \leq C'(n)\int_M |\nabla(\eta_m u)| d\mu_g.$$

According to (iii) and (v) we obtain

$$\begin{aligned}
\left(\int_M |u|^{\frac{n}{n-1}} d\mu_g\right)^{\frac{n-1}{n}} &= \left(\int_M \Big|\sum_{m=1}^{N}\eta_m u\Big|^{\frac{n}{n-1}} d\mu_g\right)^{\frac{n-1}{n}} := \Big\|\sum_{m=1}^{N}\eta_m u\Big\|_{L^{\frac{n}{n-1}}(M)} \\
&\leq \sum_{m=1}^{N}\|\eta_m u\|_{L^{\frac{n}{n-1}}(M)} := \sum_{m=1}^{N}\left(\int_M |\eta_m u|^{\frac{n}{n-1}} d\mu_g\right)^{\frac{n-1}{n}} \\
&\leq \sum_{m=1}^{N} C'(n)\int_M |\nabla(\eta_m u)| d\mu_g \\
&= C'(n)\sum_{m=1}^{N}\int_M |u\nabla\eta_m + \eta_m\nabla u| d\mu_g \\
&\leq C'(n)\sum_{m=1}^{N}\int_M |u||\nabla\eta_m| + |\eta_m||\nabla u| d\mu_g \\
&= C'(n)\left(\int_M |u|\Big(\sum_{m=1}^{N}|\nabla\eta_m|\Big) d\mu_g + \int_M |\nabla u|\Big(\sum_{m=1}^{N}\eta_m\Big) d\mu_g\right) \\
&\leq C'(n)\left(K\int_M |u| d\mu_g + \int_M |\nabla u| d\mu_g\right) \\
&\leq C''(n)\int_M |u| + |\nabla u| d\mu_g := C''(n)\|u\|_{H_1^1(M)}.
\end{aligned}$$

where we used the fact that $K := \max_{x\in M}\sum_{m=1}^{N}|\nabla\eta_m(x)| < \infty$ since for any $m = 1, 2, \ldots, N$, $\eta_m \in C^\infty(\Omega_m)$ and M is compact. $\square$

5.4. Proof of theorem 5.1.1

We will now state and prove a series of theorems which will prove theorem 5.1.1. The first step will be the following.

Theorem 5.4.1. *Let (M, g) be a complete Riemannian manifold of dimension n. Suppose that for some $1 \leq q < n$ and $1/p = 1/q - 1/n$, $H_1^q(M) \hookrightarrow L^p(M)$. Then, for any $r > 0$, there exists a positive constant $v \equiv v(M, q, r)$ such that $Vol_g(B(x,r)) \geq v$, for all $x \in M$.*

Proof. Suppose that for some $1 \leq q < n$ and $1/p = 1/q - 1/n$ the embedding $H_1^q(M) \hookrightarrow L^p(M)$ holds. Let $r > 0$, $x \in M$ such that $B(x,r) \subsetneqq M$ and pick any $u \in H_1^q(M)$ such that $u \equiv 0$ on $M \setminus B(x,r)$. Set $\alpha := p/q$ and $\beta := p/(p-q)$. A direct calculation shows that α and β are conjugate, that is, $1/\alpha + 1/\beta = 1$. By Holder's inequality we get

$$\begin{aligned}
\|u\|_{L^q(M)} &= \||u|^q\|^{1/q}_{L^1(B(x,r))} \leq \Big(\||u|^q\|_\alpha \|1\|_\beta\Big)^{1/q} \\
&= \Big(\int_{B(x,r)} |u|^p d\mu_g\Big)^{1/p}\Big(\int_{B(x,r)} d\mu_g\Big)^{\frac{p-q}{qp}} \\
&= \|u\|_{L^p(M)}\Big(Vol_g(B(x,r))\Big)^{1/n},
\end{aligned}$$

and since $H_1^q(M) \hookrightarrow L^p(M)$,

$$\begin{aligned}
&\|u\|_{L^q(M)} \leq A\Big(\|u\|_{L^q(M)} + \|\nabla u\|_{L^q(M)}\Big)\Big(Vol_g(B(x,r))\Big)^{1/n} \Longrightarrow \\
&\left(1 - A\Big(Vol_g(B(x,r))\Big)^{1/n}\right)\|u\|_{L^q(M)} \leq A\Big(Vol_g(B(x,r))\Big)^{1/n}\|\nabla u\|_{L^q(M)} \Longrightarrow \\
&\Big(Vol_g(B(x,r))\Big)^{-1/n} - A \leq A\frac{\|\nabla u\|_{L^q(M)}}{\|u\|_{L^q(M)}}.
\end{aligned}$$

for some real constant A. If $Vol_g(B(x,r)) \geq (1/2A)^n$, we have nothing to prove. If $Vol_g(B(x,r)) \leq (1/2A)^n$, we have

$$1/2\Big(Vol_g(B(x,r))\Big)^{1/n} \geq A,$$

and so, from the inequality above we obtain

$$\begin{aligned}
\frac{1}{2}\Big(Vol_g(B(x,r))\Big)^{-1/n} &= \Big(Vol_g(B(x,r))\Big)^{-1/n} - \frac{1}{2}\Big(Vol_g(B(x,r))\Big)^{-1/n} \\
&\leq \Big(Vol_g(B(x,r))\Big)^{-1/n} - A \leq A\frac{\|\nabla u\|_{L^q(M)}}{\|u\|_{L^q(M)}} \Longrightarrow
\end{aligned}$$

$$\frac{1}{2}\Big(Vol_g(B(x,r))\Big)^{-1/n} \leq A\frac{\|\nabla u\|_{L^q(M)}}{\|u\|_{L^q(M)}}. \tag{5.3}$$

Let $x \in M$ be chosen such that the function $d : M \longrightarrow \mathbb{R}$, $y \longmapsto d(y) := d(y, \cdot)$ is differentiable at y. From now on, let

$$u(y) := \begin{cases} r - d(y,x), & y \in \overline{B(x,r)}, \\ 0, & \text{otherwise} \end{cases}$$

Clearly, u is Lipschitz since $d(x,\cdot)$ is Lipschitz. As a consequence, $u \in H_1^q(M)$, $|\nabla u(y)| = |\nabla_y d(x,y)| = 1$ for $y \in B(x,r)$ and $|\nabla u(y)| = 0$ for $y \in M \setminus B(x,r)$ almost everywhere. Notice that, for $y \in B(x,r/2)$, we have

$$u(y) = r - d(x,y) \geq r - r/2 = r/2 \Longrightarrow |u(y)|^q \geq (r/2)^q.$$

Therefore,

$$\begin{aligned}
\|\nabla u\|_{L^q(M)} &= \Big(\int_{B(x,r)} d\mu_g\Big)^{1/q} = \Big(Vol_g(B(x,r))\Big)^{1/q}.\\
\|u\|_{L^q(M)} &= \Big(\int_{B(x,r)} |u(y)|^q d\mu_g(y)\Big)^{1/q} \geq \Big(\int_{B(x,\frac{r}{2})} |u(y)|^q d\mu_g(y)\Big)^{1/q}\\
&\geq \Big(\int_{B(x,\frac{r}{2})} (r/2)^q\Big)^{1/q} = \frac{r}{2}\Big(Vol_g(B(x,\frac{r}{2})\Big)^{1/q}.
\end{aligned}$$

Hence, from (5.3), we have

$$\begin{aligned}
\frac{1}{2}\Big(Vol_g(B(x,r))\Big)^{-1/n} &\leq A\frac{\Big(Vol_g(B(x,r))\Big)^{1/q}}{\frac{r}{2}\Big(Vol_g(B(x,\frac{r}{2}))\Big)^{1/q}} \Longrightarrow\\
Vol_g(B(x,r)) &\geq \Big(\frac{r}{4A}\Big)^{\frac{nq}{n+q}}\Big(Vol_g(B(x,\frac{r}{2}))\Big)^{\frac{n}{n+q}}.
\end{aligned}$$

Fix any $R > 0$ such that $B(x,R) \subsetneqq M$. By induction we get that, for any $m \in \mathbb{N} \setminus \{0\}$,

$$Vol_g(B(x,r)) \geq \Big(\frac{R}{2A}\Big)^{q\alpha(m)}\Big(\frac{1}{2}\Big)^{q\beta(m)}\Big(Vol_g(B(x,\frac{R}{2^m}))\Big)^{\gamma(m)}, \tag{5.4}$$

where the sequences $\alpha(m), \beta(m)$ and $\gamma(m)$ are given by

$$\begin{aligned}
\alpha(m) &:= \sum_{j=1}^{m}\Big(\frac{n}{n+q}\Big)^j \longrightarrow \frac{\frac{n}{n+q}}{1-\frac{n}{n+q}} = \frac{n}{q}, \quad \text{as } m \longrightarrow +\infty,\\
\beta(m) &:= \sum_{j=1}^{m} j\Big(\frac{n}{n+q}\Big)^j \longrightarrow \frac{\frac{n}{n+q}}{\big(1-\frac{n}{n+q}\big)^2} = \frac{n(n+q)}{q^2}, \quad \text{as } m \longrightarrow +\infty,\\
\gamma(m) &:= \Big(\frac{n}{n+q}\Big)^m \longrightarrow 0, \quad \text{as } m \longrightarrow +\infty.
\end{aligned}$$

Note that the local geometry indicates that (see for example [7] p.168)

$$Vol_g(B(x,r)) = b_n r^n\big(1 + o(r)\big)$$

where b_n is the volume of the unit ball in the Euclidean space $\mathbb{R}^n$. Therefore,

$$\Big(Vol_g(B(x,\frac{R}{2^m}))\Big)^{\gamma(m)} = \Big(\frac{b_n R^n}{2^{nm}}(1 + o(\frac{R}{2^m}))\Big)^{\gamma(m)} \longrightarrow 1, \text{ as } m \longrightarrow +\infty$$

Thus, letting $m \longrightarrow +\infty$ in (5.4) we obtain that, for any $R > 0$,

$$Vol_g(B(x,R)) \geq \Big(\frac{R}{2A}\Big)^{q\frac{n}{q}} \Big(\frac{1}{2}\Big)^{q\frac{n(n+q)}{q^2}} = \Big(\frac{R}{A2^{\frac{n+2p}{p}}}\Big)^n.$$

In summary, including both cases, for any $R > 0$, we have

$$Vol_g(B(x,R)) \geq v(M,q,R) := \Big(\min\Big\{\frac{1}{2A}, \frac{R}{A2^{\frac{n+2p}{p}}}\Big\}\Big)^n \text{, for all } x \in M.$$

□

Let us now present an application of theorem 5.4.1.

Example 5.4.2. *For $n \geq 3$ there exists complete Riemannian manifolds with Ricci curvature bounded from below such that $H_1^q(M) \not\subseteq L^p(M)$ for any p, q satisfying $1 \leq q < n, 1/p = 1/q - 1/n$.*

Proof. Consider the $n-$dimensional cylinder $M := \mathbb{R} \times \mathbb{S}^{n-1}$ endowed with the metric

$$g_{(x,\vartheta)} = \xi_x \oplus u(x)h_\vartheta, \quad \text{for any } (x,\vartheta) \in M,$$

where ξ_x, h_ϑ are the standard metrics on $\mathbb{R}$ and $\mathbb{S}^{n-1}$ respectively and $u : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function defined by

$$u(x) = \begin{cases} 1 & , x \leq 0 \\ e^{-2x} & , x \geq 1 \end{cases}$$

For any tangent vector $X, Y \in T_{(x,\vartheta)}M \simeq T_x\mathbb{R} \oplus T_\vartheta\mathbb{S}^{n-1}$ the metric $g_{(x,\vartheta)}$ is given by

$$g_{(x,\vartheta)}(X,Y) := \xi_x(X_1,Y_1) + u(x)h_\vartheta(X_2,Y_2),$$

where $X = X_1 + X_2$, $Y = Y_1 + Y_2$ and $X_1, Y_1 \in T_x\mathbb{R} \simeq \mathbb{R}$ and $X_2, Y_2 \in T_\vartheta\mathbb{S}^{n-1}$.

Step 1. First of all we will prove that (M, g) is complete. For any two points $y = (x,\vartheta),\ z = (\widehat{x}, \widehat{\vartheta}) \in M$ it is easy to prove that

$$d\big((x,\vartheta),(\widehat{x},\widehat{\vartheta})\big) \geq |x - \widehat{x}|. \tag{5.5}$$

Let $\gamma : [0,t_0) \longrightarrow M$ be the geodesic curve satisfying $\gamma(0) = (x,0)$, $\|\gamma'\| = 1$ which can not be extended. Then, from the theory of ODE's, we get

$$\begin{aligned} &\forall \text{ compact } K \subseteq M,\ \exists t_1 \in (0,t_0) : \gamma(t) \notin K,\ \forall t \in (t_1,t_0) \Longrightarrow \\ &\exists t_1 \in (0,t_0) : \gamma(t) \notin [-t_0,t_0] \times \mathbb{S}^{n-1},\ \forall t \in (t_1,t_0). \end{aligned}$$

But then, for any $t \in (t_1,t_0)$,

$$t_0 < |\pi_1(\gamma(t)) - \pi_1(\gamma(t_0))| \leq d_g\big(\gamma(t),\gamma(t_0)\big) \leq t_0, \text{ a contradiction.}$$

Thus, every geodesic curve can be extended for every $t \in \mathbb{R}$ which proves that (M, g) is complete according to Hopf-Rinow theorem.

Step 2. We will now prove that $H_1^q(M) \not\subseteq L^p(M)$ for any p, q satisfying $1 \leq q < n$ and $1/p = 1/q - 1/n$. Clearly, for any point $y = (x, \vartheta) \in M$, inequality (5.5) yields

$$B(y, 1) \subseteq (x-1, x+1) \times \mathbb{S}^{n-1}.$$

As a consequence, for any $x \geq 2$,

$$\begin{aligned}
Vol_g\big(B(y,1)\big) &\leq Vol_g\big((x-1,x+1)\times\mathbb{S}^{n-1}\big)\\
&= \int_{(x-1,x+1)\times\mathbb{S}^{n-1}} \sqrt{det(g_{(x,\vartheta)})}dt\wedge d\vartheta_1\wedge\cdots\wedge d\vartheta_{n-1}\\
&= \int_{(x-1,x+1)\times\mathbb{S}^{n-1}} \sqrt{det\begin{pmatrix}1 & 0\\ 0 & u(x)h_{ij}\end{pmatrix}}dt\wedge d\vartheta_1\wedge\cdots\wedge d\vartheta_{n-1}\\
&= \int_{(x-1,x+1)\times\mathbb{S}^{n-1}} \sqrt{(u(x))^{n-1}det(g_{\mathbb{S}^{n-1}})}dt\wedge d\vartheta_1\wedge\cdots\wedge d\vartheta_{n-1}\\
&= \int_{(x-1,x+1)\times\mathbb{S}^{n-1}} e^{-t(n-1)}dt\wedge\Big(\sqrt{det(h_{ij})}d\vartheta_1\wedge\cdots\wedge d\vartheta_{n-1}\Big)\\
&= \Big(\int_{x-1}^{x+1} e^{-t(n-1)}dt\Big)\Big(\int_{\mathbb{S}^{n-1}}\sqrt{det(h_{ij})}d\vartheta_1\wedge\cdots\wedge d\vartheta_{n-1}\Big)\\
&= C(n)e^{-x(n-1)} \longrightarrow 0, \text{ as } x\longrightarrow+\infty.
\end{aligned}$$

Thus,

$$\inf_{(x,\vartheta)\in M} Vol_g\big(B((x,\vartheta),1)\big) = 0$$

and the desired result follows from theorem 5.4.1.

