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Elliptic Functions and Transcendence

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Abstract Transcendental numbers form a fascinating subject: so little is known about the nature of analytic constants that more research is needed in this area. Even if one is interested only in numbers related to the classical exponential function, like π and e^{π}, one finds that elliptic functions are required to prove transcendence results and get a better understanding of the situation.

We first briefly recall some of the basic transcendence results related to the exponential function (section 1). Next in section 2 we survey the main properties of elliptic functions that are involved in transcendence theory.

We survey transcendence theory of values of elliptic functions in section 3, linear independence in section 4 and algebraic independence in section 5. This splitting is somewhat artificial but convenient. Also we restrict ourselves to elliptic functions, even when many results are only special cases of statements valid for abelian functions. A number of related topics are not considered here (e.g. heights, p-adic theory, theta functions, diophantine geometry on elliptic curves,. . .).

Keywords Transcendental numbers, elliptic functions, elliptic curves, elliptic integrals, algebraic independence, transcendence measures, measures of algebraic independence, Diophantine approximation

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1 Exponential Function and Transcendance

We start with a very brief list of some of the main transcendence results concerning numbers related to the exponential function. References are for instance [183, 80, 180, 106, 166, 12, 202, 75].

Next we point out some properties of the exponential function, the elliptic analog of which we shall later consider (§ 2.1).

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1.1 Short survey on the transcendence of numbers related to the exponential function

1.1.1 Hermite, Lindemann and Weierstraß

The first transcendence result for a number related to the exponential function is Hermite's Theorem on the transcendence of e.

Theorem 1 (Hermite, 1873)) *The number e is transcendental.*

This means that for any non-zero polynomial $P \in \mathbb{Z}[X]$, the number $P(e)$ is not zero. We denote by $\overline{\mathbb{Q}}$ the set of algebraic numbers. Hence Hermite's Theorem can be written: $e \notin \overline{\mathbb{Q}}$. A complex number is called *transcendental* if it is transcendental over $\mathbb{Q}$, or over $\overline{\mathbb{Q}}$, this is the same. Also we shall say that complex numbers are *algebraically independent* if they are algebraically independent over $\mathbb{Q}$, which is the same as over $\overline{\mathbb{Q}}$.

The second result in chronological order is Lindemann's Theorem on the transcendence of π.

Theorem 2 (Lindemann, 1882) *The number π is transcendental.*

The next result contains the transcendence of both numbers e and π:

Theorem 3 (Hermite-Lindemann, 1882) *For $\alpha \in \overline{\mathbb{Q}}^\times$, any non-zero logarithm $\log \alpha$ of α is transcendental.*

We denote by $\mathcal{L}$ the $\mathbb{Q}$-vector space of logarithms of algebraic numbers:

$$\mathcal{L} = \left\{\log \alpha \; ; \; \alpha \in \overline{\mathbb{Q}}^\times\right\} = \left\{\ell \in \mathbb{C} \; ; \; e^\ell \in \overline{\mathbb{Q}}^\times\right\} = \exp^{-1}(\overline{\mathbb{Q}}^\times).$$

Hence Theorem 3 means $\mathcal{L} \cap \overline{\mathbb{Q}} = \{0\}$. An equivalent statement is:

Theorem 4 (Hermite-Lindemann, 1882) *For any $\beta \in \overline{\mathbb{Q}}^\times$, the number e^β is transcendental.*

The first result of algebraic independence for the values of the exponential function goes back to the end of XIXth century.

Theorem 5 (Lindemann-Weierstraß, 1885) *Let $\beta_1, \ldots, \beta_n$ be $\mathbb{Q}$-linearly independent algebraic numbers. Then the numbers $e^{\beta_1}, \ldots, e^{\beta_n}$ are algebraically independent.*

Again, there is an equivalent way of stating Theorem 5: it amounts to a statement of linear independence.

Theorem 6 (Lindemann-Weierstraß, 1885) *Let $\gamma_1, \ldots, \gamma_m$ be pairwise distinct algebraic numbers. Then the numbers $e^{\gamma_1}, \ldots, e^{\gamma_m}$ are linearly independent over $\overline{\mathbb{Q}}$.*

It is not difficult to check that Theorem 6 is equivalent to Theorem 5 with the conclusion that $e^{\beta_1}, \ldots, e^{\beta_n}$ are algebraically independent over $\overline{\mathbb{Q}}$; since it is equivalent to say that $e^{\beta_1}, \ldots, e^{\beta_n}$ are algebraically independent over $\mathbb{Q}$, one does not lose anything if one changes the conclusion of Theorem 6 by stating that the numbers $e^{\gamma_1}, \ldots, e^{\gamma_m}$ are linearly independent over $\mathbb{Q}$.

1.1.2 Hilbert's seventh problem, Gel'fond and Schneider

The solution of Hilbert's seventh problem on the transcendence of α^β has been obtained by Gel'fond and Schneider in 1934 (see [80, 180]).

Theorem 7 (Gel'fond-Schneider, 1934) *For α and β algebraic numbers with $\alpha \neq 0$ and $\beta \notin \mathbb{Q}$ and for any choice of $\log\alpha \neq 0$, the number $\alpha^\beta = \exp(\beta\log\alpha)$ is transcendental.*

This means that the two algebraically independent functions e^z and $e^{\beta z}$ cannot take algebraic values at the point $\log\alpha$ (A.O. Gel'fond) and also that the two algebraically independent functions z and $\alpha^z = e^{z\log\alpha}$ cannot take algebraic values at the point $m + n\beta$ with $(m, n) \in \mathbb{Z}^2$ (Th. Schneider).

Examples, quoted by D. Hilbert in 1900, of numbers whose transcendence follow from Theorem 7 are $2^{\sqrt{2}}$ and e^π (recall $e^{i\pi} = -1$). The transcendence of e^π had been proved already in 1929 by A.O. Gel'fond.

Here is an equivalent statement to Gel'fond-Schneider Theorem 7:

Theorem 8 (Gel'fond-Schneider, 1934) *Let $\log\alpha_1, \log\alpha_2$ be two non-zero logarithms of algebraic numbers. Assume that the quotient $(\log\alpha_1)/(\log\alpha_2)$ is irrational. Then this quotient is transcendental.*

1.1.3 Linear independence of logarithms of algebraic numbers

The generalization of Theorem 8 to more than two logarithms, conjectured by A.O. Gel'fond [80], has been proved by A. Baker in 1966. His results includes not only Gel'fond-Scheider's Theorem 8 but also Hermite-Lindemann's Theorem 3.

Theorem 9 (Baker, 1966) *Let $\log\alpha_1, \dots, \log\alpha_n$ be $\mathbb{Q}$–linearly independent logarithms of algebraic numbers. Then the numbers $1, \log\alpha_1, \dots, \log\alpha_n$ are linearly independent over the field $\overline{\mathbb{Q}}$.*

1.1.4 The six exponentials Theorem and the four exponentials conjecture

The next result, which does not follow from any of the previously mentioned results, has been proved independently in the 60's by C.L. Siegel, A. Selberg, S. Lang and K. Ramachandra (see for instance [106, 165, 166, 207]; see also Problem 1 in [180] for the four exponentials conjecture).

Theorem 10 (Six exponentials Theorem) *Let $x_1, \dots, x_d$ be $\mathbb{Q}$-linearly independent complex numbers and let $y_1, \dots, y_\ell$ be $\mathbb{Q}$-linearly independent complex numbers. Assume $\ell d > \ell + d$. Then one at least of the ℓd numbers*

$$e^{x_i y_j}, \qquad (1 \le i \le d,\ 1 \le j \le \ell)$$

is transcendental.

Notice that the condition $\ell d > \ell + d$ can be written ($\ell \ge 2$ and $d \ge 3$) or ($\ell \ge 3$ and $d \ge 2$); it suffices to consider the case $\ell d = 6$ (hence the name of the result). Therefore Theorem 10 can be stated in another equivalent form:

Theorem 11 (Six exponentials Theorem - logarithmic form) *Let*

$$M = \begin{pmatrix} \log\alpha_1 & \log\alpha_2 & \log\alpha_3 \\ \log\beta_1 & \log\beta_2 & \log\beta_3 \end{pmatrix}$$

be a 2 by 3 matrix whose entries are logarithms of algebraic numbers. Assume the three columns are linearly independent over $\mathbb{Q}$ and the two rows are also linearly independent over $\mathbb{Q}$. Then the matrix M has rank 2.

It is expected that the condition $d\ell > d + \ell$ in Theorem 10 is too restrictive and that the same conclusion holds in case $d = \ell = 2$. We state this conjecture in the logarithmic form:

Conjecture 12 (Four exponentials conjecture - logarithmic form) Let

$$M = \begin{pmatrix} \log\alpha_1 & \log\alpha_2 \\ \log\beta_1 & \log\beta_2 \end{pmatrix}$$

be a 2 by 2 matrix whose entries are logarithms of algebraic numbers. Assume the two columns are linearly independent over $\mathbb{Q}$ and the two rows are also linearly independent over $\mathbb{Q}$. Then the matrix M has rank 2.

1.1.5 Algebraic independence

In 1949 A.O. Gel'fond extended his solution of Hilbert's seventh problem to a result of algebraic independence [80]. One of his theorems is that the two numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$ are algebraically independent. His general statements can be seen as extensions of Theorem 10 into a result of algebraic independence (in spite of the fact that the six exponentials Theorem 10 was stated and proved only several years later). In his original work Gel'fond needed a stronger assumption, namely a measure of linear independence of the x_i's as well as of the y_j's. This assumption was removed in the early 70's by R. Tijdeman (see for instance [31, 204, 205, 211]).

Theorem 13 *Let $x_1, \dots, x_d$ be $\mathbb{Q}$-linearly independent complex numbers and let $y_1, \dots, y_\ell$ be $\mathbb{Q}$-linearly independent complex numbers.*
1. If $d\ell \geq 2(d+\ell)$, then two at least of the $d\ell$ numbers

$$e^{x_i y_j}, \qquad (1 \leq i \leq d,\ 1 \leq j \leq \ell)$$

are algebraically independent.
2. If $d\ell \geq d + 2\ell$, then two at least of the $d\ell + d$ numbers

$$x_i,\ e^{x_i y_j}, \qquad (1 \leq i \leq d,\ 1 \leq j \leq \ell)$$

are algebraically independent.
3. If $d\ell > d + \ell$, then two at least of the $d\ell + d + \ell$ numbers

$$x_i,\ y_j,\ e^{x_i y_j}, \qquad (1 \leq i \leq d,\ 1 \leq j \leq \ell)$$

are algebraically independent.
4. If $d = \ell = 2$ and if the two numbers $e^{x_1 y_1}$ and $e^{x_1 y_2}$ are algebraic, then two at least of the 6 numbers

$$x_1,\ x_2,\ \ y_1,\ y_2,\ \ e^{x_2 y_1},\ \ e^{x_2 y_2}$$

are algebraically independent.

From the last part of Theorem 13, taking $x_1 = y_1 = i\pi$ and $x_2 = y_2 = 1$, one deduces that at least one of the two following statements is true:
(i) *The number e^{π^2} is transcendental.*
(ii) *The two numbers e and π are algebraically independent.*

One expects that both statements are true.

If it were possible to prove that, under the assumptions of Theorem 13, two at least of the 8 numbers

$$x_1,\ x_2,\ \ y_1,\ y_2,\ \ e^{x_1 y_1},\ \ e^{x_1 y_2},\ \ e^{x_2 y_1},\ \ e^{x_2 y_2}$$

are algebraically independent, one would deduce the algebraic independence or the two numbers π and e^π (take $x_1 = 1$, $x_2 = i$, $y_1 = \pi$, $y_2 = i\pi$; see Corollary 48 below).

For results concerning *large transcendence degree*, see § 5.3 below.