Step 3. We will now show that the Ricci curvature of (M, g) is bounded from below. Notice that it is sufficient to show $Ric_{(M,g)} \geq -(n-1)g$ for any $x > 1$. Consider the conformally related metric

$$g'_{(x,\vartheta)} := e^{2x}g_{(x,\vartheta)} = e^{2x}\xi_x + h_\vartheta,\ (x,\vartheta)\in(1,+\infty)\times\mathbb{S}^{n-1}.$$

Pick any $X, Y \in T_{(x,\vartheta)}(\mathbb{R}\times\mathbb{S}^{n-1})$ and $\{Z^1, \cdots, Z^n\}$ an orthonormal base for $T_{(x,\vartheta)}(\mathbb{R}\times\mathbb{S}^{n-1})$. Write $X = X_1 + X_2$, $Y = Y_1 + Y_2$, and for any $i = 1, 2, \cdots, n$, $Z^i = Z_1^i + Z_2^i$ where

$X_1, Y_1, Z_1^i \in T_x\mathbb{R} \simeq \mathbb{R}$ and $X_2, Y_2, Z_2^i \in T_\vartheta \mathbb{S}^{n-1}$. We calculate

$$
\begin{aligned}
Ric_{(M,g')}(X,Y) &:= \sum_{k=1}^{n} g'\big(R(X,Z_k)Y, Z_k\big) \\
&= \sum_{k=1}^{n} g'\big(R^{\mathbb{R}}(X_1,Z_1^1)Y_1 + R^{\mathbb{S}^{n-1}}(X_2,Z_2^2)Y_2, Z_1^k + Z_2^k\big) \\
&= \sum_{k=1}^{n} e^{2x}\xi_x\big(R_{\mathbb{R}}(X_1,Z_1^k)Y_1, Z_1^k\big) + \sum_{k=1}^{n} h_\vartheta\big(R_{\mathbb{S}^{n-1}}(X_2,Z_2^k)Y_2, Z_2^k\big) \\
&= e^{2x}\underbrace{Ric_{(\mathbb{R},\xi_x)}(X_1,Y_1)}_{=0} + Ric_{(\mathbb{S}^{n-1},h_\vartheta)}(X_2,Y_2) \\
&= (n-2)h_\vartheta(X_2,Y_2).
\end{aligned}
$$

Let R_{ij} be the components of $Ric_{(M,g)}$ in some product chart $(\mathbb{R}\times\Omega, id\times\varphi)$, and R'_{ij} be the components of $Ric_{(M,g')}$ in the same chart and let $\{\frac{d}{dx} = \frac{\partial}{\partial\vartheta_1}, \frac{\partial}{\partial\vartheta_2}, \cdots, \frac{\partial}{\partial\vartheta_n}\}$ be the standard base for the tangent bundle $T(\mathbb{R}\times\Omega)$. We calculate

$$
R'_{ij} := Ric_{(M,g')}\Big(\frac{\partial}{\partial\vartheta_i}, \frac{\partial}{\partial\vartheta_j}\Big) = \begin{cases} Ric_{(M,g')}(\frac{d}{dx}+0, 0+\frac{\partial}{\partial\vartheta_j}) = 0, & i=1,\ j=1,2,\dots,n \\ Ric_{(M,g')}(0+\frac{\partial}{\partial\vartheta_i}, \frac{d}{dx}+0) = 0, & i=1,2,\dots,n,\ j=1 \\ Ric_{(\mathbb{S}^{n-1},h_\vartheta)}(\frac{\partial}{\partial\vartheta_i}, \frac{\partial}{\partial\vartheta_j}) = (n-2)h_{ij}, & i,j=2,3,\dots,n \end{cases}
$$

Independently, under the conformal rescaling $g' = e^f g$, the components R_{ij} of the Ricci tensor $Ric_{(M,g')}$ are related to the components R'_{ij} of $Ric_{(M,g)}$ via

$$
R_{ij} = R'_{ij} + \frac{n-2}{2}(\nabla^2 f)_{ij} - \frac{n-2}{4}(\nabla f)_i(\nabla f)_j + \frac{1}{2}(-\Delta_g f + \frac{n-2}{2}|\nabla f|^2)g_{ij},
$$

in the same product chart. Here, $f(x,\vartheta_1,\dots,\vartheta_{n-1}) = 2x,\ (x,\vartheta)\in(1,+\infty)\times\mathbb{S}^{n-1}$. Set $x_1 = x$ and $x_i = \vartheta_i$ $(i=2,3,\dots,n-1)$. For any $x>1$, we calculate,

$$
\begin{aligned}
&(\nabla f)_1 = \frac{\partial f}{\partial x_1} = 2,\ (\nabla f)_i := \frac{\partial f}{\partial x_i} = 0\ (i=2,3,\dots,n), \\
&(\nabla^2 f)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{k=1}^{n}\Gamma_{ij}^k\frac{\partial f}{\partial x_k} = -\Gamma_{ij}^1\frac{\partial f}{\partial x_1} = -2(\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j})_1 = 0, \\
&|\nabla f|^2 = \sum_{i,j=1}^{n} g^{ij}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} = g^{11}\Big(\frac{\partial f}{\partial x_1}\Big)^2 + 2\frac{\partial f}{\partial x_1}\sum_{j=2}^{n} g^{1j}\frac{\partial f}{\partial x_j} + \sum_{i,j=2}^{n} g^{ij}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} = 4, \\
&\Delta_g f := trace_g(\nabla^2 f) = \sum_{i,j=1}^{n} g^{ij}(\nabla^2 f)_{ij} = 0,
\end{aligned}
$$

and so, according to the formula above, we get

$$
\begin{aligned}
&R_{11} = 0 \\
&R_{1j} = (n-2)g_{1j}\ (j=2,3,\dots,n), \\
&R_{ij} = (n-2)(1+e^{2x})g_{ij}\ (i,j=2,3,\dots,n), \\
&R_{i1} = (n-2)g_{i1}\ (i=2,3,\dots,n).
\end{aligned}
$$

As a consequence, $Ric_{(M,g)} \geq -(n-1)g,\ x>1.$ □

Our next result will be the following: The lower bound of the volume of the geodesic balls $B(x,1)$ on complete Riemannian manifolds with Ricci curvature bounded from below implies the validity of the classical isoperimetric inequality for small domains of (M,g).

Theorem 5.4.3. *Let (M,g) be a complete $n-$dimensional Riemannian manifold with $Rig_{(M,g)} \geq kg$ for some real constant $k \in \mathbb{R}$. Suppose that there exists a positive constant v such that $Vol_g(B(x,1)) \geq v$, $\forall x \in M$. There exists two positive constants $C \equiv C(n,k,v)$, $\eta \equiv \eta(n,k,v)$ such that for any open subset Ω, of M with smooth boundary and compact closure, if $Vol_g(\Omega) \leq \eta$, then $\Big(Vol_g(\Omega)\Big)^{\frac{n-1}{n}} \leq C Area(\partial\Omega)$.*

Proof. By the remark 5.1.2, for all $x \in M$ and for all $0 < r < 1$, we have

$$Vol_g(B(x,r)) \geq C(n,k,v)r^n := ve^{-(n-1)\sqrt{|k|}}r^n. \tag{5.6}$$

Let $\eta \equiv \eta(n,k,v) := C(n,k,v)/16$ and pick any open subset $\Omega \subseteq M$ with smooth boundary and compact closure such that $Vol_g(\Omega) \leq \eta$. Also, let any $\epsilon > 0$. We define the function

$$u_\epsilon(x) := \begin{cases} 1, & x \in \Omega, \\ 0, & x \in M \setminus \Omega \ , \ d(x,\partial\Omega) \geq \epsilon, \\ 1 - \frac{1}{\epsilon} d(x,\partial\Omega), & x \in M \setminus \Omega \ , \ d(x,\partial\Omega) < \epsilon. \end{cases}$$

Clearly, u_ϵ is Lipschitz on M and therefore $u_\epsilon \in H_1^p(M)$ for any $p \geq 1$. We calculate

$$|\nabla u_\epsilon(x)| = \begin{cases} 1/\epsilon, & x \in M \setminus \overline{\Omega} \ , \ d(x,\partial\Omega) < \epsilon, \\ 0, & \text{otherwise} \end{cases}$$

Since Ω has compact closure and (M,g) is complete we have that Ω is bounded and hence $\text{diam } \Omega := \sup\{d(x,y) : x,y \in \Omega\} < +\infty$. Since

$$|u_\epsilon| \leq 1_{B(x_0, diam\Omega + \epsilon)} \in L^1(M),$$

dominated convergence theorem implies,

$$\lim_{\epsilon \longrightarrow 0} \int_M u_\epsilon(x) d\mu_g(x) = \int_M \lim_{\epsilon \longrightarrow 0} u_\epsilon(x) d\mu_g(x) = \int_M 1_\Omega(x) d\mu_g(x) = Vol_g(\Omega). \tag{5.7}$$

Furthermore, Coarea formula yields,

$$\begin{aligned} \lim_{\epsilon \longrightarrow 0} \int_M |\nabla u_\epsilon(x)| d\mu_g(x) &= \lim_{\epsilon \longrightarrow 0} \frac{1}{\epsilon} \int_{\{x \in M \setminus \Omega : d(x,\partial\Omega) < \epsilon\}} |\nabla d(x,\partial\Omega)| d\mu_g(x) \\ &= \lim_{\epsilon \longrightarrow 0} \frac{1}{\epsilon} \int_0^\epsilon \int_{\{x \in M \setminus \Omega : d(x,\partial\Omega) = t\}} dA(t) dt \\ &= \lim_{\epsilon \longrightarrow 0} \frac{1}{\epsilon} \int_0^\epsilon Area\Big(\{x \in M \setminus \Omega : d(x,\partial\Omega) = t\}\Big) dt \\ &= Area_g\Big(\{x \in M \setminus \Omega : d(x,\partial\Omega) = 0\}\Big) \\ &= Area_g(\partial\Omega). \end{aligned}$$

Recall that $\overline{u}_{\epsilon,r}(x)$ denotes that average of u_ϵ over the ball $B(x,r)$, that is

$$\overline{u}_{\epsilon,r}(x) := \frac{1}{Vol_g(B(x,r))}\int_{B(x,r)} u_\epsilon(y)d\mu_g(y),\ x\in M,\ r>0.$$

For any $r>0$, set

$$A(r) := \{x\in M : \overline{u}_{\epsilon,r}(x)\geq \frac{1}{2}\},$$
$$B(r) := \{x\in M : |u_\epsilon(x)-\overline{u}_{\epsilon,r}(x)|\geq \frac{1}{2}\}.$$

Since for any $x\in M$ and any $r>0$, $u_\epsilon(x)\leq |u_\epsilon(x)-\overline{u}_{\epsilon,r}(x)|+\overline{u}_{\epsilon,r}(x)$, we have

$$\begin{aligned}&A^c(r)\cap B^c(r)\subseteq \{x\in M : u_\epsilon(x)<1\}=\Omega^c \Longrightarrow\\ &\Omega\subseteq A(r)\cup B(r)\Longrightarrow Vol_g(\Omega)\leq Vol_g(A(r))+Vol_g(B(r)).\end{aligned} \tag{5.8}$$

According to (5.7), there exists some $\delta>0$ such that for any $0<\epsilon<\delta$ and any $r>0$,

$$\int_M u_\epsilon(y)d\mu_g(y)-Vol_g(\Omega)\leq \left|\int_M u_\epsilon(y)d\mu_g(y)-Vol_g(\Omega)\right|<Vol_g(\Omega)\Longrightarrow$$
$$\int_{B(x,r)} u_\epsilon(y)d\mu_g(y)\leq \int_M u_\epsilon(y)d\mu_g(y)<2Vol_g(\Omega)\Longrightarrow$$
$$\frac{1}{Vol_g(B(x,r))}\int_{B(x,r)} u_\epsilon(y)d\mu_g(y)<\frac{2Vol_g(\Omega)}{Vol_g(B(x,r))}.$$

As a consequence, for any $0<\epsilon<\delta$ and $r>0$,

$$\overline{u}_{\epsilon,r}(x)\leq \frac{2Vol_g(\Omega)}{Vol_g(B(x,r))} \tag{5.9}$$

Until now, the constant $r>0$ was arbitrary. Choose $r:=\left(\frac{8Vol_g(\Omega)}{C(n,k,v)}\right)^{1/n}$. First of all notice that $0<r<1$, since

$$0<r:=\left(\frac{8Vol_g(\Omega)}{C(n,k,v)}\right)^{1/n}\leq\left(\frac{8\eta}{C(n,k,v)}\right)^{1/n}=\left(\frac{8\frac{C(n,k,v)}{16}}{C(n,k,v)}\right)^{1/n}=2^{-1/n}<1.$$

Furthermore, for any $0<\epsilon<\delta$, with this particular choice of r, we have $A(r):=\{x\in M:\overline{u}_{\epsilon,r}(x)\geq 1/2\}=\emptyset$, since based on (5.9) and (5.6),

$$\overline{u}_{\epsilon,r}(x)\leq\frac{2Vol_g(\Omega)}{Vol_g(B(x,r))}\leq\frac{2Vol_g(\Omega)}{C(n,k,v)r^n}=\frac{2Vol_g(\Omega)}{C(n,k,v)\frac{8Vol_g(\Omega)}{C(n,k,v)}}=\frac{1}{4}.$$

Applying (5.8), Chebyshev's inequality and Poincare's inequality on M (see theorem 4.4.1) we get

$$\begin{aligned}Vol_g(\Omega)&\leq Vol_g(B(r))=Vol_g\Big(\{x\in M : |u_\epsilon(x)-\overline{u}_{\epsilon,r}(x)|\geq\frac{1}{2}\}\Big)\\&\leq 2\int_M |u_\epsilon(x)-\overline{u}_{\epsilon,r}(x)|d\mu_g(x)\\&\leq 2C'r\int_M |\nabla u_\epsilon(x)|d\mu_g(x).\end{aligned}$$

Letting $\epsilon \longrightarrow 0$,

$$Vol_g(\Omega) \leq 2C' r Area(\partial\Omega) = 2C'\Big(\frac{8Vol_g(\Omega)}{C(n,k,v)}\Big)^{1/n} Area(\partial\Omega) \Longrightarrow$$
$$\Big(Vol_g(\Omega)\Big)^{1-1/n} \leq C'' Area(\partial\Omega).$$

□

As a consequence of this theorem one can prove the following.

Theorem 5.4.4. *Let (M,g) be a complete Riemannian n-manifold with $Ric_{(M,g)} \geq kg$ for some real constant k. Suppose that there exists $v > 0$ such that $Vol_g(B(x,1)) \geq v$ for any $x \in M$. There exist two positive constants $\delta \equiv \delta(n,k,v)$ and $A \equiv A(n,k,v)$ such that for any $x \in M$ and any $u \in C_c^\infty(B(x,\delta))$,*

$$\|u\|_{L^{\frac{n}{n-1}}(M)} \leq A\|\nabla u\|_{L^1(M)}.$$

Proof. Pick the constant $\eta \equiv \eta(n,k,v)$ as in the previous theorem and the constant $\delta \equiv \delta(n,k,v)$ such that $V_k(\delta) = \eta$, where $V_k(\delta)$ denotes the volume of a ball of radius δ in the complete simply connected Riemannian n-manifold of constant curvature k. By Bishop-Gromov's theorem we have

$$\forall x \in M, \quad Vol_g(B(x,\delta)) \leq V_k(\delta) = \eta.$$

Let $x \in M$, $u \in C_c^\infty(B(x,\delta))$ and set

$$\Omega(t) := \{y \in M : |u(y)| > t\}, \text{ for all } t > 0.$$

Since $u \equiv 0$ on $M \setminus B(x,\delta)$, we have

$$\Omega(t) = \{y \in B(x,\delta) : |u(y)| > t\} \subseteq B(x,\delta) \Longrightarrow Vol_g(\Omega(t)) \leq Vol_g(B(x,\delta)) \leq \eta.$$

Now, Coarea formula and the theorem 5.4.3 implies,

$$\begin{aligned}\int_M |\nabla u(x)| d\mu_g(x) &= \int_0^\infty \int_{\{x\in M : |u(x)|=t\}} dA(t)dt \\ &= \int_0^\infty Area\big(\{x \in M : |u(x)| = t\}\big)dt = \int_0^\infty Area(\partial\Omega)dt \\ &\geq \frac{1}{C}\int_0^\infty \Big(Vol_g(\Omega(t))\Big)^{\frac{n-1}{n}} dt.\end{aligned}$$

Furthermore,

$$\int_M |u(x)|^{\frac{n}{n-1}} d\mu_g(x) = \int_M \left(\int_0^{|u(x)|} \frac{n}{n-1} t^{\frac{1}{n-1}} dt\right) d\mu_g(x) = \frac{n}{n-1}\int_E t^{\frac{1}{n-1}} d(m_{\mathbb{R}} \otimes \mu_g)(t,x),$$

where $m_{\mathbb{R}}$ stands for the Lebesque's measure on $\mathbb{R}$ and

$$E := \{(t,x) : x \in M, \quad 0 \leq t \leq |u(x)|\} = \{(t,x) : 0 \leq t \leq \infty, \quad x \in \Omega(t)\}.$$

Fubini-Toneli's theorem yields

$$\int_M |u(x)|^{\frac{n}{n-1}} d\mu_g(x) = \frac{n}{n-1}\int_0^\infty \left(\int_{\Omega(t)} t^{\frac{1}{n-1}} d\mu_g(x)\right) dt = \frac{n}{n-1}\int_0^\infty t^{\frac{1}{n-1}} Vol_g(\Omega(t))dt \Longrightarrow$$
$$\|u\|_{L^{\frac{n}{n-1}}(M)} := \left(\int_M |u(x)|^{\frac{n}{n-1}} d\mu_g(x)\right)^{\frac{n-1}{n}} = \left(\frac{n}{n-1}\int_0^\infty t^{\frac{1}{n-1}} Vol_g(\Omega(t))dt\right)^{\frac{n-1}{n}}.$$

Now, it is sufficient to show that

$$\int_0^\infty \big(Vol_g(\Omega(t))\big)^{\frac{n-1}{n}} dt \geq \left(\frac{n}{n-1}\int_0^\infty t^{\frac{1}{n-1}} Vol_g(\Omega(t))dt\right)^{\frac{n-1}{n}}. \tag{5.10}$$

In order to do so, for any $s > 0$, we define the functions

$$F(s) := \int_0^s \big(Vol_g(\Omega(t))\big)^{\frac{n-1}{n}} dt, \quad G(s) := \left(\frac{n}{n-1}\int_0^s t^{\frac{1}{n-1}} Vol_g(\Omega(t))dt\right)^{\frac{n-1}{n}}.$$

For any $s > 0$ we have,

$$\begin{aligned}
G'(s) &= \left(\frac{n}{n-1}\int_0^s t^{\frac{1}{n-1}} Vol_g(\Omega(t))dt\right)^{-1/n} s^{\frac{1}{n-1}} Vol_g(\Omega(s)) \\
&\leq \left(\frac{n}{n-1}\int_0^s t^{\frac{1}{n-1}} Vol_g(\Omega(s))dt\right)^{-1/n} s^{\frac{1}{n-1}} Vol_g(\Omega(s)) \\
&= \left(\frac{n}{n-1}\int_0^s t^{\frac{1}{n-1}} dt\right)^{-1/n} s^{\frac{1}{n-1}} \Big(Vol_g(\Omega(s))\Big)^{1-1/n} \\
&= \left(\frac{n}{n-1}\right)^{-1/n} \left(\frac{n}{n-1} s^{\frac{n}{n-1}}\right)^{-1/n} s^{\frac{1}{n-1}} \Big(Vol_g(\Omega(s))\Big)^{\frac{n-1}{n}} \\
&= \Big(Vol_s(\Omega(s))\Big)^{\frac{n-1}{n}} = F'(s).
\end{aligned}$$

Integrate this inequality for $0 < s < +\infty$ to find the desired result. □

With such a result we are now in position to prove theorem 5.1.1.

Theorem 5.4.5. *Let (M, g) be a complete $n-$dimensional Riemannian manifold satisfying $Ric_{(M,g)} \geq kg$ for some real constant k. Assume that*

$$\inf_{x\in M} Vol_g(B(x,1)) > 0.$$

Then all Sobolev embeddings are valid for (M, g).

Proof. As already mentioned, by theorem 5.2.1, we just have to prove that $H_1^1(M) \hookrightarrow L^{\frac{n}{n-1}}(M)$. We assume that there exists a positive constant v such that, $Vol_g(B(x,1)) > v$, for all $x \in M$. Pick $\delta \equiv \delta(n,k,v)$ and $\eta \equiv \eta(n,k,v) := ve^{-(n-1)\sqrt{|k|}}/16$ as in theorems 5.4.3 and 5.4.4. By the covering lemma 3.4.1 there exists a sequence of points $\{x_i\}_{i=1}^{\infty} \subseteq M$ such that

(i) $M = \bigcup_{i=1}^{\infty} B(x_i, \delta/2)$

(ii) $B(x_i, \delta/4) \cap B(x_j, \delta/4) = \emptyset$ for all $i,j = 1,2,\dots,n$ with $i \neq j$

(iii) There exists a positive constant $N = N(n,k,v)$ such that each point of M has a neighbourhood which intersects at most N of the $B(x_i,\delta)$'s.

We define the function $\alpha_i : M \longrightarrow [0,1]$ by

$$\alpha_i(x) := \varrho\big(d(x_i,x)\big),$$

where $\varrho : [0,\infty] \longrightarrow [0,1]$ is given by

$$\varrho(t) := \begin{cases} 1, & 0 \le t \le \delta/2, \\ 3 - 4\delta/t, & \delta/2 \le t \le 3\delta/4, \\ 0, & 3\delta/4 \le t < \infty. \end{cases}$$

Clearly, α_i is Lipschitz (since ϱ and d are Lipschitz) with compact support, and as a consequence, $\alpha_i \in H_1^1(B(x_i,\delta))$. Since supp $\alpha_i \subseteq B(x_i, 3\delta/4)$ one can easily prove that α_i belongs in the closure of $C_c^{\infty}\big((B(x_i,\delta))\big)$ in $H_1^1\big((B(x_i,\delta))\big)$. We calculate

$$|\nabla \alpha_i| = |\varrho'\big(d(x_i,x)\big)| \cdot |\nabla d(x_i,x)| = \varrho'\big(d(x_i,x)\big) \le 4/\delta,$$

almost everywhere in M. Let

$$\eta_i := \frac{\alpha_i}{\sum_{m=1}^{\infty} \alpha_m}$$

for any $i \in \mathbb{N}$. For all $i \in \mathbb{N}$, the function η_i is well defined since M is covered by $B(x_i, \delta/2)$. Furthermore, it is not difficult to verify that by (ii) we can get

(iv) $\{\eta\}_{i=1}^{\infty}$ is a partition of unity subordinate to the covering $\{B(x_i,\delta)\}_{i=1}^{\infty}$,

(v) η_i belongs in the closure of $C_c^{\infty}\big((B(x_i,\delta))\big)$ in $H_1^1\big((B(x_i,\delta))\big)$,

(vi) Almost everywhere on M, the differential $\nabla \eta_i$ exists and there exists a positive constant $H \equiv H(n,k,v)$ such that $|\nabla \eta_i| \le H$.

Let $u \in C_c^\infty(M)$. Applying theorem 5.4.4 to $\eta_i u \in C_c^\infty(B(x_i,\delta))$ we get

$$\begin{aligned}
\|u\|_{L^{\frac{n}{n-1}}(M)} &:= \left(\int_M |u|^{\frac{n}{n-1}} d\mu_g\right)^{\frac{n-1}{n}} = \left(\int_M \left|\sum_{i=1}^\infty \eta_i u\right|^{\frac{n}{n-1}} d\mu_g\right)^{\frac{n-1}{n}} \\
&\leq \left(\sum_{i=1}^\infty \int_M |\eta_i u|^{\frac{n}{n-1}} d\mu_g\right)^{\frac{n-1}{n}} \\
&\leq \sum_{i=1}^\infty \left(\int_M |\eta_i u|^{\frac{n}{n-1}} d\mu_g\right)^{\frac{n-1}{n}} \quad (\text{since } \frac{n-1}{n} < 1) \\
&\leq A \sum_{i=1}^\infty \int_M |\nabla(\eta_i u)| d\mu_g \\
&= A \sum_{i=1}^\infty \int_M \eta_i |\nabla u| d\mu_g + A \sum_{i=1}^\infty \int_M |u||\nabla \eta_i| d\mu_g \\
&\leq A \int_M \left(\sum_{i=1}^\infty \eta_i\right) |\nabla u| d\mu_g + AN \int_M |u| H d\mu_g \\
&= A \int_M |\nabla u| d\mu_g + ANH \int_M |u| d\mu_g \\
&\leq C(n,k,v) \int_M |u| + |\nabla u| d\mu_g := C(n,k,v) \|u\|_{H_1^1(M)},
\end{aligned}$$

where $C(n,k,v) := \max\{A, ANH\}$. Since $C_c^\infty(M)$ is dense in $H_1^1(M)$, the result follows by a standard approximation argument. □

5.4.1. The embedding $\exists q_0 \in [1,n) : H_1^{q_0}(M) \hookrightarrow L^{\frac{nq_0}{n-q_0}}(M)$ implies all Sobolev embeddings

Working as in the proof of lemma 5.2.1 one can prove that

$$\exists q_0 \in [1,n) : H_1^{q_0}(M) \hookrightarrow L^{\frac{nq_0}{n-q_0}}(M) \Longrightarrow \forall q \in [q_0,n) : H_1^q(M) \hookrightarrow L^{\frac{nq}{n-q}}(M),$$

without any assumption on the curvature. But, if the Ricci curvature is bounded from below, then

$$\exists q_0 \in [1,n) : H_1^{q_0}(M) \hookrightarrow L^{\frac{nq_0}{n-q_0}}(M) \Longrightarrow \text{ All Sobolev embeddings are valid for } (M,g),$$

since

$$\begin{aligned}
\exists q_0 \in [1,n) : H_1^{q_0}(M) \hookrightarrow L^{\frac{nq_0}{n-q_0}}(M) &\Longrightarrow \inf_{x\in M} Vol_g\big(B(x,1)\big) > 0 \quad (\text{theorem 5.4.1}) \\
&\Longrightarrow H_1^1(M) \hookrightarrow L^{\frac{n}{n-1}}(M) \quad (\text{theorem 5.4.5}) \\
&\Longrightarrow \text{All Sobolev embeddings are valid for } (M,g). \quad (\text{lemma 5.2.1})
\end{aligned}$$

Part II

On the best constant in the Euclidean-type Sobolev Inequality

In this part, we will study the connection of the best Sobolev constant in Euclidean-type Sobolev inequalities to the geometry and topology of a manifold with non negative or asymptotically non-negative Ricci curvature. The $n-$dimensional Riemannian manifold (M, g) is said to support a Euclidean-type Sobolev inequality if for some $1 \leq q < n$ there exists a positive constant $C_M \equiv C_M(n, q)$ such that

$$\left(\int_M |u|^{q^\star} d\mu_g \right)^{1/q^\star} \leq C_M \left(\int_M |\nabla u|^q d\mu_g \right)^{1/q}, \quad \forall u \in C_c^\infty(M), \tag{5.11}$$

where $q^\star := nq/(n-q)$. It is a classical fact that if (5.11) holds on a $n-$dimensional Riemannian manifold for some $1 \leq q < n$, then $C_M \geq K(n, q)$, where $K(n, q)$ is the optimal Sobolev constant in the same inequality on $\mathbb{R}^n$ (see chapter 1). For a detailed proof of this result, we refer the reader to [8] proposition 4.5, p.62.

In the following pages, we will show that for complete Riemannian manifolds with non-negative Ricci curvature which support a Euclidean-type Sobolev inequality,

(1) if C_M is sufficiently close to $K(n, q)$, then M is diffeomorphic to $\mathbb{R}^n$ and

(2) if $C_M = K(n, q)$, then M is isometric to $\mathbb{R}^n$.

Later, we will prove that (1) holds even when the Ricci curvature is asymptotically non-negative.

Chapter 6

When the Ricci curvature is non-negative

As already mentioned, in this chapter, we will see that the validity of the Euclidean-type Sobolev inequality as well as the optimal value of the Sobolev constant C_M have deep connections with the geometry and the topology of the underlying manifold. Namely, for complete Riemannian manifolds with non-negative Ricci curvature, a Euclidean-type Sobolev inequality forces M to be diffeomorphic to $\mathbb{R}^n$, if C_M is sufficiently close to $K(n,q)$ and isometric to $\mathbb{R}^n$, if $C_M = K(n,q)$. Let $V_0(r) = b_n r^n$ be the volume of a Euclidean ball of radius r. In the case of diffeomorphism, we will use the following result of Cheeger and Colding (see [5] p.459, theorem A.1.11).

Theorem 6.0.6. (**Cheeger – Colding**) *Given an integer $n \geq 2$, there exists a positive constant $\delta \equiv \delta(n)$ such that any $n-$dimensional complete Riemannian manifold (M,g) with non-negative Ricci curvature satisfying*

$$Vol_g(B(x,r)) \geq \big(1-\delta(n)\big)V_0(r),\ \forall x \in M,\ r>0,$$

is diffeomorphic to $\mathbb{R}^n$.

For the case of isometry, we will use the following lemma.

Lemma 6.0.7. (**Equality in Bishop – Gromov′s theorem**) *If a complete Riemannian manifold with non-negative Ricci curvature satisfies*

$$Vol_g(B(x_0,r)) = V_0(r) \text{ for all } x_0 \in M,\ r>0,$$

then M is isometric to $\mathbb{R}^n$.

In both cases the essential result is the following volume comparison theorem.

6.1. A volume comparison theorem

This result is due to C.Xia [20].

Theorem 6.1.1. *Let (M,g) be a complete Riemannian manifold of dimension $n \geq 2$ with non-negative Ricci curvature satisfying a Euclidean-type Sobolev inequality, that is*

$$\left(\int_M |u|^{q^\star} d\mu_g\right)^{1/q^\star} \leq C_M \left(\int_M |\nabla u|^q d\mu_g\right)^{1/q},\quad \forall u \in C_c^\infty(M),$$

for some $1 \le q < n$, $C_M > 0$ *and* $q^\star := nq/(n-q)$. *Then, for all* $x_0 \in M$ *and* $r > 0$,

$$\left(\frac{K(n,q)}{C_M}\right)^n V_0(r) \le Vol_g(B(x_0,r)) \le V_0(r), \tag{6.1}$$

where $V_0(r) = b_n r^n$ *denotes the volume of a Euclidean ball of radius* r.