1.2 The Exponential function

The exponential function

$$\begin{aligned} \exp : \mathbb{C} &\to \mathbb{C}^\times \\ z &\mapsto e^z \end{aligned}$$

satisfies both a differential equation and an addition Theorem:

$$\frac{d}{dz}e^z = e^z, \qquad e^{z_1+z_2} = e^{z_1}e^{z_2}.$$

It is a homomorphism of the additive group $\mathbb{C}$ of complex numbers onto the multiplicative group $\mathbb{C}^\times$ of nonzero complex numbers, with kernel

$$\ker \exp = 2i\pi\mathbb{Z},$$

hence it yields an isomorphism between the quotient additive group $\mathbb{C}/2i\pi\mathbb{Z}$ and the multiplicative group $\mathbb{C}^\times$.

The group $\mathbb{C}^\times$ is the group of complex points of the multiplicative group $\mathbb{G}_m$; $z \mapsto e^z$ is the exponential function of the multiplicative group $\mathbb{G}_m$. We shall replace this algebraic group by an elliptic curve. We could replace it also by other commutative algebraic groups. For instance the exponential function of the additive group $\mathbb{G}_a$ is

$$\begin{aligned} \mathbb{C} &\to \mathbb{C} \\ z &\mapsto z \end{aligned}$$

Further examples are commutative linear algebraic groups (over an algebraically closed field, these are nothing else than products of several copies of the additive and multiplicative group), Abelian varieties and in full generality they are extensions of Abelian varieties by commutative linear algebraic groups. See for instance [106, 202, 137].

2 Elliptic curves and elliptic functions

Among many references for this section are the books by S. Lang [112], K. Chandrasekharan [39] and J. Silverman [184, 185]. See also the book by M. Hindry and J. Silverman [87].

2.1 Basic concepts

An elliptic curve may be defined as

- a connected compact Lie group of dimension 1,
- a complex torus $\mathbb{C}/\Omega$ where Ω is a lattice in $\mathbb{C}$,
- a Riemann surface of genus 1,
- a nonsingular cubic in $\mathbb{P}_2(\mathbb{C})$,
- an algebraic group of dimension 1, with underlying projective algebraic variety.

We shall use the Weierstraß form

$$E = \left\{(u : x : y)\ ;\ y^2u = 4x^3 - g_2xu^2 - g_3u^3\right\} \subset \mathbb{P}_2.$$

Here g_2 and g_3 are complex numbers, with the only assumption $g_2^3 \neq 27g_3^2$, which means that the discriminant of the polynomial $4X^3 - g_2X - g_3$ does not vanish.

A parametrization of the complex points $E(\mathbb{C})$ of E is given by means of *Weierstraß elliptic function* $\wp$, which satisfies the differential equation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3, \tag{1}$$

has a double pole at the origin with principal part $1/z^2$ and satisfies also an addition theorem

$$\wp(z_1+z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4}\cdot\left(\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right)^2.$$

The exponential map of the Lie group $E(\mathbb{C})$ is

$$\begin{array}{rcl}\exp_E : \mathbb{C} & \to & E(\mathbb{C}) \\ z & \mapsto & \bigl(1 : \wp(z) : \wp'(z)\bigr).\end{array}$$

The kernel of this map is a *lattice* in $\mathbb{C}$ (that means a discrete rank 2 subgroup),

$$\Omega = \ker \exp_E = \{\omega \in \mathbb{C}\ ;\ \wp(z+\omega) = \wp(z)\} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$

Hence $\exp_E$ induces an isomorphism between the quotient additive group $\mathbb{C}/\Omega$ and $E(\mathbb{C})$. The elements of Ω are the *periods* of $\wp$. A pair (ω_1, ω_2) of fundamental periods is given by

$$\omega_i = 2\int_{e_i}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad (i = 1, 2)$$

where

$$4t^3 - g_2 t - g_3 = 4(t-e_1)(t-e_2)(t-e_3).$$

Indeed, since $\wp'$ is periodic and odd, it vanishes at $\omega_1/2$, $\omega_2/2$ and $(\omega_1+\omega_2)/2$, hence the values of $\wp$ at these points are the three pairwise distinct complex numbers e_1, e_2 and e_3.

Conversely, given a lattice Ω, there is a unique Weierstraß elliptic function $\wp_\Omega$ whose period lattice is Ω (see § 2.5). We denote its invariants in the differential equation (1) by $g_2(\Omega)$ and $g_3(\Omega)$.

We shall be interested mainly (but not only) with elliptic curves which are defined over the field of algebraic numbers: they have a Weierstraß equation with algebraic g_2 and g_3. However we shall also use the Weierstraß elliptic function associated with the lattice $\lambda\Omega$ where $\lambda \in \mathbb{C}^\times$ may be transcendental; the relations are

$$\wp_{\lambda\Omega}(\lambda z) = \lambda^{-2}\wp_\Omega(z), \qquad g_2(\lambda\Omega) = \lambda^{-4} g_2(\Omega), \qquad g_3(\lambda\Omega) = \lambda^{-6} g_3(\Omega). \tag{2}$$

The lattice $\Omega = \mathbb{Z} + \mathbb{Z}\tau$, where τ is a complex number with positive imaginary part, has

$$g_2(\mathbb{Z}+\mathbb{Z}\tau) = 60 G_2(\tau) \quad \text{and} \quad g_3(\mathbb{Z}+\mathbb{Z}\tau) = 140 G_3(\tau),$$

where $G_k(\tau)$ are the Eisenstein series

$$G_k(\tau) = \tau^{2k} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m+n\tau)^{-2k}. \tag{3}$$

2.2 Morphisms between elliptic curves. The modular invariant

If Ω and Ω' are two lattices in $\mathbb{C}$ and if $f : \mathbb{C}/\Omega \to \mathbb{C}/\Omega'$ is an analytic homomorphism, then the map $\mathbb{C} \to \mathbb{C}/\Omega \to \mathbb{C}/\Omega'$ factors through a homothety $\mathbb{C} \to \mathbb{C}$ given by some $\lambda \in \mathbb{C}$ such that $\lambda\Omega \subset \Omega'$:

$$\begin{array}{ccc}\mathbb{C} & \xrightarrow{\lambda} & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C}/\Omega & \xrightarrow[f]{} & \mathbb{C}/\Omega'\end{array}$$

If $f \neq 0$, then $\lambda \in \mathbb{C}^\times$ and f is surjective.

Conversely, if there exists $\lambda \in \mathbb{C}$ such that $\lambda\Omega \subset \Omega'$, then $f_\lambda(x+\Omega) = \lambda x + \Omega'$ defines an analytic surjective homomorphism $f_\lambda : \mathbb{C}/\Omega \to \mathbb{C}/\Omega'$. In this case $\lambda\Omega$ is a subgroup of finite index in Ω', hence the kernel of f_λ is finite and there exists $\mu\in\mathbb{C}^\times$ with $\mu\Omega'\subset\Omega$: the two elliptic curves $\mathbb{C}/\Omega$ and $\mathbb{C}/\Omega'$ are *isogeneous*.

If Ω and Ω^* are two lattices, $\wp$ and $\wp^*$ the associated Weierstraß elliptic functions and g_2, g_3 the invariants of $\wp$, the following are equivalent:
(i) There is a square 2×2 matrix with rational coefficients which maps a basis of Ω to a basis of Ω^*.
(ii) There exists $\lambda\in\mathbb{Q}^\times$ such that $\lambda\Omega\subset\Omega^*$.
(iii) There exists $\lambda\in\mathbb{Z}\setminus\{0\}$ such that $\lambda\Omega\subset\Omega^*$.
(iv) The two functions $\wp$ and $\wp^*$ are algebraically dependent over the field $\mathbb{Q}(g_2,g_3)$.
(v) The two functions $\wp$ and $\wp^*$ are algebraically dependent over $\mathbb{C}$.

The map f_λ is an isomorphism if and only if $\lambda\Omega=\Omega'$.

The number
$$j=\frac{1728g_2^3}{g_2^3-27g_3^2}$$
is the *modular invariant* of the elliptic curve E. Two elliptic curves over $\mathbb{C}$ are isomorphic if and only if they have the same modular invariant.

Set $\tau=\omega_2/\omega_1$, $q=e^{2i\pi\tau}$ and $J(e^{2i\pi\tau})=j(\tau)$. Then
$$\begin{aligned}J(q)&=q^{-1}\left(1+240\sum_{m=1}^{\infty}m^3\frac{q^m}{1-q^m}\right)^3\prod_{n=1}^{\infty}(1-q^n)^{-24}\\&=\quad\frac{1}{q}+744+196884\,q+21493760\,q^2+\cdots\end{aligned}$$

2.3 Endomorphisms of an elliptic curve; complex multiplication

Let Ω be a lattice in $\mathbb{C}$. The set of analytic endomorphisms of $\mathbb{C}/\Omega$ is the subring
$$\mathrm{End}(\mathbb{C}/\Omega)=\bigl\{f_\lambda\;;\;\lambda\in\mathbb{C}\;\text{ with }\;\lambda\Omega\subset\Omega\bigr\}$$
of $\mathbb{C}$. We call it also the ring of endomorphisms of the associated elliptic curve, or of the corresponding Weierstraß $\wp$ function and we identify it with the subring
$$\bigl\{\lambda\in\mathbb{C}\;\;;\;\lambda\Omega\subset\Omega\bigr\}$$
of $\mathbb{C}$. The *field of endomorphisms* is the quotient field $\mathrm{End}(\mathbb{C}/\Omega)\otimes_{\mathbb{Z}}\mathbb{Q}$ of this ring.

If $\lambda\in\mathbb{C}$ satisfies $\lambda\Omega\subset\Omega$, then λ is either a rational integer or else an algebraic integer in an imaginary quadratic field. In this case $\wp_\Omega(\lambda z)$ is a rational function of $\wp_\Omega(z)$, such that the degree of the numerator is λ^2 if $\lambda\in\mathbb{Z}$ and $N(\lambda)$ otherwise (where N is the norm of the imaginary quadratic field); the degree of the denominator is λ^2-1 if $\lambda\in\mathbb{Z}$ and $N(\lambda)-1$ otherwise.

Let E be the elliptic curve attached to the Weierstraß $\wp$ function. The ring of endomorphisms $\mathrm{End}(E)$ of E is either $\mathbb{Z}$ or else an order in an imaginary quadratic field k. The latter case arises if and only if the quotient $\tau=\omega_2/\omega_1$ of a pair of fundamental periods is a quadratic number; in this case the field of endomorphisms of E is $k=\mathbb{Q}(\tau)$ and the curve E has *complex multiplication*. This means also that the two functions $\wp(z)$ and $\wp(\tau z)$ are algebraically dependent. In this case, the value $j(\tau)$ of the modular invariant j is an algebraic integer of degree the class number h of the quadratic field $k=\mathbb{Q}(\tau)$.

Remark 14 From Theorem 7 one deduces the transcendence of the number
$$e^{\pi\sqrt{163}}=262\,537\,412\,640\,768\,743.999\,999\,999\,999\,2\ldots.$$

If we set
$$\tau=\frac{1+i\sqrt{163}}{2},\quad q=e^{2i\pi\tau}=-e^{-\pi\sqrt{163}}$$

then the class number h of the imaginary quadratic field $\mathbb{Q}(\tau)$ is 1, we have $j(\tau) = -(640\,320)^3$ and

$$\left| j(\tau) - \frac{1}{q} - 744 \right| < 10^{-12}.$$

Let $\wp$ be a Weierstraß elliptic function with field of endomorphisms k. Hence $k = \mathbb{Q}$ if the associated elliptic curve has no complex multiplication, while k is an imaginary quadratic field otherwise, namely $k = \mathbb{Q}(\tau)$, where τ is the quotient of two linearly independent periods of $\wp$. Let $u_1, \ldots, u_d$ be nonzero complex numbers. Then the functions $\wp(u_1 z), \ldots, \wp(u_d z)$ are algebraically independent (over $\mathbb{C}$ or over $\mathbb{Q}(g_2, g_3)$, this is equivalent) if and only if the numbers $u_1, \ldots, u_d$ are linearly independent over k. This generalizes the fact that $\wp(z)$ and $\wp(\tau z)$ are algebraically dependent if and only if the elliptic curve has complex multiplication. Much more general and deeper results of algebraic independence of functions (exponential and elliptic functions, zeta functions,…) have been proved by W.D. Brownawell and K.K. Kubota [33].