Proof. In the case $q = 1$, inequality (5.11) is equivalent to the isoperimetric inequality

$$\Big(Vol_g(\Omega)\Big)^{\frac{n-1}{n}} \le C_M Area_g(\partial\Omega),$$

where Ω is an open and bounded subset of M. A detailed proof of this equivalence can be found in [4], p.131. For any fixed $x \in M$ and $r > 0$ we have

$$\Big(Vol_g(B(x_0,r))\Big)^{\frac{n-1}{n}} \le C_M Area_g(\partial B(x_0,r)) = C_M \frac{d}{dr}\Big(Vol_g(B(x_0,r))\Big).$$

Solving this differential inequality we find

$$\Big(Vol_g(B(x_0,r))\Big)^{1/n} \ge \frac{r}{nC} \Longrightarrow Vol_g(B(x_0,r)) \ge \Big(\frac{r}{nC_M}\Big)^n = \Big(\frac{K(n,1)}{C_M}\Big)^n V_0(r).$$

For the general case $1 < q < n$ let any $x_0 \in M$, $r > 0$, $\beta > 1$ and consider the functions

$$\begin{aligned} f(x) &:= \frac{1}{\beta} d(x,x_0), \ x \in M \\ F(\lambda) &:= \frac{1}{n-1}\int_M \frac{1}{(\lambda + f^{\frac{q}{q-1}}(x))^{n-1}} d\mu_g(x), \ \lambda > 0. \end{aligned}$$

Recall from chapter 1 that the extremal functions in the Sobolev inequality with $C_M = K(n,q)$ in $\mathbb{R}^n$ are the functions $(\lambda + |x|^{\frac{q}{q-1}})^{1-n/q}$, $\lambda > 0$. Here, we will use similar functions on M,

$$f_\lambda(x) := (\lambda + f^{\frac{q}{q-1}})^{1-n/q}, \ \lambda > 0.$$

For any $\lambda > 0$, coarea formula and integration by parts imply

$$\begin{aligned} F(\lambda) &:= \frac{1}{n-1}\int_M \frac{1}{(\lambda + f^{\frac{q}{q-1}}(x))^{n-1}} d\mu_g(x) \\ &= \frac{1}{n-1}\int_0^\infty \int_{\{x\in M: f(x)=t\}} \frac{dA(t)dt}{|\nabla f|(\lambda + f^{\frac{q}{q-1}}(x))^{n-1}} \\ &= \frac{\beta}{n-1}\int_0^\infty \int_{\{x\in M: f(x)=t\}} \frac{dA(t)dt}{(\lambda + t^{\frac{q}{q-1}}(x))^{n-1}} \\ &= \frac{\beta}{n-1}\int_0^\infty \frac{Area(\partial B(x_0,\beta t))}{(\lambda + t^{\frac{q}{q-1}}(x))^{n-1}} dt \\ &= \frac{1}{n-1}\int_0^\infty \frac{d\Big(Vol_g(B(x_0,\beta t))\Big)}{(\lambda + t^{\frac{q}{q-1}}(x))^{n-1}} \\ &= \frac{1}{n-1}\left\{ \frac{Vol_g(B(x_0,\beta t))}{(\lambda + t^{\frac{q}{q-1}})^{n-1}}\Bigg|_0^\infty - \int_0^\infty Vol_d(B(x_0,\beta t)) d\Big(\frac{1}{(\lambda + t^{\frac{q}{q-1}})^{n-1}}\Big)\right\} \end{aligned}$$

By Bishop-Gromov's result (see theorem 3.3.1 (i)), we have

$$\frac{Vol_g(B(x_0,\beta t))}{(\lambda+t^{\frac{q}{q-1}})^{n-1}} \leq \frac{V_0(\beta t)}{(\lambda+t^{\frac{q}{q-1}})^{n-1}} = \frac{b_n\beta^n t^n}{(\lambda+t^{\frac{q}{q-1}})^{n-1}} \longrightarrow 0, \text{ as } t \longrightarrow +\infty,$$

since $n < \frac{q(n-1)}{q-1}$. Hence,

$$F(\lambda) = \frac{q}{q-1}\int_0^\infty Vol_g(B(x_0,\beta t))\frac{t^{1/(q-1)}dt}{(\lambda+t^{\frac{q}{q-1}})^n}. \tag{6.2}$$

From this we obtain that F is well defined and differentiable for all $\lambda > 0$. Indeed; from Bishop-Gromov's theorem,

$$0 \leq F(\lambda) \leq \frac{q}{q-1}\int_0^\infty V_0(t)\frac{t^{1/(q-1)}dt}{(\lambda+t^{\frac{q}{q-1}})^n} = \frac{q}{q-1}\int_0^\infty b_n\beta^n\frac{t^{n+1/(q-1)}dt}{(\lambda+t^{\frac{q}{q-1}})^n} < +\infty.$$

Notice that f_λ do not necessarily belong to $C_c^\infty(M)$ but by a simple approximation procedure we can apply (5.11) to f_λ to get

$$\left(\int_M |f_\lambda|^{q^\star}d\mu_g\right)^{q/q^\star} \leq C_M^q\int_M |\nabla f_\lambda|^q d\mu_g \Longrightarrow$$

$$\left(\int_M (\lambda+f^{\frac{q}{q-1}})^{-n}d\mu_g\right)^{1-q/n} \leq C_M^q\int_M |1-\frac{n}{q}|^q(\lambda+f^{\frac{q}{q-1}})^{-n}\Big(\frac{q}{q-1}\Big)^q f^{\frac{q}{q-1}}|\nabla f|^q d\mu_g \Longrightarrow$$

$$\left(\int_M \frac{1}{(\lambda+f^{\frac{q}{q-1}})^n}d\mu_g\right)^{1-q/n} \leq \Big(\frac{C_M}{\beta}\frac{n-q}{q-1}\Big)^q\int_M \frac{f^{\frac{q}{q-1}}}{(\lambda+f^{\frac{q}{q-1}})^n}d\mu_g$$

$$= \Big(\frac{C_M}{\beta}\frac{n-q}{q-1}\Big)^q\left\{\int_M \frac{1}{(\lambda+f^{\frac{q}{q-1}})^{n-1}}d\mu_g - \lambda\int_M \frac{1}{(\lambda+f^{\frac{q}{q-1}})^n}d\mu_g\right\} \Longrightarrow$$

$$\big(-F'(\lambda)\big)^{1-q/n} \leq \Big(C_M\frac{n-q}{q-1}\Big)^q\Big((n-1)F(\lambda)+\lambda F'(\lambda)\Big).$$

Thus, the function $F(\lambda)$ satisfies the differential inequality

$$d\big(-F'(\lambda)\big)^{1-q/n} + (n-1)\big(-F(\lambda)\big) \leq \lambda F'(\lambda),\ \lambda > 0 \tag{6.3}$$

where we have set

$$d := \Big(C_M\frac{n-q}{q-1}\Big)^{-q}.$$

Now, the idea is to compare the solutions of (6.3) to the solutions of

$$d\big(-H'(\lambda)\big)^{1-q/n} + (n-1)\big(-H(\lambda)\big) = \lambda H'(\lambda),\ \lambda > 0. \tag{6.4}$$

It is easy to see that a particular solution of (6.4) is the function

$$H_1(\lambda) := \frac{B}{\lambda^{-1+n/q}} \tag{6.5}$$

if and only if

$$B = \frac{q}{n-q}\Big(\frac{d(n-q)}{n(q-1)}\Big)^{n/q}. \tag{6.6}$$

From now on, we choose the constant B as in (6.6). Also, a direct calculation shows that this constant can be written as

$$B = \Big(\frac{K(n,q)}{C_M}\Big)^n \frac{1}{n-1}\int_{\mathbb{R}^n} \frac{1}{(1+|x|^{\frac{q}{q-1}})^{n-1}}dx. \tag{6.7}$$

Let

$$H_0(\lambda) := \frac{1}{n-1}\int_{\mathbb{R}^n} \frac{1}{(\lambda+|x|^{\frac{q}{q-1}})^{n-1}}. \tag{6.8}$$

Working in the same way as in (6.2) we may write

$$H_0(\lambda) = \frac{q}{q-1}\int_0^\infty b_n t^n \frac{t^{\frac{1}{n-1}}}{(\lambda+t^{\frac{q}{q-1}})^n}dt. \tag{6.9}$$

One can use (6.7) to obtain, for any $\lambda > 0$,

$$H_1(\lambda) = \Big(\frac{K(n,q)}{C_M}\Big)^n H_0(\lambda), \tag{6.10}$$

This can be easily proved as follows: For any $\lambda > 0$,

$$\begin{aligned} H_1(\lambda) &:= \Big(\frac{K(n,q)}{C_M}\Big)^n \frac{1}{n-1}\frac{1}{\lambda^{-1+n/q}}\int_{\mathbb{R}^n} \frac{1}{(1+|x|^{\frac{q}{q-1}})^{n-1}}dx \\ &= \Big(\frac{K(n,q)}{C_M}\Big)^n \frac{1}{n-1}\frac{1}{\lambda^{-1+n/q}}\int_{\mathbb{R}^n} \frac{1}{(\lambda+|t|^{\frac{q}{q-1}})}dt \quad (x=\lambda^{-\frac{q-1}{q}}t) \\ &:= \Big(\frac{K(n,q)}{C_M}\Big)^n H_0(\lambda). \end{aligned}$$

Now, the local geometry indicates that

$$\liminf_{\lambda\longrightarrow 0} \frac{F(\lambda)}{H_0(\lambda)} > 1. \tag{6.11}$$

Indeed; since (see [7])

$$\lim_{s\longrightarrow 0} \frac{Vol_g(B(x_0,s))}{V_0(s)} = 1,$$

we have

$$(\forall\epsilon>0)(\exists\delta>0)(\forall s\in(0,\delta)) \;\; (1-\epsilon)V_0(s) < Vol_g(B(x_0,s)). \tag{6.12}$$

Let $\epsilon > 0$. For any $\lambda > 0$, using (6.2), (6.12) and (6.9), we have

$$\begin{aligned}
F(\lambda) &= \frac{q}{q-1}\int_0^\infty Vol_g(B(x_0,s))\frac{t^{\frac{1}{q-1}}}{(\lambda+t^{\frac{q}{q-1}})}dt \quad (set\ \beta t = s)\\
&= \frac{q}{q-1}\beta^{\frac{(n-1)q}{q-1}}\int_0^\infty Vol_g(B(x_0,s))\frac{s^{\frac{1}{q-1}}}{(\lambda\beta^{\frac{q}{q-1}}+s^{\frac{q}{q-1}})^n}ds\\
&\geq \frac{q}{q-1}\beta^{\frac{(n-1)q}{q-1}}\int_0^\delta Vol_g(B(x_0,s))\frac{s^{\frac{1}{q-1}}}{(\lambda\beta^{\frac{q}{q-1}}+s^{\frac{q}{q-1}})^n}ds\\
&\geq \frac{q}{q-1}\beta^{\frac{(n-1)q}{q-1}}\int_0^\delta (1-\epsilon)V_0(s)\frac{s^{\frac{1}{q-1}}}{(\lambda\beta^{\frac{q}{q-1}}+s^{\frac{q}{q-1}})^n}ds \quad (set\ s=\beta\lambda^{\frac{q-1}{q}}t)\\
&= (1-\epsilon)\beta^n\frac{q}{q-1}\lambda^{1-\frac{n}{q}}\int_0^{\delta/(\beta\lambda^{\frac{q-1}{q}})}V_0(t)\frac{t^{\frac{1}{q-1}}}{(1+t^{\frac{q}{q-1}})^n}dt \Longrightarrow\\
\frac{F(\lambda)}{H_0(\lambda)} &\geq (1-\epsilon)\beta^n\lambda^{1-\frac{n}{q}}\frac{\int_0^{\delta/(\beta\lambda^{\frac{q}{q-1}})}V_0(t)t^{\frac{1}{q-1}}(1+t^{\frac{q}{q-1}})^{-n}dt}{\int_0^\infty b_n s^n s^{\frac{1}{q-1}}(\lambda+s^{\frac{q}{q-1}})^{-n}ds} \quad (set\ s=\lambda^{\frac{q-1}{q}}t)\\
&= (1-\epsilon)\beta^n\frac{\int_0^{\delta/(\beta\lambda^{\frac{q-1}{q}})}V_0(t)t^{\frac{1}{q-1}}(1+t^{\frac{q}{q-1}})^{-n}dt}{\int_0^\infty V_0(t)t^{\frac{1}{q-1}}(1+t^{\frac{q}{q-1}})^{-n}}dt \Longrightarrow \liminf_{\lambda\longrightarrow 0}\frac{F(\lambda)}{H_0(\lambda)} \geq (1-\epsilon)\beta^n
\end{aligned}$$

which proves (6.11), since $\beta > 1$. Now, (6.11) and (6.10) give

$$\liminf_{\lambda\longrightarrow 0}\frac{F(\lambda)}{H_1(\lambda)} = \Big(\frac{C_M}{K(n,q)}\Big)^n\liminf_{\lambda\longrightarrow 0}\frac{F(\lambda)}{H_0(\lambda)} > 1,$$

since, according to that we said above, $C_M \geq K(n,q)$. Thus, for small values of λ we have $F(\lambda) \geq H_1(\lambda)$. We will now prove that this estimate holds for all $\lambda > 0$. Specifically, we claim that

$$\exists\lambda_0 > 0 : F(\lambda_0) < H_1(\lambda_0) \Longrightarrow \forall\lambda \leq \lambda_0 : F(\lambda) < H_1(\lambda) \tag{6.13}$$

If this is not the case, let

$$\lambda_1 := \sup\{\lambda < \lambda_0 : F(\lambda) = H_0(\lambda)\}.$$

For every $\lambda > 0$, the function

$$\varphi_\lambda(X) := \delta X^{q/q^\star} + \lambda X, \quad X \geq 0,$$

is strictly increasing in $X \geq 0$ and so (6.3) reads as

$$\begin{aligned}
&d\big(-F'(\lambda)\big)^{q/q^\star} + \lambda\big(-F(\lambda)\big) \leq (n-1)F(\lambda) \Longrightarrow\\
&\varphi_\lambda\big(-F'(\lambda)\big) \leq (n-1)F(\lambda) \Longrightarrow -F'(\lambda) \leq \varphi_\lambda^{-1}\big((n-1)F(\lambda)\big).
\end{aligned}$$

Similarly,

$$-H'(\lambda) = \varphi_\lambda^{-1}\big((n-1)H(\lambda)\big). \tag{6.14}$$

On the set $\{\lambda > 0 : F(\lambda) \leq H_1(\lambda)\}$ we have

$$\varphi_\lambda^{-1}\big((n-1)F(\lambda)\big) \leq \varphi_\lambda^{-1}\big((n-1)H_1(\lambda)\big)$$

and so

$$\big(F(\lambda) - H_1(\lambda)\big)' = -\varphi_\lambda^{-1}\big((n-1)F(\lambda)\big) + \varphi_\lambda^{-1}\big((n-1)H_1(\lambda)\big) \geq 0.$$

But this leads to a contradiction, since

$$\begin{aligned} &F - H_1 \text{ is increasing on } \{\lambda > 0 : F(\lambda) \leq H_1(\lambda)\} \Longrightarrow \\ &F - H_1 \text{ is increasing on } [\lambda_1, \lambda_0] \Longrightarrow \\ &0 = F(\lambda_1) - H_1(\lambda_1) \leq F(\lambda_0) - H_1(\lambda_0) < 0. \end{aligned}$$

Hence, (6.11) and (6.13) give $F(\lambda) \geq H_1(\lambda),\ \forall \lambda > 0$ which, in turn, yields

$$\int_0^\infty \left(Vol_g(B(x_0, \beta t)) - \Big(\frac{K(n,q)}{C_M}\Big)^n V_0(t) \right) \frac{t^{\frac{1}{q-1}}}{(\lambda + t^{\frac{q}{q-1}})^n} dt \geq 0, \quad \forall \lambda > 0, \tag{6.15}$$

according to (6.2), (6.10) and (6.9). Now, let

$$L := \Big(\frac{K(n,q)}{C_M}\Big)^n, \quad L_\infty := \lim_{s \longrightarrow \infty} \frac{Vol_g(B(x_0, s))}{V_0(s)}.$$

By Bishop-Gromov's theorem, the function

$$\frac{Vol_g(B(x_0, s))}{V_0(s)}, \ s > 0$$

is decreasing with respect to s, and so, in order to prove

$$Vol_g(B(x_0, r)) \geq \Big(\frac{K(n,q)}{C_M}\Big)^n V_0(r), \quad \forall r > 0,$$

it is sufficient to prove that $L_\infty \geq L$. In contrary, assume that $L_\infty < L$, i.e there exist a positive constant ϵ_0 such that $L_\infty = L - \epsilon_0$. By the definition of L_∞ we have

$$(\exists \delta > 0)(\forall s \geq \delta) \ \ \frac{Vol_g(B(x_0, s))}{V_0(s)} \leq L - \frac{\epsilon_0}{2}. \tag{6.16}$$