If $\wp$ is a Weierstraß elliptic function with algebraic invariants g_2 and g_3, if E is the associated elliptic curve and if k denotes its field of endomorphisms, then the set

$$\mathcal{L}_E = \Omega \cup \left\{ u \in \mathbb{C} \setminus \Omega \; ; \; \wp(u) \in \overline{\mathbb{Q}} \right\}$$

is a k-vector subspace of $\mathbb{C}$: this is the set of *elliptic logarithms of algebraic points on E*. It plays a role with respect to E similar to the role of $\mathcal{L}$ for the multiplicative group $\mathbb{G}_m$.

Let $k = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field with class number $h(d) = h$. There are h non-isomorphic elliptic curves $E_1, \ldots, E_h$ with ring of endomorphisms the ring of integers of k. The numbers $j(E_i)$ are conjugate algebraic integers of degree h, each of them generates the Hilbert class field H of k (maximal unramified Abelian extension of k). The Galois group of H/k is isomorphic to the ideal class group of the ring of integers of k.

Since the group of roots of units of an imaginary quadratic field is $\{-1, +1\}$ except for $\mathbb{Q}(i)$ and $\mathbb{Q}(\varrho)$, where $\varrho = e^{2i\pi/3}$, it follows that there are exactly two elliptic curves over $\mathbb{Q}$ (up to isomorphism) having an automorphism group bigger than $\{-1, +1\}$. They correspond to Weierstraß elliptic functions $\wp$ for which there exists a complex number $\lambda \neq \pm 1$ with $\lambda^2 \wp(\lambda z) = \wp(z)$.

The first one has $g_3 = 0$ and $j = 1728$. A pair of fundamental periods of the elliptic curve

$$y^2 t = 4x^3 - 4xt^2.$$

is given by

$$\omega_1 = \int_1^\infty \frac{dt}{\sqrt{t^3 - t}} = \frac{1}{2} B(1/4, 1/2) = \frac{\Gamma(1/4)^2}{2^{3/2} \pi^{1/2}} \quad \text{and} \quad \omega_2 = i\omega_1. \tag{4}$$

The lattice $\mathbb{Z}[i]$ has $g_2 = 4\omega_1^4$, hence

$$\sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} (m + ni)^{-4} = \frac{\Gamma(1/4)^8}{2^6 \cdot 3 \cdot 5 \cdot \pi^2}.$$

The second one has $g_2 = 0$ and $j = 0$. A pair of fundamental periods of the elliptic curve

$$y^2 t = 4x^3 - 4t^3.$$

is

$$\omega_1 = \int_1^\infty \frac{dt}{\sqrt{t^3 - 1}} = \frac{1}{3} B(1/6, 1/2) = \frac{\Gamma(1/3)^3}{2^{4/3} \pi} \quad \text{and} \quad \omega_2 = \varrho\omega_1. \tag{5}$$

The lattice $\mathbb{Z}[\varrho]$ has $g_3 = 4\omega_1^6$, hence

$$\sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} (m + n\varrho)^{-6} = \frac{\Gamma(1/3)^{18}}{2^8 \pi^6}.$$

These two examples involve special values of Euler Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^z \cdot \frac{dt}{t} = e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}, \tag{6}$$

while Euler Beta function is

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1} dx.$$

More generally, the formula of Chowla and Selberg (1966) [40] (see also [85,103,102,86,203] for related results) expresses periods of elliptic curves with complex multiplication as products of Gamma values: *if k is an imaginary quadratic field and $\mathcal{O}$ an order in k, if E is an elliptic curve with complex multiplication by $\mathcal{O}$, then the corresponding lattice Ω determines a vector space $\Omega \otimes_{\mathbb{Z}} \mathbb{Q}$ which is invariant under the action of k and thus has the form $k \cdot \omega$ for some $\omega \in \mathbb{C}^\times$ defined up to elements in $k^\times$. In particular if $\mathcal{O}$ is the ring of integers $\mathbb{Z}_k$ of k, then*

$$\omega = \alpha \sqrt{\pi} \prod_{\substack{0<a<d \\ (a,d)=1}} \Gamma(a/d)^{w\epsilon(a)/4h},$$

where α is a nonzero algebraic number, w is the number of roots of unity in k, h is the class number of k, ϵ is the Dirichlet character modulo the discriminant d of k.

2.4 Standard relations among Gamma values

Euler Gamma function satisfies the following relations
(Translation):

$$\Gamma(z+1) = z\Gamma(z)$$

(Reflexion):

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

(Multiplication): For any positive integer n,

$$\prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{-nz+(1/2)} \Gamma(nz).$$

D. Rohrlich conjectured that any multiplicative relation among Gamma values is a consequence of these standard relations, while S. Lang was more optimistic (see for instance [212]).

Conjecture 15 (D. Rohrlich) Any multiplicative relation

$$\pi^{b/2} \prod_{a \in \mathbb{Q}} \Gamma(a)^{m_a} \in \overline{\mathbb{Q}}$$

with b and m_a in $\mathbb{Z}$ is a consequence of the standard relations.

Conjecture 16 (S. Lang) Any algebraic dependence relation among the numbers $(2\pi)^{-1/2}\Gamma(a)$ with $a \in \mathbb{Q}$ is in the ideal generated by the standard relations.

2.5 Quasiperiods of elliptic curves and elliptic integrals of the second kind

Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$. The *Weierstraß canonical product* attached to this lattice is the entire function σ_Ω defined by

$$\sigma_\Omega(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{z^2}{2\omega^2}}.$$

It has a simple zero at any point of Ω.

Hence Weierstraß sigma function plays, for the lattice Ω, the role of the function $e^{-\gamma z}\Gamma(-z)^{-1}$ for the natural integers $\mathbb{N} = \{0, 1, 2, \dots\}$ (see the infinite product (6) for Euler Gamma function), and also the role of the function

$$\pi^{-1}\sin(\pi z) = z \prod_{n \in \mathbb{Z}} \left(1 - \frac{z}{n}\right) e^{z/n}$$

for the set $\mathbb{Z}$ of rational integers.

The Weierstraß sigma function σ associated with a lattice in $\mathbb{C}$ is an entire function of *order* 2:

$$\limsup_{r \to \infty} \frac{1}{\log r} \cdot \log\log \sup_{|z|=r} |\sigma(z)| = 2;$$

the product $\sigma^2\wp$ is also an entire function of order 2 (this can be checked using infinite products, it is easier to use the quasiperiodicity of σ – see formula (7) below).

The logarithmic derivative of the sigma function is *Weierstraß zeta function* $\zeta = \sigma'/\sigma$ whose Laurent expansion at the origin is

$$\zeta(z) = \frac{1}{z} + \sum_{k \geq 2} s_k z^{2k-1},$$

where

$$s_k = s_k(\Omega) = \sum_{\substack{\omega \in \Omega \\ \omega \neq 0}} \omega^{-2k} = \omega_2^{-2k} G_k(\tau)$$

for $m \in \mathbb{Z}$, $m > 2$ (recall (3)).

The derivative of ζ is $-\wp$. From

$$\wp'' = 6\wp^2 - (g_2/2)$$

one deduces that $s_k(\Omega)$ is a homogenous polynomial in $\mathbb{Q}[g_2, g_3]$ of weight $2k$ when g_2 has weight 4 and g_3 weight 6.

As a side remark, we notice that for any $u \in \mathbb{C} \setminus \Omega$ we have

$$\mathbb{Q}(g_2, g_3) \subset \mathbb{Q}\big(\wp(u), \wp'(u), \wp''(u)\big).$$

Since its derivative is periodic, the function ζ is *quasiperiodic*: for each $\omega \in \Omega$ there is a $\eta = \eta(\omega)$ such that

$$\zeta(z + \omega) = \zeta(z) + \eta.$$

These numbers η are the *quasiperiods* of the elliptic curve. If (ω_1, ω_2) is a pair of fundamental periods and if we set $\eta_1 = \eta(\omega_1)$ and $\eta_2 = \eta(\omega_2)$, then, for $(a, b) \in \mathbb{Z}^2$,

$$\eta(a\omega_1 + b\omega_2) = a\eta_1 + b\eta_2.$$

Coming back to the sigma function, one deduces

$$\sigma(z + \omega_i) = -\sigma(z) \exp\Big(\eta_i\big(z + (\omega_i/2)\big)\Big) \qquad (i = 1, 2). \tag{7}$$

The zeta function also satisfies an addition Theorem:

$$\zeta(z_1+z_2)=\zeta(z_1)+\zeta(z_2)+\frac{1}{2}\cdot\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\cdot$$

Legendre relation relating the periods and the quasiperiods

$$\omega_2\eta_1-\omega_1\eta_2=\pm 2i\pi$$

can be obtained by integrating $\zeta(z)$ on the boundary of a fundamental parallelogram.

In the case of complex multiplication, if τ is the quotient of a pair of fundamental periods of $\wp$, then the function $\zeta(\tau z)$ is algebraic over the field $\mathbb{Q}\bigl(g_2,g_3,z,\wp(z),\zeta(z)\bigr)$.

Examples. For the curve $y^2t=4x^3-4xt^2$ the quasiperiods attached to the above mentioned pair of fundamental periods (4) are

$$\eta_1=\frac{\pi}{\omega_1}=\frac{(2\pi)^{3/2}}{\Gamma(1/4)^2},\qquad \eta_2=-i\eta_1; \tag{8}$$

it follows that the fields $\mathbb{Q}(\omega_1,\omega_2,\eta_1,\eta_2)$ and $\mathbb{Q}\bigl(\pi,\Gamma(1/4)\bigr)$ have the same algebraic closure over $\mathbb{Q}$, hence the same transcendence degree. For the curve $y^2t=4x^3-4t^3$ with periods (5) they are

$$\eta_1=\frac{2\pi}{\sqrt{3}\omega_1}=\frac{2^{7/3}\pi^2}{3^{1/2}\Gamma(1/3)^3},\qquad \eta_2=\varrho^2\eta_1. \tag{9}$$

In this case the fields $\mathbb{Q}(\omega_1,\omega_2,\eta_1,\eta_2)$ and $\mathbb{Q}\bigl(\pi,\Gamma(1/3)\bigr)$ have the same algebraic closure over $\mathbb{Q}$, hence the same transcendence degree.

2.6 Elliptic integrals

Let

$$\mathcal{E}=\{(x:y:t)\in\mathbb{P}_2; y^2t=4x^3-g_2xt^2-g_3t^3\}$$

be an elliptic curve. The field of rational (meromorphic) functions on $\mathcal{E}$ over $\mathbb{C}$ is $\mathbb{C}(\mathcal{E})=\mathbb{C}(\wp,\wp')=\mathbb{C}(x,y)$ where x and y are related by the cubic equation $y^2=4x^3-g_2x-g_3$. Under the isomorphism $\mathbb{C}/\Omega\to\mathcal{E}(\mathbb{C})$ given by $(1:\wp:\wp')$, the differential form dz is mapped to dx/y. The holomorphic differential forms on $\mathbb{C}/\Omega$ are λdz with $\lambda\in\mathbb{C}$.

The differential form $d\zeta=\zeta'/\zeta$ is mapped to $-xdx/y$. The differential forms of second kind on $\mathcal{E}(\mathbb{C})$ are $adz+bd\zeta+d\chi$, where a and b are complex numbers and $\chi\in\mathbb{C}(x,y)$ is a meromorphic function on $\mathcal{E}$.

Assume the elliptic curve $\mathcal{E}$ is defined over $\overline{\mathbb{Q}}$: the invariants g_2 and g_3 are algebraic. We shall be interested with differential forms which are defined over $\overline{\mathbb{Q}}$. Those of second kind are $adz+bd\zeta+d\chi$, where a and b are algebraic numbers and $\chi\in\overline{\mathbb{Q}}(x,y)$.