Substituting (6.16) into (6.15) we obtain that, for all $\lambda > 0$,

$$\begin{aligned}
0 &\leq \int_0^\infty \left(Vol_g(B(x_0,\beta t)) - \Big(\frac{K(n,q)}{C_M}\Big)^n V_0(t) \right) \frac{t^{\frac{1}{q-1}}}{(\lambda + t^{\frac{q}{q-1}})^n} dt \\
&= \int_0^\infty \left(\frac{Vol_g(B(x_0,\beta t))}{V_0(t)} - L \right) V_0(t) \frac{t^{\frac{1}{q-1}}}{(\lambda + t^{\frac{q}{q-1}})^n} dt \\
&= b_n \int_0^\infty \frac{Vol_g(B(x_0,\beta t))}{V_0(t)} \frac{t^{n+\frac{1}{q-1}}}{(\lambda + t^{\frac{q}{q-1}})^n} dt - L b_n \int_0^\infty \frac{t^{n+\frac{1}{q-1}}}{(\lambda + t^{\frac{q}{q-1}})^n} dt \\
&= b_n \int_0^\delta \frac{Vol_g(B(x_0,\beta t))}{V_0(t)} \frac{t^{n+\frac{1}{q-1}}}{(\lambda + t^{\frac{q}{q-1}})^n} dt + b_n \int_\delta^\infty \frac{Vol_g(B(x_0,\beta t))}{V_0(t)} \frac{t^{n+\frac{1}{q-1}}}{(\lambda + t^{\frac{q}{q-1}})^n} dt \\
&\qquad - L b_n \int_0^\infty \frac{t^{n+\frac{1}{q-1}}}{(\lambda + t^{\frac{q}{q-1}})^n} dt \\
&\leq \int_0^\delta \frac{Vol_g(B(x_0,\beta t))}{V_0(t)} \frac{b_n t^{n+\frac{1}{q-1}}}{(\lambda + t^{\frac{q}{q-1}})^n} dt + \int_\delta^\infty \big(L - \frac{\epsilon_0}{2}\big) \frac{b_n t^{n+\frac{1}{q-1}}}{(\lambda + t^{\frac{q}{q-1}})^n} dt \\
&\qquad - L b_n \int_0^\infty \frac{t^{n+\frac{1}{q-1}}}{(\lambda + t^{\frac{q}{q-1}})^n} dt \\
&= \int_0^\delta \frac{Vol_g(B(x_0,\beta t))}{V_0(t)} \frac{b_n t^{n+\frac{1}{q-1}}}{(\lambda + t^{\frac{q}{q-1}})^n} dt + \int_0^\infty \big(L - \frac{\epsilon_0}{2}\big) \frac{b_n t^{n+\frac{1}{q-1}}}{(\lambda + t^{\frac{q}{q-1}})^n} dt \\
&\qquad - \int_0^\delta \big(L - \frac{\epsilon_0}{2}\big) \frac{b_n t^{n+\frac{1}{q-1}}}{(\lambda + t^{\frac{q}{q-1}})^n} dt - L b_n \int_0^\infty \frac{t^{n+\frac{1}{q-1}}}{(\lambda + t^{\frac{q}{q-1}})^n} dt \\
&= \int_0^\delta \left(\frac{Vol_g(B(x_0,t))}{V_0(t)} - L + \frac{\epsilon_0}{2} \right) \frac{b_n t^{n+\frac{1}{q-1}}}{(\lambda + t^{\frac{q}{q-1}})^n} dt - \frac{\epsilon_0}{2} \int_0^\infty \frac{b_n t^{n+\frac{1}{q-1}}}{(\lambda + t^{\frac{q}{q-1}})^n} dt.
\end{aligned}$$

Note that all these integrals converge. Using (6.9), (6.10) and (6.5), we may write

$$\int_0^\infty \frac{b_n t^{n+\frac{1}{q-1}}}{(\lambda + t^{\frac{q}{q-1}})^n} dt = \frac{q-1}{q} H_0(\lambda) = \frac{q-1}{q} \Big(\frac{C_M}{K(n,q)}\Big)^n H_1(\lambda) = \frac{q-1}{q} \Big(\frac{C_M}{K(n,q)}\Big)^n B \lambda^{1-\frac{n}{q}}.$$

As a consequence, for all $\lambda > 0$,

$$\begin{aligned}
0 &\leq \int_0^\delta \left(\frac{Vol_g(B(x_0,t))}{V_0(t)} - L - \frac{\epsilon_0}{2} \right) \frac{b_n t^{n+\frac{1}{q-1}}}{(\lambda + t^{\frac{q}{q-1}})^n} dt - \frac{\epsilon_0}{2} \frac{q}{q-1} \Big(\frac{C_M}{K(n,q)}\Big)^n B \lambda^{1-\frac{n}{q}} \\
&\leq \int_0^\delta \Big(1 - L + \frac{\epsilon_0}{2}\Big) \frac{b_n t^{n+\frac{1}{q-1}}}{\lambda^n} dt - \frac{\epsilon_0}{2} \frac{q}{q-1} \Big(\frac{C_M}{K(n,q)}\Big)^n B \lambda^{1-\frac{n}{q}} \\
&= \Big(1 - L + \frac{\epsilon_0}{2}\Big) \frac{b_n}{\lambda^n} \frac{\delta^{n+\frac{1}{q-1}+1}}{n + \frac{1}{q-1} + 1} - \frac{\epsilon_0}{2} \frac{q}{q-1} \Big(\frac{C_M}{K(n,q)}\Big)^n B \lambda^{1-\frac{n}{q}} \Longrightarrow \\
0 &\leq \Big(1 - L + \frac{\epsilon_0}{2}\Big) \frac{b_n}{\lambda^{n+\frac{n}{q}-1}} \frac{\delta^{n+\frac{1}{q-1}+1}}{n + \frac{1}{q-1} + 1} - \frac{\epsilon_0}{2} \frac{q}{q-1} \Big(\frac{C_M}{K(n,q)}\Big)^n B.
\end{aligned}$$

Letting $\lambda \longrightarrow \infty$, we find

$$0 \leq -\frac{\epsilon_0}{2}\frac{q-1}{q}\Big(\frac{C_M}{K(n,q)}\Big)^n B,$$

since $n + \frac{n}{q} - 1 > 0$. This concludes the proof of the left-hand side inequality in (6.1). The right-hand side inequality is a consequence of Bishop-Gromov's result (see theorem 3.3.1 (i)). □

6.2. If C_M is sufficiently close to $K(n,q)$, then M is diffeomorphic to $\mathbb{R}^n$

This result is due to Cheeger and Colding based on Xia's volume comparison theorem [20].

Theorem 6.2.1. *Given an integer $n \geq 2$ and $1 \leq q < n$, $q^\star := nq/(n-q)$, there exists a positive constant $\epsilon \equiv \epsilon(n,q)$ such that any $n-$dimensional complete Riemannian manifold (M,g) with non-negative Ricci curvature which satisfies*

$$\Big(\int_M |f|^{q^\star} d\mu_g\Big)^{1/q^\star} \leq \big(K(n,q)+\epsilon\big)\Big(\int_M |\nabla f|^q d\mu_g\Big)^{1/q}, \quad \forall f \in C_c^\infty(M)$$

is diffeomorphic to $\mathbb{R}^n$.

Proof. Theorem 6.2.1 is a consequence of theorems 6.1.1 and 6.0.6. □

6.3. If $C_M = K(n,q)$, then M is isometric to $\mathbb{R}^n$

This result is due to Ledoux [12]. Ledoux's proof is quite similar with those of Xia. Here, we will give a proof based on Xia's result [20].

Theorem 6.3.1. *Any $n-$dimensional Riemannian manifold (M,g) with non-negative Ricci curvature which satisfies*

$$\Big(\int_M |f|^{q^\star} d\mu_g\Big)^{1/q^\star} \leq K(n,q)\Big(\int_M |\nabla f|^q d\mu_g\Big)^{1/q}, \quad \forall f \in C_c^\infty(M),$$

for some $1 \leq q < n$, $q^\star := nq/(n-q)$, is isometric to $\mathbb{R}^n$.

Proof. Theorem 6.1.1, implies

$$Vol_g(B(x_0,r)) \geq V_0(r) \text{ for all } x_0 \in M,\ r > 0. \tag{6.17}$$

If we assume that

$$\exists p_0 \in M : \quad Ric_{(M,g)}(p_0) > 0,$$

then, by the continuity of the function $Ric_{(M,g)} : M \longrightarrow \mathbb{R}$, we have

$$\exists r_0 > 0 \text{ and } \exists c > 0 : \quad Ric_{(M,g)}(p) > cg(p) \text{ for all } p \in B(p_0, r_0).$$

But then, if $(\mathbb{S}^n, h)$ is the sphere of sectional curvature $c > 0$ and $B^{\mathbb{S}^n}(p_0, r_0)$ denotes the geodesic ball of center p_0 and radius r_0 on $(\mathbb{S}^n, h)$, by Bishop-gromov's theorem,

$$Vol_g(B(p_0, r_0)) \leq Vol_h(B^{\mathbb{S}^n}(p_0, r_0)) < V_0(r_0)$$

which contradicts with (6.17). Thus, $Ric_{(M,g)} \equiv 0$ and the result follows from Schur's theorem. □

Chapter 7

When the Ricci curvature is asymptotically non-negative

Continuing our study, we will now prove that for complete Riemannian manifolds supporting a Euclidean-type Sobolev inequality,

"if $C_M$is sufficiently close to $K(n,q)$,then M is diffeomorphic to $\mathbb{R}^n$",

still holds if the manifold has asymptotically non-negative Ricci curvature. For this case, we will use the following result of Zhu:

Theorem 7.0.2. (**Zhu**) *Let (M,g) be a complete Riemannian manifold of dimension $n \geq 3$, with asymptotically non-negative Ricci curvature, that is*

$$Ric_{(M,g)}(x) \geq -(n-1)G\big(d(x)\big), \quad \forall x \in M$$

for some non-negative function $G \in C^0\big([0,\infty)\big)$. Assume that G satisfies the integrability condition

$$\int_0^\infty tG(t)dt = b < +\infty$$

and that a Euclidean Sobolev inequality (5.11) holds on M for some $1 \leq q < n$. There exists an explicit positive constant $\omega_0(b)$ satisfying $\omega_0(b) \longrightarrow 1/2$ as $b \longrightarrow 0$ and such that, if $Vol_g\big(B(0,r)\big) \geq cV_0(r)$, $\forall r > 0$ for some $c \geq \omega_0(b)$, then M is diffeomorphic to $\mathbb{R}^n$.

As before, the essential result is the following volume comparison theorem.

7.1. A volume comparison theorem

First, we will prove an extension of Xia's result to complete manifolds with asymptotically non-negative Ricci curvature due to Pigola and Veronelli [15]. Having fixed a reference point $0 \in M$, $d(x)$ will denote the geodesic distance from 0 to $x \in M$.

Theorem 7.1.1. *Let (M,g) be a complete Riemannian manifold of dimension $n \geq 3$, with*

$$Ric_{(M,g)}(x) \geq -(n-1)G\big(d(x)\big), \quad \forall x \in M$$

for some non-negative function $G \in C^0\big([0,\infty)\big)$. Assume that G satisfies the integrability condition

$$\int_0^\infty tG(t)dt = b < +\infty$$

and that a Euclidean Sobolev inequality (5.11) holds on M for some $1 \le q < n$. Then,

$$e^{-(n-1)b}\left(\Big(\frac{C_M}{K(n,q)}\Big)^q + C_M^q C\right)^{-n/q} V_0(t) \le Vol_g(B(x,t)) \le e^{(n-1)b}\ V_0(t) \tag{7.1}$$

where

$$0 < C \equiv C(n,q,b) \longrightarrow 0,\ \ as\ b \longrightarrow 0.$$

Before proceeding to the proof of this theorem, it is necessary to give some preliminaries [16]. Let $h \in C^2\big([0.\infty)\big)$ be the unique solution of the initial value problem

$$\begin{cases} h''(s) - G(s)h(s) = 0, \quad s \ge 0 \\ h(0) = 0,\ h'(0) = 1 \end{cases} \tag{7.2}$$

Consider the $n-$dimensional model Riemannian manifold

$$M_h := \big(\mathbb{R} \times \mathbb{S}^{n-1},\ g_h := ds^2 + h^2(s)d\vartheta^2\big), \tag{7.3}$$

where $ds^2 := ds \otimes ds,\ d\vartheta^2 := d\vartheta \otimes d\vartheta$ are the standard metrics on $\mathbb{R}$ and $\mathbb{S}^{n-1}$ respectively. It's sectional curvature is given by

$$K_h(\sigma) = -h''(s)/h(s) = -G(s) \le 0,$$

for all two dimensional subspaces $\sigma \subseteq T_{(s,\vartheta)}M_h$ and it's Ricci curvature is given by

$$Ric_{(M_h,g_h)}(s,\vartheta) = -(n-1)G(s).$$

Furthermore, if v is a tangent vector with $\|v\| = 1$, $\{e_1, \dots, e_{n-1}, v\}$ is an orthonormal base for $T_{(s,\vartheta)}M_h$ and $\gamma(t) = exp_{(s,\vartheta)}(tv),\ t \in [0,l]$, then the Jacobi field Y_i along γ satisfying the initial conditions $Y_i(0) = 0,\ Y_i'(0) = e_i$ is given by

$$Y_i(s) = h(s)e_i,\ \text{for all } i = 1, 2, \dots, n-1. \tag{7.4}$$

We shall use an index h to denote quantities referred to M_h. Namely, we will denote by $B^h(0,t)$ and $\partial B^h(0,t)$ the geodesic ball and sphere of radius $t > 0$ in M_h. Our first preliminary result will be the following [16].

Lemma 7.1.2. *For all $t \ge 0$,*

$$Vol_{g_h}(B^h(t)) \ge V_0(t)$$

where $V_0(t) = b_n t^n$ stands for the volume of a Euclidean ball of radius t in $\mathbb{R}^n$.

Proof. We will first prove that $h(s) \geq s$, for all $s \geq 0$. Integrating the differential inequality satisfied by h we have, for all $s \geq 0$,

$$\begin{aligned}
&h''(s) - G(s)h(s) = 0 \Longrightarrow \\
&h'(s) = 1 + \int_0^s G(s)h(s)ds \Longrightarrow \\
&h'(s) > 0, \text{ on the set } \{s > 0 : h(s) > 0\},
\end{aligned}$$

since G is non-negative. From this we can conclude that $h(s) > 0$, $\forall s > 0$, since $h(0) = 0$, $h'(0) = 1 > 0$ and $h \in C^2([0.\infty))$. Let $\varphi(s) = s$, $s \geq 0$. We calculate

$$\big(h'(s)\varphi(s) - h(s)\varphi'(s)\big)' = h''(s)\varphi(s) - h(s)\varphi''(s) = sG(s)h(s) \geq 0,$$

for all $s \geq 0$. Integrate this inequality to find that, for all $s > 0$,

$$\begin{aligned}
&h'(s)\varphi(s) \geq h(s)\varphi(s) \Longrightarrow \frac{h'(s)}{h(s)} \geq \frac{1}{s} \Longrightarrow \\
&\int_\epsilon^s \frac{dh(s)}{h(s)} \geq \int_\epsilon^s \frac{ds}{s} \Longrightarrow \quad (0 < \epsilon < s < \infty) \\
&\log\left(\frac{h(s)}{h(\epsilon)}\right) \geq \log\left(\frac{s}{\epsilon}\right) \Longrightarrow \\
&\frac{s}{h(s)} \leq \frac{\epsilon}{h(\epsilon)} = \frac{\varphi(\epsilon) - \varphi(0)}{h(\epsilon) - h(0)} \longrightarrow \frac{\varphi'(0)}{h'(0)} = 1, \text{ as } \epsilon \longrightarrow 0^+.
\end{aligned}$$

According to (7.4) and theorem 3.2.2, for all $t > 0$, we have

$$\begin{aligned}
Vol_{g_h}(B^h(t)) &= \int_{\|\xi\|=1} \int_0^t (h(s))^{n-1} ds d\xi = b_n \int_0^t (h(s))^{n-1} ds \\
&\geq b_n \int_0^t s^{n-1} ds = \int_{\|\xi\|=1} \int_0^t s^{n-1} ds d\xi = V_0(t).
\end{aligned}$$

□

Our next preliminary result is to prove that a Euclidean type Sobolev inequality is sufficient to imply a lower bound for the volume growth of M [17].

Theorem 7.1.3. *Let (M, g) be a $n-$dimensional Riemannian manifold (not necessarily complete) satisfying*

$$\|f\|_{L^{q^\star}(M)} \leq C_M \|\nabla f\|_{L^q(M)} \text{ ,for any } f \in C_c^\infty(M)$$

for some $1 \leq q < n$, $q^\star := nq/(n-q)$ and $C_M \geq 0$. Then, for some positive constant $c \equiv c(n, q, C_M)$,

$$Vol_g(B(x, r)) \geq cV_0(r), \text{ for any } r > 0 \text{ and } x \in M.$$

where $V_0(r) := b_n r^n$ denotes the volume of a Euclidean ball of radius r in $\mathbb{R}^n$.