An elliptic integral is an integral

$$\int R(x,y)dx$$

where R is a rational function of x and y, while y^2 is a polynomial in x of degree 3 or 4 without multiple roots. One may transform this integral as follows: one reduces to an integral of $dx/\sqrt{P(x)}$ where P is a polynomial of 3rd or 4th degree; in case P has degree 4 one replaces it with a degree 3 polynomial by sending one root to infinity; finally one reduces to a Weierstraß equation by means of a birational transformation. The value of the integral is not modified.

For transcendence purposes, if the initial integral is defined over $\overline{\mathbb{Q}}$, then all these transformations involve only algebraic numbers.

3 Transcendence results of numbers related with elliptic functions

3.1 Elliptic analog of Lindemann's Theorem on the transcendence of π and of Hermite-Lindemann Theorem on the transcendence of $\log\alpha$.

The first transcendence result on periods of elliptic functions was proved by C.L. Siegel [182] as early as 1932.

Theorem 17 (Siegel, 1932) *Let $\wp$ be a Weierstraß elliptic function with period lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Assume that the invariants g_2 and g_3 of $\wp$ are algebraic. Then one at least of the two numbers ω_1, ω_2 is transcendental.*

One main feature of Siegel's proof is that he used Dirichlet's box principle (the so-called Thue-Siegel Lemma which occurs in his 1929 paper) to construct an auxiliary function. This idea turned out to be critical for the solution of Hilbert's seventh problem by Gel'fond and Schneider two years later.

In the case of complex multiplication, it follows from Theorem 17 that *any non-zero period of $\wp$ is transcendental.*

From formulae (4) and (5) it follows as a consequence of Siegel's 1932 result [182] that both numbers $\Gamma(1/4)^4/\pi$ and $\Gamma(1/3)^3/\pi$ are transcendental.

Other consequences of Siegel's result concern the length of an arc of an ellipse [183, 180]:

$$2\int_{-b}^{b}\sqrt{1+\frac{a^2x^2}{b^4-b^2x^2}}\,dx$$

as well as the perimeter of the lemniscate $(x^2+y^2)^2 = 2a^2(x^2-y^2)$.

A further example [183] is the transcendence of values of hypergeometric series related to elliptic integrals

$$\begin{aligned} K(z) &= \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-z^2x^2)}} \\ &= \frac{\pi}{2}\cdot {}_2F_1\left(1/2,\ 1/2\,;\ 1 \,\middle|\, z^2\right), \end{aligned}$$

where ${}_2F_1$ denotes Gauss hypergeometric series

$${}_2F_1\left(a,\ b\,;\ c \,\middle|\, z\right) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\cdot\frac{z^n}{n!}$$

with $(a)_n = a(a+1)\cdots(a+n-1)$.

Further results on this topic were obtained by Th. Schneider [176] in 1934 and then in a joint work by K. Mahler and J. Popken [164] in 1935 using Siegel's method. These results were superseded by Th. Schneider's fundamental memoir [177] in 1936 where he proved a number of definitive results on the subject, including:

Theorem 18 (Schneider, 1936) *Assume that the invariants g_2 and g_3 of $\wp$ are algebraic. Then for any nonzero period ω of $\wp$, the numbers ω and $\eta(\omega)$ are transcendental.*

It follows from Theorem 18 that any non-zero period of an elliptic integral of the first or second kind is transcendental:

Corollary 19 *Let $\mathcal{E}$ be an elliptic curve over $\overline{\mathbb{Q}}$, p_1 and p_2 two algebraic points on $\mathcal{E}(\overline{\mathbb{Q}})$, w a differential form of first or second kind on $\mathcal{E}$ which is defined over $\overline{\mathbb{Q}}$, holomorphic at p_1 and p_2 and is not the differential of a rational function. Let γ be a path on $\mathcal{E}$ of origin p_1 and end p_2. In case $p_1 = p_2$ one assumes that γ is not homologous to 0. Then the number*

$$\int_\gamma w$$

is transcendental.

Examples: Using Corollary 19 and formulae (8) and (9), one deduces that the numbers

$$\Gamma(1/4)^4/\pi^3 \quad \text{and} \quad \Gamma(1/3)^3/\pi^2$$

are transcendental.

The main results of Schneider's 1936 paper [177] are as follows (see also [180]):

Theorem 20 (Schneider, 1936)
1. *If $\wp$ is a Weierstraß elliptic function with algebraic invariants g_2, g_3 and if β is a non-zero algebraic number, then β is not a pole of $\wp$ and $\wp(\beta)$ is transcendental.*
More generally, if a and b are two algebraic numbers with $(a,b) \neq (0,0)$, then for any $u \in \mathbb{C} \setminus \Omega$ one at least of the two numbers $\wp(u)$, $au + b\zeta(u)$ is transcendental.
2. *If $\wp$ and $\wp^*$ are two algebraically independent elliptic functions with algebraic invariants g_2, g_3, g_2^*, g_3^*, if $t \in \mathbb{C}$ is a pole neither of $\wp$ nor of $\wp^*$, then one at least of the two numbers $\wp(t)$ and $\wp^*(t)$ is transcendental.*
3. *If $\wp$ is a Weierstraß elliptic functions with algebraic invariants g_2, g_3, for any $t \in \mathbb{C} \setminus \Omega$ one at least of the two numbers $\wp(t)$, e^t is transcendental.*

It follows from Theorem 20.2 that the quotient of an elliptic integral of the first kind (between algebraic points) by a nonzero period is either in the field of endomorphisms (hence a rational number, or a quadratic number in the field of complex multiplication), or else a transcendental number.

Here is another important consequence of Theorem 20.2.

Corollary 21 (Schneider, 1936) *Let $\tau \in \mathcal{H}$ be a complex number in the upper half plane $\Im m(\tau) > 0$ such that $j(\tau)$ is algebraic. Then τ is algebraic if and only if τ is imaginary quadratic.*

In this connection we quote Schneider's second problem in [180], which is still open (see Wakabayashi's papers [196–198]):

Open problem: *Prove Corollary 21 without using elliptic functions.*

Sketch of proof of Corollary 21 as a consequence of part 2 of Theorem 20.