Proof. The proof is similar to the proof of theorem 5.4.1. Let $x \in M$ be chosen such that the function $d : M \longrightarrow \mathbb{R},\ y \longmapsto d(y) := d(y, \cdot)$ is differentiable at y and consider the function

$$f(y) := \begin{cases} r - d(y,x) & , y \in \overline{B(x,r)}, \\ 0 & , \text{otherwise} \end{cases}$$

Clearly, $f \in C_c^\infty(M)$ almost everywhere. Proceeding as in the proof of theorem 5.4.1, we find

$$\|f\|_{L^{q^\star}(M)} \geq \frac{r}{2}\Big(Vol_g(B(x,\frac{r}{2}))\Big)^{1/q^\star}$$
$$\|\nabla f\|_{L^q(M)} \leq \Big(Vol_g(B(x,r))\Big)^{1/q}$$

Use the Euclidean-type Sobolev inequality to find

$$Vol_g(B(x,t)) \geq \Big(\frac{r}{2C_M}\Big)^q \Big(Vol_g(B(x,\frac{r}{2}))\Big)^{q/q^\star}$$

and induction yields, for any $m \in \mathbb{N} \setminus \{0\},\ r > 0,\ x \in M$,

$$\begin{aligned} Vol_g(B(x,r)) &\geq (2C_M)^{-q^\star \sum_{j=1}^m (q/q^\star)^j} r^{q^\star \sum_{j=1}^m (q/q^\star)^j} 2^{q^\star \sum_{j=1}^m (j-1)(q/q^\star)^j} \Big(Vol_g(B(x,\frac{r}{2^m})\Big)^{(q/q^\star)^m} \\ &\geq \liminf_{m \longrightarrow \infty} \Big\{ (2C_M)^{-q^\star \sum_{j=1}^m (q/q^\star)^j} r^{q^\star \sum_{j=1}^m (q/q^\star)^j} 2^{q^\star \sum_{j=1}^m (j-1)(q/q^\star)^j} \Big(Vol_g(B(x,\frac{r}{2^m})\Big)^{(q/q^\star)^m} \Big\} \\ &\geq (2C_M)^{-n} r^n 2^{-n^2/q^\star} \liminf_{m \longrightarrow \infty} \Big(Vol_g(B(x,\frac{r}{2^m})\Big)^{(q/q^\star)^m} \\ &\geq \frac{r^n}{C_M^n 2^{n+n^2/q^\star}} := c(n,p,C_M)V_0(r). \end{aligned}$$

□

Laplacian comparison theorem states that

Theorem 7.1.4. *Let (M,g) be a complete Riemannian manifold of dimension n. Assume that the redial Ricci curvature satisfy*

$$Ric_{(M,g)}(\nabla d, \nabla d) \geq -(n-1)G(d),$$

for some non-negative function $G \in C^0([0,\infty))$ and let $h \in C^2([0,\infty))$ be a solution of the problem

$$\begin{cases} h''(s) - G(s)h(s) \geq 0, & s \geq 0 \\ h(0) = 0,\ h'(0) = 1 \end{cases}$$

Then, the inequality

$$\Delta d(x) \leq (n-1)\frac{h'(d(x))}{h(d(x))}$$

holds pointwise on $M \setminus \big(Cut(0) \cup \{0\}\big)$, and weakly on all of M.

Proof. For a detailed proof of this result see [16] theorem 2.4, p.32. □

The following result is a somewhat generalized version of Bishop-Gromov's result [16].

Theorem 7.1.5. *Let (M, g) be a complete Riemannian manifold of dimension n satisfying*

$$Ric_{(M,g)}(x) \geq -(n-1)G(d(x)), \quad x \in M$$

for some non-negative function $G \in C^0([0\infty))$. Also, let $h(s)$ be the non-negative solution of the problem

$$\begin{cases} h''(s) - G(s)h(s) = 0, & s \geq 0 \\ h(0) = 0, \ h'(0) = 1 \end{cases}$$

Then,

(i) For almost every $r > 1$, the function

$$\frac{Vol_g(\partial B(0,r))}{Vol_{g_h}(\partial B^h(0,r))}$$

is decreasing with respect to r,

(ii) For almost every $r > 0$, the function

$$\frac{Vol_g(B(0,r))}{Vol_{g_h}(B^h(0,r))}$$

is decreasing with respect to r,

(iii) For almost every $r > 1$, $Vol_g(\partial B(0,r)) \leq Vol_{g_h}(\partial B^h(0,r))$.

Proof. Applying Green's identity and Laplacian comparison theorem to a positive, smooth function φ with compact support, we get

$$\begin{aligned} -\int_M g(\nabla d(x), \nabla \varphi(x)) d\mu_g(x) &= \int_M \varphi(x) \Delta d(x) d\mu_g(x) \\ &\leq (n-1) \int_M \varphi \frac{h'(d(x))}{h(d(x))} d\mu_g(x). \end{aligned} \tag{7.5}$$

Let $\epsilon > 0$ and $0 < r \leq R$ be given. We define the radial function

$$\varphi_\epsilon(x) := \varrho_\epsilon(d(x))\big(h(d(x))\big)^{1-n}.$$

where ϱ_ϵ is given by

$$\varrho_\epsilon(t) := \begin{cases} 0, & 0 \leq t < r \\ (t-r)/\epsilon, & r \leq t < r+\epsilon \\ 1, & r+\epsilon \leq t < R-\epsilon \\ (R-t)/\epsilon, & R-\epsilon \leq t < R \\ 0, & R \leq t < \infty. \end{cases}$$

For almost all $x \in M$, we calculate

$$\begin{aligned}
\nabla\varphi_\epsilon(x) &= \varrho_\epsilon'(d(x))\nabla d(x)\big(h(d(x))\big)^{1-n} + \varrho_\epsilon(d(x))(1-n)\big(h(d(x))\big)^{-n}h'(d(x))\nabla d(x)\\
&= \left\{\varrho_\epsilon'(d(x)) - (n-1)\varrho_\epsilon(d(x))\frac{h'(d(x))}{h(d(x))}\right\}\big(h(d(x))\big)^{1-n}\nabla d(x)\\
&= \Bigg\{ -\frac{1}{\epsilon}\mathbb{1}_{[R-\epsilon,R)}(d(x)) + \frac{1}{\epsilon}\mathbb{1}_{[r,r+\epsilon)}(d(x)) +\\
&\qquad\qquad - (n-1)\varrho_\epsilon(d(x))\frac{h'(d(x))}{h(d(x))}\Bigg\}\big(h(d(x))\big)^{1-n}\nabla d(x)\\
&= \Bigg\{ -\frac{1}{\epsilon}\mathbb{1}_{B(0,R)\setminus B(0,R-\epsilon)}(x) + \frac{1}{\epsilon}\mathbb{1}_{B(0,r+\epsilon)\setminus B(0,r)}(x) +\\
&\qquad\qquad - (n-1)\varrho_\epsilon(d(x))\frac{h'(d(x))}{h(d(x))}\Bigg\}\big(h(d(x))\big)^{1-n}\nabla d(x).
\end{aligned}$$

Notice that φ_ϵ do not necessarily belongs in $C_c^\infty(M)$. But since it is Lipschitz on M, by a standard approximation argument, we can use φ_ϵ into (7.5) to get

$$\frac{1}{\epsilon}\int_{B(0,R)\setminus B(0,R-\epsilon)} \big(h(d(x))\big)^{1-n} d\mu_g(x) \leq \frac{1}{\epsilon}\int_{B(0,r+\epsilon)\setminus B(0,r)} \big(h(d(x))\big)^{1-n} d\mu_g(x),$$

since $g(\nabla d(x), \nabla d(x)) = |\nabla d(x)|^2 = 1$, almost everywhere in M. Now, coarea formula implies

$$\frac{1}{\epsilon}\int_{R-\epsilon}^{R} \big(h(t)\big)^{1-n} Area_g\big(\partial B(0,t)\big)dt \leq \frac{1}{\epsilon}\int_{r}^{r+\epsilon} \big(h(t)\big)^{1-n} Area_g\big(\partial B(0,t)\big)dt.$$

Letting $\epsilon \longrightarrow 0$, we obtain

$$\begin{aligned}
\big(h(R)\big)^{1-n} Area_g\big(\partial B(0,R)\big) &\leq \big(h(r)\big)^{1-n} Area_g\big(\partial B(0,r)\big) \Longrightarrow\\
\frac{Area_g\big(\partial B(0,R)\big)}{b_n h^{n-1}(R)} &\leq \frac{Area_g\big(\partial B(0,r)\big)}{b_n h^{n-1}(r)} \Longrightarrow\\
\frac{Area_g\big(\partial B(0,R)\big)}{Area_{g_h}\big(\partial B^h(0,R)\big)} &\leq \frac{Area_g\big(\partial B(0,r)\big)}{Area_{g_h}\big(\partial B^h(0,r)\big)},
\end{aligned}$$

which proves (i), while (iii) follows by letting $r \longrightarrow 0$ in the inequality above. Finally, (ii) is a consequence of (i) and lemma 3.3.2. □

The next preliminary result is also due to Pigola and Veronelli [16].

Lemma 7.1.6. *Maintaining the notation of the theorem 7.1.1, for any $t > 0$, we have*

$$\begin{aligned}
&(i)\ Area_g\big(\partial B(0,t)\big) \leq Area_g\big(\partial B^h(0,t)\big) \leq e^{b(n-1)}A_0(t),\\
&(ii)\ Vol_g\big(B(0,t)\big) \leq Vol_{g_h}\big(B^h(0,t)\big) \leq e^{b(n-1)}V_0(t)
\end{aligned}$$

where $A_0(t)$, $V_0(t)$ are the area of a Euclidean sphere and the volume of a Euclidean ball in $\mathbb{R}^n$.

Proof. Clearly, (ii) follows from (i) by integration. As for (i), according to what we said above, we just have to prove the right hand side inequality,

$$Area_{g_h}\big(\partial B^h(0,t)\big) \leq e^{b(n-1)} A_0(t), \;\; \forall t \geq 0.$$

Let

$$\psi(s) := \int_0^s e^{\int_0^r uG(u)du} dr, \quad s > 0.$$

Clearly, for all $s > 0$,

$$\psi(s) \leq \int_0^s e^{\int_0^s uG(u)du} dr = s e^{\int_0^s uG(u)du}.$$

Thus, ψ is a solution to the problem

$$\begin{cases} \psi''(s) - G(s)\psi(s) \geq 0 \\ \psi(0) = 0, \;\; \psi'(0) = 1 \end{cases}$$

while h satisfies

$$\begin{cases} h''(s) - G(s)h(s) = 0 \\ h(0) = 0, \;\; h'(0) = 1 \end{cases}$$

As a consequence, for all $s > 0$, we have

$$h(s) \leq \psi(s) \leq \int_0^s e^{\int_0^\infty uG(u)du} dr = \int_0^s e^b dr = s e^b \Longrightarrow h(s) \leq s e^b.$$

According to (7.4) and theorem 3.2.2, for all $t > 0$, we have

$$Area_{g_h}\big(\partial B^h(0,t)\big) = \int_0^t (h(s))^{n-1} ds \leq e^{b(n-1)} \int_0^s s^{n-1} ds = e^{b(n-1)} A_0(t).$$

□

With such results we are now in position to prove theorem 7.1.1.

Proof. For any $\lambda > 0$, we consider the functions

(1) $\varphi_\lambda(t) := \beta(n,q)\lambda^{\frac{n-q}{q^2}} / \big(\lambda + t^{\frac{q}{q-1}}\big)^{\frac{n}{q}-1}, \; t \geq 0.$

(2) $\phi_\lambda(x) := \varphi_\lambda(|x|), \; x \in \mathbb{R}^n.$

(3) $\widehat{\phi_\lambda}(x) := \varphi_\lambda(d(x)), \; x \in M.$

(4) $\phi_{\lambda,h}(s,\vartheta) := \varphi_\lambda(s), \; (s,\vartheta) \in M_h.$

Recall, from chapter 1, that the functions defined in (2) realize the best constant $C_{\mathbb{R}^n} = K(n,q)$ in the inequality (5.11), in $\mathbb{R}^n$. Namely,

$$\frac{1}{\big(K(n,q)\big)^q} = \inf_{f \in C_c^\infty(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla f|^q dx}{\Big(\int_{\mathbb{R}^n} |f|^{q^\star} dx\Big)^{q/q^\star}} = \frac{\int_{\mathbb{R}^n} |\nabla \phi_\lambda|^q dx}{\Big(\int_{\mathbb{R}^n} |\phi_\lambda|^{q^\star} dx\Big)^{q/q^\star}}.$$

Choosing the constant $\beta(n,q)$ such that $\|\phi_\lambda\|_{L^{q^\star}(\mathbb{R}^n)} = 1$, we obtain

$$\frac{1}{\big(K(n,q)\big)^q} = \int_{\mathbb{R}^n} |\varphi'_\lambda(|x|)|^q dx. \tag{7.6}$$

Coarea formula and lemma 7.1.6 imply

$$\begin{cases} (i) & \int_M \widehat{\phi_\lambda}^{q^\star} d\mu_g \le \int_{M_h} \phi_{\lambda,h}^{q^\star} d\mu_{g_h} \le e^{b(n-1)} \\ (ii) & |\nabla\widehat{\phi_\lambda}| \in L^q(M) \\ (iii) & \frac{1}{d(x)}\widehat{\phi_\lambda}|\nabla\widehat{\phi_\lambda}|^{q-1} \in L^1(M) \\ (iv) & \lim_{R\longrightarrow+\infty} \frac{1}{R}\int_{B(0,R)} \widehat{\phi_\lambda}|\nabla\widehat{\phi_\lambda}|^{q-1} d\mu_g = 0 \end{cases} \tag{7.7}$$

For (7.7) (i), we have

$$\begin{aligned} \int_M \widehat{\phi_\lambda}^{q^\star} d\mu_g &= \int_0^\infty \int_{\{x:d(x)=t\}} \big(\varphi_\lambda(d(x))\big)^{q^\star} dA(t)dt \\ &= \int_0^\infty \big(\varphi_\lambda(t)\big)^{q^\star} Area_g\big(\partial B(0,t)\big)dt \\ &\le \int_0^\infty \big(\varphi_\lambda(t)\big)^{q^\star} Area_{g_h}\big(\partial B^h(0,t)\big)dt \\ &= \int_0^\infty \int_{\{x:d(x)=t\}} \big(\varphi_\lambda(d(x))\big)^{q^\star} dA(t)dt \\ &= \int_{M_h} \phi_{\lambda,h}^{q^\star} d\mu_{g_h}, \end{aligned}$$

and

$$\begin{aligned} \int_{M_h} \phi_{\lambda,h}^{q^\star} d\mu_{g_h} &= \int_0^\infty \big(\varphi_\lambda(t)\big)^{q^\star} Area_{g_h}\big(\partial B^h(0,t)\big)dt \\ &\le \int_0^\infty \big(\varphi_\lambda(t)\big)^{q^\star} e^{b(n-1)} A_0(t)dt \\ &= \int_{\mathbb{R}^n} \big(\phi_\lambda(t)\big)^{q^\star} e^{b(n-1)}dt = e^{b(n-1)}. \end{aligned}$$

For (7.7) (ii), we calculate

$$\begin{aligned} \int_M |\nabla\widehat{\phi_\lambda}(x)|^q d\mu_g(x) &= \int_M |\varphi'_\lambda(d(x))\nabla d(x)| d\mu_g(x) \\ &= \int_M |\varphi'_\lambda(d(x))| d\mu_g(x) \\ &= \int_0^\infty \int_{\{x:d(x)=t\}} |\varphi'_\lambda(d(x))|^q dA(t)dt \\ &= \int_0^\infty |\varphi'_\lambda(t)|^q Area_g\big(\partial B(0,t)\big)dt \\ &\le \int_0^\infty |\varphi'_\lambda(t)|^q e^{b(n-1)} A_0(t)dt < \infty. \end{aligned}$$

As for (7.7) (iii), we have

$$\begin{aligned}\int_M \frac{1}{d(x)}\widehat{\phi_\lambda}(x)|\nabla\widehat{\phi_\lambda}(x)|^{q-1}d\mu_g(x) &= \int_0^\infty\int_{\{x:d(x)=t\}}\frac{1}{d(x)}\widehat{\phi_\lambda}(x)|\nabla\widehat{\phi_\lambda}(x)|^{q-1}dA(t)dt\\ &= \int_0^\infty \frac{1}{t}\varphi_\lambda(t)|\varphi'_\lambda(t)|^{q-1}|\nabla d|^{q-1}Area_g\big(\partial B(0,t)\big)dt\\ &\le \int_0^\infty \frac{1}{t}\varphi_\lambda(t)|\varphi'_\lambda(t)|^{q-1}e^{b(n-1)}A_0(t)dt<\infty.\end{aligned}$$

Finally, for (7.7) (iv), we have

$$\begin{aligned}&\lim_{R\longrightarrow 0}\frac{1}{R}\int_{B(0,t)}\widehat{\phi_\lambda}(x)|\nabla\widehat{\phi_\lambda}(x)|^{q-1}d\mu_g(x)=\\ &\lim_{R\longrightarrow 0}\frac{1}{R}\int_0^R\int_{\{x:d(x)=t\}}\varphi_\lambda(d(x))|\nabla\varphi_\lambda(d(x))|^{q-1}dA(t)dt=\\ &\lim_{R\longrightarrow 0}\frac{1}{R}\int_0^R\int_{\{x:d(x)=t\}}\varphi_\lambda(d(x))|\varphi'_\lambda(d(x))|^{q-1}|\nabla d|^{q-1}dA(t)dt=\\ &\lim_{R\longrightarrow 0}\frac{1}{R}\int_0^R\varphi_\lambda(t)|\varphi'_\lambda(t)|^{q-1}Area_g\big(\partial B(0,t)\big)dt=\\ &\lim_{R\longrightarrow 0}\frac{1}{R}\int_0^R\varphi_\lambda(t)|\varphi'_\lambda(t)|^{q-1}e^{b(n-1)}A_0(t)dt=\\ &C(n,b,q,\lambda)\lim_{R\longrightarrow 0}\frac{1}{R}\int_0^R\frac{t^n dt}{\big(\lambda+t^{\frac{q}{q-1}}\big)^{n-1}}\le\lim_{R\longrightarrow 0}\frac{1}{R}\int_0^R\frac{t^n dt}{t^{\frac{q(n-1)}{q-1}}}=\\ &\frac{C(n,b,q,\lambda)}{1+\frac{q-n}{q-1}}\lim_{R\longrightarrow 0}R^{\frac{q-n}{q-1}}=0.\end{aligned}$$

For any $0\le\xi\in H_1^2(M)$ (to be chosen later), let

$$\eta=-(\widehat{\xi\phi_\lambda})|\varphi'_\lambda(d)|^{q-2}\varphi'_\lambda(d).$$

Notice that $0\le\eta\in H_1^2(M)$ with compact support. Applying Green's identity and Laplacian comparison theorem (see theorem 7.1.4) we get

$$-\int_M g^{-1}(\nabla d,\nabla\eta)d\mu_g\le\int_M\eta\Delta d\,d\mu_g\le\int_M\eta\frac{e^{b(n-1)}}{d}d\mu_g. \tag{7.8}$$

Substituting η in (7.8) we obtain

$$\int_M -(\widehat{\xi\phi_\lambda})|\varphi_\lambda'(d)|^{q-2}\varphi_\lambda'(d)\frac{e^b(n-1)}{d}d\mu_g \geq$$
$$\int_M g^{-1}\Big(\nabla d, \nabla\Big((\widehat{\xi\phi_\lambda})|\varphi_\lambda'(d)|^{q-2}\varphi_\lambda'(d)\Big)\Big)d\mu_g =$$
$$\int_M g^{-1}\Big(\nabla d, |\varphi_\lambda'(d)|^{q-2}\varphi_\lambda'(d)\nabla(\widehat{\xi\phi_\lambda}) + (\widehat{\xi\phi_\lambda})\nabla(|\varphi_\lambda'(d)|^{q-2}\varphi_\lambda'(d))\Big) =$$
$$\int_M g^{-1}\Big(\nabla d, |\varphi_\lambda'(d)|^{q-2}\varphi_\lambda'(d)\nabla(\widehat{\xi\phi_\lambda})\Big)d\mu_g + \int_M g^{-1}\Big(\nabla d, (\widehat{\xi\phi_\lambda})\nabla(-|\varphi_\lambda'(d)|^{q-1})\Big)d\mu_g =$$
$$\int_M |\varphi_\lambda'(d)|^{q-2}\varphi_\lambda'(d)g^{-1}\Big(\nabla d, \nabla(\widehat{\xi\phi_\lambda})\Big)d\mu_g - \int_M (\widehat{\xi\phi_\lambda})g^{-1}\Big(\nabla d, (q-1)\varphi_\lambda''(d)\varphi_\lambda'(d)|\varphi_\lambda'(d)|^{q-3}\nabla d\Big)d\mu_g =$$
$$\int_M |\varphi_\lambda'(d)|^{q-2}\varphi_\lambda'(d)g^{-1}\Big(\nabla d, \nabla(\widehat{\xi\phi_\lambda})\Big)d\mu_g + (q-1)\int_M (\widehat{\xi\phi_\lambda})\varphi_\lambda''(d)|\varphi_\lambda'(d)|^{q-2}|\nabla d|^2 d\mu_g \Longrightarrow$$
$$\int_M |\varphi_\lambda'(d)|^{q-2}\varphi_\lambda'(d)g^{-1}\Big(\nabla d, \nabla(\widehat{\xi\phi_\lambda})\Big)d\mu_g \leq$$
$$-(q-1)\int_M (\widehat{\xi\phi_\lambda})\varphi_\lambda''(d)|\varphi_\lambda'(d)|^{q-2}d\mu_g - \int_M (\widehat{\xi\phi_\lambda})|\varphi_\lambda'(d)|^{q-2}\varphi_\lambda'(d)\frac{e^b(n-1)}{d}d\mu_g.$$

Using same arguments as in chapter 1, we can prove that the extremal functions ϕ_λ obey the non-linear Yamabe-type equation

$$-\Delta_q\phi_\lambda = \frac{1}{\big(K(n,q)\big)^q}\phi_\lambda^{q^\star-1},$$

where

$$\Delta_q u := div\big(|\nabla u|^{q-2}\nabla u\big)$$

stands for the $q-$Laplacian in $\mathbb{R}^n$ of a given function u. Substituting ϕ_λ into this differential equation we obtain

$$-|\varphi_\lambda'(t)|^{q-2}\Big\{(q-1)\varphi_\lambda''(t) + \frac{n-1}{t}\varphi_\lambda'(t)\Big\} = \frac{1}{\big(K(n,q)\big)^q}\varphi_\lambda^{q^\star-1}(t),$$

and so, from the inequality above, we have

$$\int_M |\varphi'_\lambda(d)|^{q-2}\varphi'_\lambda(d) g^{-1}\Big(\nabla d, \nabla(\xi\widehat{\phi_\lambda})\Big) d\mu_g \leq$$
$$\int_M (\xi\widehat{\phi_\lambda}) \frac{1}{\big(K(n,q)\big)^q} \widehat{\phi_\lambda}^{q^\star-1} d\mu_g + \int_M (\xi\widehat{\phi_\lambda}) \frac{n-1}{d} |\varphi'_\lambda(d)|^{q-2}\varphi'_\lambda(d) d\mu_g$$
$$- \int_M (\xi\widehat{\phi_\lambda}) |\varphi'_\lambda(d)|^{q-2}\varphi'_\lambda(d) \frac{e^b(n-1)}{d} d\mu_g$$
$$= \int_M (\xi\widehat{\phi_\lambda}) \frac{1}{\big(K(n,q)\big)^q} \widehat{\phi_\lambda}^{q^\star-1} d\mu_g + \int_M (\xi\widehat{\varphi_\lambda}) |\varphi'_\lambda(d)|^{q-2}\varphi'_\lambda(d) \frac{(n-1)(1-e^b)}{d} d\mu_g \Longrightarrow$$
$$- \int_M (\xi\widehat{\phi_\lambda}) \frac{1}{\big(K(n,q)\big)^q} \widehat{\phi_\lambda}^{q^\star-1} d\mu_g + \int_M (\xi\widehat{\varphi_\lambda}) |\varphi'_\lambda(d)|^{q-2}\varphi'_\lambda(d) \frac{(n-1)(e^b-1)}{d} d\mu_g$$
$$\leq - \int_M |\varphi'_\lambda(d)|^{q-2}\varphi'_\lambda(d) g^{-1}\Big(\nabla d, \nabla(\xi\widehat{\phi_\lambda})\Big) d\mu_g$$
$$= - \int_M |\varphi'_\lambda(d)|^{q-2}\varphi'_\lambda(d) g^{-1}\Big(\nabla d, \widehat{\phi_\lambda}\nabla\xi + \xi\varphi'_\lambda(d)\nabla d\Big) d\mu_g$$
$$= \int_M |\varphi'_\lambda(d)|^{q-1}\widehat{\phi_\lambda} g^{-1}\Big(\nabla d, \nabla\xi\Big) d\mu_g - \int_M |\varphi'_\lambda(d)|^q \xi |\nabla d|^2 d\mu_g$$
$$\leq \int_M |\varphi'_\lambda(d)|^{q-1}\widehat{\phi_\lambda} |\nabla d||\nabla\xi| d\mu_g - \int_M |\varphi'_\lambda(d)|^q \xi d\mu_g,$$

where in the last line we used Cauchy-Shwartz inequality. Finally, we proved that

$$- \int_M \xi \frac{1}{\big(K(n,q)\big)^q} \widehat{\phi_\lambda}^{q^\star} d\mu_g + \int_M |\nabla\widehat{\phi_\lambda}|^q \xi d\mu_g \leq$$
$$\int_M |\nabla\widehat{\phi_\lambda}|^{q-1}\widehat{\phi_\lambda}|\nabla\xi| d\mu_g + \int_M \xi\widehat{\varphi_\lambda}|\varphi'_\lambda(d)|^{q-1} \frac{(n-1)(e^b-1)}{d} d\mu_g. \tag{7.9}$$

Now, fix $R > 0$ and choose

$$\xi := \begin{cases} 1, & \text{on } B(0,R) \\ 0, & \text{on } M \setminus B(0,2R) \end{cases}$$

such that $|\nabla\xi| < 2/R$. Clearly, $\xi \longrightarrow 1$, as $R \longrightarrow +\infty$. According to (7.7) (iv), we have

$$\int_M |\nabla\widehat{\phi_\lambda}|^{q-1}\widehat{\phi_\lambda}|\nabla\xi| d\mu_g \leq \frac{2}{R}\int_{B(0,2R)} |\nabla\widehat{\phi_\lambda}|^{q-1}\widehat{\phi_\lambda} d\mu_g \longrightarrow 0 \ \text{ as } R \longrightarrow +\infty.$$

Thus, taking the limits as $R \longrightarrow +\infty$ in (7.9), we obtain

$$- \int_M \frac{1}{\big(K(n,q)\big)^q} \widehat{\phi_\lambda}^{q^\star} d\mu_g + \int_M |\nabla\widehat{\phi_\lambda}(d)|^q d\mu_g \leq \int_M \widehat{\varphi_\lambda}|\varphi'_\lambda(d)|^{q-1} \frac{(n-1)(e^b-1)}{d} d\mu_g =$$
$$(e^b-1)(n-1)\big(\beta(n,q)\big)^q \lambda^{\frac{n-q}{q}} \Big(\frac{n-q}{q-1}\Big)^{q-1} \int_M \frac{1}{\big(\lambda + d^{\frac{q}{q-1}}\big)^{n-1}} d\mu_g,$$

and so,

$$\frac{\int |\nabla\widehat{\phi_\lambda}|^q d\mu_g}{\int_M \widehat{\phi_\lambda}^{q^\star} d\mu_g} \leq \frac{1}{\big(K(n,q)\big)^q} + C(b,n,q,\lambda)\frac{\int_M \big(\lambda + d^{\frac{q}{q-1}}\big)^{1-n} d\mu_g}{\int_M \widehat{\phi_\lambda}^{q^\star} d\mu_g}, \tag{7.10}$$

where the constant $C(b,n,q,\lambda)$ above is given by

$$C(b,n,q,\lambda) := (e^b - 1)(n-1)\big(\beta(n,q)\big)^q \lambda^{\frac{n-q}{q}} \Big(\frac{n-q}{q-1}\Big)^{q-1}. \tag{7.11}$$

Let us now estimate the integrals in the right hand side of (7.10). Coarea formula and lemma 7.1.6 imply

$$\begin{aligned}
\int_M \big(\lambda + d^{\frac{q}{q-1}}\big)^{1-n} d\mu_g &= \int_0^\infty \int_{\{x:d(x)=t\}} \frac{dA(tdt)}{|\nabla d|\big(\lambda + d^{\frac{q}{q-1}}\big)^{n-1}} \\
&= \int_0^\infty \frac{Area_g\big(\partial B(0,t)\big)}{\big(\lambda + t^{\frac{q}{q-1}}\big)^{n-1}} dt \\
&\leq \int_0^\infty e^{b(n-1)} A_0(t) \frac{dt}{\big(\lambda + t^{\frac{q}{q-1}}\big)^{n-1}} \\
&= e^{b(n-1)} \int_0^\infty n b_n \frac{t^{n-1} dt}{\big(\lambda + t^{\frac{q}{q-1}}\big)^{n-1}} \\
&= e^{b(n-1)} A_0(1) \frac{\Gamma(n - \frac{n}{q})\Gamma(\frac{n}{q} - 1)}{\lambda^{\frac{n-q}{q}} \frac{q}{q-1}\Gamma(n-1)},
\end{aligned}$$

where Γ stands for the Euler Gamma function. On the other hand, Coarea formula and integration by parts imply

$$\begin{aligned}
\int_M \widehat{\phi_\lambda}^{q^\star} d\mu_g &= \int_0^\infty \int_{\{x:d(x)=t\}} \big(\varphi_\lambda(d)\big)^{q^\star} dA(t)dt \\
&= \int_0^\infty \big(\varphi_\lambda(d)\big)^{q^\star} Area_g\big(\partial B(0,t)\big) dt \\
&= \big(\beta(n,q)\big)^{q^\star} \lambda^{\frac{n-q}{q^2} q^\star} \int_0^\infty \frac{Area_g\big(\partial B(0,t)\big)}{\big(\lambda + t^{\frac{q}{q-1}}\big)^{(\frac{n}{q}-1)q^\star}} dt \\
&= \big(\beta(n,q)\big)^{\frac{nq}{n-q}} \lambda^{\frac{n}{q}} \int_0^\infty \frac{d\Big(Vol_g\big(B(0,t)\big)\Big)}{\big(\lambda + t^{\frac{q}{q-1}}\big)^n} \\
&= \big(\beta(n,q)\big)^{\frac{nq}{n-q}} \lambda^{\frac{n}{q}} \Bigg\{ \frac{Vol_g\big(B(0,t)\big)}{\big(\lambda + t^{\frac{q}{q-1}}\big)^n}\Bigg|_0^\infty - \int_0^\infty Vol_g\big(B(0,t)\big) d\Big(\big(\lambda + t^{\frac{q}{q-1}}\big)^{-n}\Big) \Bigg\}.
\end{aligned}$$

Notice that, by lemma 7.1.6,

$$\frac{Vol_g\big(B(0,t)\big)}{\big(\lambda + t^{\frac{q}{q-1}}\big)^n} \leq \frac{Vol_{g_h}\big(B^h(0,t)\big)}{\big(\lambda + t^{\frac{q}{q-1}}\big)^n} \leq \frac{e^{b(n-1)} V_0(t)}{\big(\lambda + t^{\frac{q}{q-1}}\big)^n} = e^{b(n-1)} \frac{b_n t^n}{\big(\lambda + t^{\frac{q}{q-1}}\big)^n} \longrightarrow 0, \tag{7.12}$$

as $t \longrightarrow +\infty$, since $nq/(q-1) > n$. Therefore,

$$\int_M \widehat{\phi_\lambda}^{q^\star} d\mu_g = \big(\beta(n,q)\big)^{\frac{nq}{n-q}} \lambda^{\frac{n}{q}} \int_0^\infty Vol_g\big(B(0,t)\big) d\Big(-\big(\lambda + t^{\frac{q}{q-1}}\big)^{-n}\Big).$$

Now, theorem 7.1.3 implies

$$\int_M \widehat{\phi_\lambda}^{q^\star} d\mu_g \geq \big(\beta(n,q)\big)^{\frac{nq}{n-q}} \lambda^{\frac{n}{q}} \int_0^\infty c(n,q,C_M) V_0(t) d\Big(-\big(\lambda + t^{\frac{q}{q-1}}\big)^{-n}\Big)$$

for some positive constant $c(n,q,C_M)$. Integrate by parts again and use once more (7.12) to find

$$\begin{aligned}\int_M \widehat{\phi_\lambda}^{q^\star} d\mu_g &\geq \big(\beta(n,q)\big)^{\frac{nq}{n-q}} \lambda^{\frac{n}{q}} c(n,q,C_M) \int_0^\infty A_0(t)\big(\lambda + t^{\frac{q}{q-1}}\big)^{-n} dt \\ &= \big(\beta(n,q)\big)^{\frac{nq}{n-q}} c(n,q,C_M) A_0(1) \frac{\Gamma(n-\frac{n}{q})\Gamma(\frac{n}{q})}{\frac{q}{q-1}\Gamma(n)}.\end{aligned}$$

Inserting these estimates into (7.10) and using the fact that $\Gamma(n) = (n-1)\Gamma(n-1)$ we get

$$\frac{\int |\nabla \widehat{\phi_\lambda}|^q d\mu_g}{\int_M \widehat{\phi_\lambda}^{q^\star} d\mu_g} \leq \frac{1}{\big(K(n,q)\big)^q} + C'(n,q,b,C_M)$$

where

$$C'(n,q,b,C_M) := \frac{(n-1)^2 q}{n-q} \Big(\frac{n-q}{q-1}\Big)^{q-1} \big(\beta(n,q)\big)^{-\frac{q^2}{n-q}} (e^b - 1) \frac{e^{b(n-1)}}{c(n,q,C_M)}. \tag{7.13}$$

On the other hand, because of (7.7), we can use $\widehat{\phi_\lambda}$ into the Euclidean type Sobolev inequality and get

$$\begin{aligned}&\Big(\int_M |\widehat{\phi_\lambda}|^{q^\star} d\mu_g\Big)^{q/q^\star} \leq C_M^q \int_M |\nabla \widehat{\phi_\lambda}|^q d\mu_g \Longrightarrow \\ &\frac{1}{C_M^q} \frac{\Big(\int_M |\widehat{\phi_\lambda}|^{q^\star} d\mu_g\Big)^{q/q^\star}}{\int_M |\widehat{\phi_\lambda}|^{q^\star} d\mu_g} \leq \frac{\int_M |\nabla \widehat{\phi_\lambda}|^q d\mu_g}{\int_M |\widehat{\phi_\lambda}|^{q^\star} d\mu_g} \leq \frac{1}{\big(K(n,q)\big)^q} + C'(n,q,b,C_M) \Longrightarrow \\ &\frac{1}{\Big(\int_M |\widehat{\phi_\lambda}|^{q^\star} d\mu_g\Big)^{1-\frac{q}{q^\star}}} \leq \Big(\frac{C_M}{K(n,q)}\Big)^q + C_M^q C'(n,q,b,C_M),\end{aligned}$$

which proves that

$$\int_M |\widehat{\phi_\lambda}|^{q^\star} d\mu_g \geq \frac{1}{C''(n,b,q,C_M)}, \tag{7.14}$$

where

$$C''(n,b,q,C_M) := \left(\Big(\frac{C_M}{K(n,q)}\Big)^q + C'(n,q,b,C_M) C_M^q\right)^{n/q}. \tag{7.15}$$

Now, by (7.14), (7.7) (i), Coarea formula, integration by parts and (7.12), we have

$$\begin{aligned}
0 &\leq C''(n,b,q,C_M)e^{b(n-1)}\int_M \widehat{\phi_\lambda}^{q^\star}d\mu_g - \int_{M_h}\phi_{\lambda,h}^{q^\star}d\mu_{g_h}\\
&= C''(n,b,q,C_M)e^{b(n-1)}\int_0^\infty \big(\varphi_\lambda(t)\big)^{q^\star}Area_g\big(\partial B(0,t)\big)dt - \int_0^\infty \big(\varphi_\lambda(t)\big)^{q^\star}Area_{g_h}\big(\partial B^h(0,t)\big)dt\\
&= C''(n,b,q,C_M)e^{b(n-1)}\int_0^\infty Vol_g\big(B(0,t)\big)d\big(-\varphi_\lambda^{q^\star}(t)\big) - \int_0^\infty Vol_{g_h}\big(B^h(0,t)\big)d\big(-\varphi_\lambda^{q^\star}(t)\big)\\
&= \int_0^\infty \Big(C''(n,b,q,C_M)e^{b(n-1)}Vol_g\big(B(0,t)\big) - Vol_{g_h}\big(B^h(0,t)\big)\Big)d\big(-\varphi_\lambda^{q^\star}(t)\big),
\end{aligned}$$

and so, it follows that

$$\int_0^\infty v_{M,h}(t)Vol_{g_h}\big(B^h(0,t)\big)\big(-\varphi_\lambda^{q^\star}(t)\big)'dt \geq 0, \tag{7.16}$$

where

$$v_{M,h}(t) := C''(n,b,q,C_M)e^{b(n-1)}\frac{Vol_g\big(B(0,t)\big)}{Vol_{g_h}\big(B^h(0,t)\big)} - 1, \quad t\geq 0. \tag{7.17}$$

Observe that, in order to prove the left hand side of (7.1), it is enough to show that

$$\lim_{t\longrightarrow+\infty} v_{M,h}(t) \geq 0.$$

Indeed; from theorem 7.1.5 (ii), the function $v_{M,h}$ is decreasing. Thus, for all $t\geq 0$,

$$\begin{aligned}
&v_{M,h}(t) \geq \lim_{t\longrightarrow+\infty} v_{M,h}(t) \geq 0 \Longrightarrow\\
&C''(n,b,q,C_M)e^{b(n-1)}\frac{Vol_g\big(B(0,t)\big)}{Vol_{g_h}\big(B^h(0,t)\big)} - 1 \geq 0 \Longrightarrow\\
&Vol_g\big(B(0,t)\big) \geq \frac{1}{C''(n,b,q,C_M)e^{b(n-1)}}Vol_{g_h}\big(B^h(0,t)\big),
\end{aligned}$$

and according to the lemma 7.1.2,

$$Vol_g\big(B(0,t)\big) \geq e^{-b(n-1)}\left(\Big(\frac{C_M}{K(n,q)}\Big)^q + C'C_M^q\right)^{-n/q}V_0(t).$$

where, in view of (7.13),

$$0 < C' \equiv C'(n,q,b,C_M) \longrightarrow 0, \quad \text{as } b\longrightarrow 0$$

By contradiction, assume that there exists two positive numbers ϵ and T such that

$$v_{M,h}(t) \leq -\epsilon, \quad \forall t > T.$$

Let $T_0 := \sup\{0 \leq t < T : v_{M,h}(t) \geq 0\}$. Clearly,

$$\begin{cases} (i) & v_{M,h}(t) \leq 0, \ \forall t \in [T_0, T) \\ (ii) & v_{M,h}(t) \leq v_{M,h}(0), \ \forall t \in [0, T_0) \\ (iii) & v(T_0) = 0 \\ (iv) & 0 < T_0 < T. \end{cases} \tag{7.18}$$

Then, for any $\lambda > 0$, from (7.16) and (7.18), we have

$$\begin{aligned} 0 &\leq \int_0^\infty v_{M,h}(t) Vol_{g_h}\big(B^h(0,t)\big)\big(-\varphi_\lambda^{q^\star}(t)\big)' dt \\ &= \int_0^{T_0} v_{M,h}(t) Vol_{g_h}\big(B^h(0,t)\big)\big(-\varphi_\lambda^{q^\star}(t)\big)' dt + \int_{T_0}^{T} v_{M,h}(t) Vol_{g_h}\big(B^h(0,t)\big)\big(-\varphi_\lambda^{q^\star}(t)\big)' dt \\ &\qquad + \int_T^\infty v_{M,h}(t) Vol_{g_h}\big(B^h(0,t)\big)\big(-\varphi_\lambda^{q^\star}(t)\big)' dt \\ &\leq \int_0^{T_0} v_{M,h}(0) Vol_{g_h}\big(B^h(0,t)\big)\big(-\varphi_\lambda^{q^\star}(t)\big)' dt + \int_T^\infty -\epsilon Vol_{g_h}\big(B^h(0,t)\big)\big(-\varphi_\lambda^{q^\star}(t)\big)' dt \Longrightarrow \end{aligned}$$

$$\epsilon \int_T^\infty Vol_{g_h}\big(B^h(0,t)\big)\big(-\varphi_\lambda^{q^\star}(t)\big)' dt \leq v_{M,h}(0) \int_0^{T_0} Vol_{g_h}\big(B^h(0,t)\big)\big(-\varphi_\lambda^{q^\star}(t)\big)' dt. \tag{7.19}$$

Note that the function

$$V_0(t)\big(-\varphi_\lambda(t)\big)' = b_n \frac{nq}{q-1}\big(\beta(n,q)\big)^{q^\star} \lambda^{n/q} \frac{t^{n+\frac{1}{q-1}}}{\big(\lambda + t^{\frac{q}{q-1}}\big)^{n+1}}$$

is decreasing with respect to λ, provided $\lambda \gg 1$. Applying lemma 7.1.6 and dominated convergence theorem we deduce

$$\begin{aligned} &\lim_{\lambda \longrightarrow +\infty} \int_0^{T_0} Vol_{g_h}\big(B^h(0,t)\big)\big(-\varphi_\lambda^{q^\star}(t)\big)' dt \leq \\ &\lim_{\lambda \longrightarrow +\infty} \int_0^{T_0} e^{b(n-1)} V_0(t)\big(-\varphi_\lambda^{q^\star}(t)\big)' dt = \\ &\int_0^{T_0} e^{b(n-1)} V_0(t) \lim_{\lambda \longrightarrow +\infty} \big(-\varphi_\lambda^{q^\star}(t)\big)' dt = \\ &e^{b(n-1)} b_n \frac{nq}{q-1}\big(\beta(n,q)\big)^{q^\star} \int_0^{T_0} t^{n+\frac{1}{q-1}} \lim_{\lambda \longrightarrow +\infty} \frac{\lambda^{n/q}}{\big(\lambda + t^{\frac{q}{q-1}}\big)^{n+1}} dt = 0 \Longrightarrow \\ &\lim_{\lambda \longrightarrow +\infty} \int_0^{T_0} Vol_{g_h}\big(B^h(0,t)\big)\big(-\varphi_\lambda^{q^\star}(t)\big)' dt = 0. \end{aligned} \tag{7.20}$$

Furthermore, using lemma 7.1.2, integration by parts, (7.12) and Coarea formula, we get

$$\begin{aligned} &\int_0^\infty Vol_{g_h}\big(B^h(0,t)\big)\big(-\varphi_\lambda(t)\big)' dt \geq \int_0^\infty V_0(t)\big(-\varphi_\lambda(t)\big)' dt = \\ &\int_0^\infty \varphi_\lambda(t) A_0(t) dt = \int_{\mathbb{R}^n} \phi_\lambda^{q^\star}(x) dx = \|\phi_\lambda\|_{L^{q^\star}(\mathbb{R}^n)}^{q^\star} = 1, \end{aligned}$$

and so, letting $\lambda \longrightarrow +\infty$ and using (7.20),

$$\lim_{\lambda \longrightarrow +\infty} \int_0^\infty Vol_{g_h}\big(B^h(0,t)\big)\big(-\varphi_\lambda(t)\big)' dt \geq 1 \Longrightarrow$$
$$\lim_{\lambda \longrightarrow +\infty} \int_T^\infty Vol_{g_h}\big(B^h(0,t)\big)\big(-\varphi_\lambda(t)\big)' dt \geq 1. \tag{7.21}$$

Inserting (7.20) and (7.21) into (7.19) we obtain $\epsilon < 0$, a contradiction. Thus, we have proven the validity of the left hand side of (7.1). Lemma 7.1.6 concludes the proof. □

7.2. If C_M is sufficiently close to $K(n,q)$, then M is diffeomorphic to $\mathbb{R}^n$

This result is due to Zhu based on the result of Pigola and Veronelli [15].

Theorem 7.2.1. *Let (M,g) be an $n-$dimensional complete Riemannian manifold supporting a Euclidean type Sobolev inequality*

$$\left(\int_M |f|^{q^\star} d\mu_g\right)^{1/q^\star} \leq \big(K(n,q)+\epsilon\big)\left(\int_M |\nabla f|^q d\mu_g\right)^{1/q}, \quad \forall f \in C_c^\infty(M)$$

for some $0 < \epsilon \leq \epsilon_0$ and such that

$$Ric_{(M,g)} \geq -(n-1)G(d(x)), \ \text{on } M$$

where $G \in C^0([0,\infty))$ is a non-negative function which satisfy the integrability condition

$$\int_0^\infty tG(t)dt = b < +\infty$$

for some $0 < b \leq b_0$. Then, M is diffeomorphic to $\mathbb{R}^n$.

Proof. Theorem 7.2.1 is a consequence of theorems 7.0.2 and 7.1.1. □

Bibliography

[1] T. Aubin, *Problemes isoperimetriques et espaces de Sobolev,* Journal Differential Geometry p.573-598, 1976.

[2] P. Buser, *A note on the isoperimetric constant,* Ann. Sci. Ecole Norm. Sup. (4) 15, no. 2, 213-230, 1982.

[3] I. Chavel, *Riemannian geometry, A modern introduction,* Cambridge University Press, 1993.

[4] I. Chavel, *Isoperimetric Inequalities Differential Geometric and Analytic Perspectives,* Cambridge University Press, 2001.

[5] J. Cheeger and T. Colding, *On the structure of spaces with Ricci curvature bounded below,* J. Differential Geom. 46, p.406-480. 1997.

[6] L. C. Evans , *Partial Differential Equations,* Second Edition, Graduate Studies in Mathematics, 2000.

[7] Gallot, Hulin and Lafontaine, *Riemannian geometry,* 2nd edition, Universitext, Springer-Verlag, 1993.

[8] E. Hebey, *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities,* Courant Lecture Notes, 2000.

[9] E. Hebey, *Sobolev Space on Riemannian Manifolds,* Lecture Notes in Mathematics, Springer 1996.

[10] D. Hoffman and J. Spruck, *Sobolev and isoperimetric inequalities for Riemannian submanifolds,* Comm. Pure Appl. Math. 27: 715-727, 1974.

[11] J. Jost, *Riemannian geometry and geometric analysis,* Universitext, Springer, Berlin-New York, 1995.

[12] M. Ledoux, *On manifolds with nonnegative Ricci curvature and Sobolev inequalities,* Comm. Anal. Geom. 7, p.347-353. 1999.

[13] V. Mazya, *Lectures on Isoperimetric and Isocapacitary Inequalities in the Theory of Sobolev Spaces.* Contemporary Mathematics, 338, Heat Kernels and Analysis on Manifolds, Graphs, and Metric Spaces, American Math. Society, p. 307?340. 2003.

[14] P. Petersen, *Riemannian Geometry,* First edition. Springer-Verlag, 1998.

[15] S. Pigola, G. Veronelli, *Lower volume estimates and Sobolev inequalities,* Proc.

Amer. Math. Soc. 138, p. 4479-4486, 2010.

[16] S.Pigola, M. Rigoli and A. Setti, *Vanishing and Finiteness Results in Geometric Analysis, A Generalization of the Bochner Technique,* 2010.

[17] L. Saloff-Coste, *Aspects of Sobolev-Type Inequalities,* Cambridge University Press 2001.

[18] G. Talenti, *Best Constant in Sobolev Inequality,* Ann. Mat. Pura Appl. 110, p.353?372, 1976.

[19] G. Talenti, *Inequalities in rearrangement invariant function spaces,* Nonlinear Analysis, Function Spaces and Applications, Vol. 5, Prometheus, Praha. pp. 177-230 1995.

[20] C. Xia, *Complete manifolds with nonnegative Ricci curvature and almost best Sobolev constant,* Illinois J. Math. 45, 2001.

[21] Qi S. Zhang, *Sobolev Inequalities, Heat Kernels under Ricci Flow and the Poincare Conjecture,* CRC Press, 2011.

[22] S.H. Zhu, *A volume comparison theorem for manifolds with asymptotically nonnegative curvature and its applications,* Amer. J. Math. 116, p.669-682 1994.