Assume that both $\tau \in \mathcal{H}$ and $j(\tau)$ are algebraic. There exists an elliptic function with algebraic invariants g_2, g_3 and periods ω_1, ω_2 such that

$$\tau = \frac{\omega_2}{\omega_1} \quad \text{and} \quad j(\tau) = \frac{1728 g_2^3}{g_2^3 - 27 g_3^2}\cdot$$

Set $\wp^*(z) = \tau^2 \wp(\tau z)$. Then $\wp^*$ is a Weierstraß function with algebraic invariants g_2^*, g_3^*. For $u = \omega_1/2$ the two numbers $\wp(u)$ and $\wp^*(u)$ are algebraic. Hence the two functions $\wp(z)$ and $\wp^*(z)$ are algebraically dependent. It follows that the corresponding elliptic curve has non trivial endomorphisms, therefore τ is quadratic. $\square$

A quantitative refinement of Schneider's Theorem on the transcendence of $j(\tau)$ given by A. Faisant and G. Philibert in 1984 [68] will be useful 10 years later in connection with Nesterenko's result (see § 5). See also [69].

We shall not review the results related to Abelian integrals, we only quote the first result on this topic, which involves the Jacobian of a Fermat curve: in 1941 Schneider [178] proved that *for a and b in $\mathbb{Q}$ with a, b and $a+b$ not in $\mathbb{Z}$, the number*

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is transcendental. We notice that in his 1932 paper [182], C.L. Siegel already announced partial results on the values of the Euler Gamma function (see also [18]).

Schneider's above mentioned results deal with elliptic (and Abelian) integrals of the first or second kind. His method can be extended to deal with elliptic (and Abelian) integrals of the third kind (this is Schneider's third problem in [180]).

As pointed out by J-P. Serre in 1979 [202], it follows from the quasiperiodicity of Weierstraß sigma function (7) that the function

$$F_u(z) = \frac{\sigma(z+u)}{\sigma(z)\sigma(u)} e^{-z\zeta(u)}$$

satisfies

$$F_u(z+\omega_i) = F_u(z) e^{\eta_i u - \omega_i \zeta(u)}.$$

Theorem 22 *Let u_1 and u_2 be two nonzero complex numbers. Assume g_2, g_3, $\wp(u_1)$, $\wp(u_2)$, β are algebraic and $\mathbb{Z}u_1 \cap \Omega = \{0\}$. Then the number*

$$\frac{\sigma(u_1+u_2)}{\sigma(u_1)\sigma(u_2)} e^{\bigl(\beta - \zeta(u_1)\bigr)u_2}$$

is transcendental.

From the next corollary, one can deduce that nonzero periods of elliptic integrals of the third kind are transcendental (see [201]).

Corollary 23 *For any nonzero period ω and for any $u \in \mathbb{C} \setminus \Omega$ the number $e^{\omega\zeta(u) - \eta u + \beta\omega}$ is transcendental.*

Further results on elliptic integrals are due to M. Laurent [113]. See also his papers [115–118].

Ya. M. Kholyavka wrote several papers devoted to the approximation of transcendental numbers related with elliptic functions [101,100,99,98,97,96,94,95,93]

Quantitative estimates (measures of transcendence) related to the results of this section have been derived by N.I. Fel'dman [70–74] — see also the papers by S. Lang [105], N.D. Nagaev [144], E. Reyssat [168,169,171,172], M. Laurent [114], R. Tubbs [189], G. Diaz [58], N. Saradha [175], P. Grinspan [84].

3.2 Elliptic analogs to the six exponentials Theorem

Elliptic analogs of the six exponentials Theorem 10 have been considered by S. Lang [106] and K. Ramachandra [165] in the 1960's.

Let d_1 and d_2, m be nonnegative integers with $m > 0$, let $x_1, \dots, x_{d_1}$ be complex numbers which are linearly independent over $\mathbb{Q}$, let $y_1, \dots, y_m$ be complex numbers which are linearly independent over $\mathbb{Q}$ and let $u_1, \dots, u_{d_2}$ be nonzero complex numbers. We consider Weierstraß elliptic functions $\wp_1, \dots, \wp_{d_2}$ and we denote by K_0 the field generated over $\mathbb{Q}$ by their invariants $g_{2,k}$ and $g_{3,k}$ ($1 \le k \le d_2$). We assume that the d_2 functions $\wp_1(u_1 z), \dots, \wp_{d_2}(u_{d_2} z)$ are algebraically independent. We denote by K_1 the field generated over K_0 by the numbers $\exp(x_i y_j)$, ($1 \le i \le d_1$, $1 \le j \le m$) together with the numbers $\wp_k(u_k y_j)$, ($1 \le k \le d_2$, $1 \le j \le m$). Next define

$$K_2 = K_1(y_1, \dots, y_m), \qquad K_3 = K_1\bigl(x_1, \dots, x_{d_1},\ u_1, \dots, u_{d_2}\bigr),$$

and let K_4 be the compositum of K_2 and K_3:

$$K_4 = K_1\bigl(y_1, \dots, y_m,\ x_1, \dots, x_{d_1},\ u_1, \dots, u_{d_2}\bigr).$$

The theorems of Hermite-Lindemann (Theorem 3), Gel'fond-Schneider (Theorem 7), the six exponentials Theorem 10 and their elliptic analogs due to Schneider, Lang and Ramachandra can be stated as follows.

Theorem 24
1. *Assume* $(d_1 + d_2)m > m + d_1 + 2d_2$*. Then the field* K_1 *has transcendence degree* ≥ 1 *over* $\mathbb{Q}$*.*
2. *Assume either* $d_1 \geq 1$ *and* $m \geq 2$*, or* $d_2 \geq 1$ *and* $m \geq 3$*. Then* K_2 *has transcendence degree* ≥ 1 *over* $\mathbb{Q}$*.*
3. *Assume* $d_1 + d_2 \geq 2$*. Then* K_3 *has transcendence degree* ≥ 1 *over* $\mathbb{Q}$*.*
4. *Assume* $d_1 + d_2 \geq 1$*. Then* K_4 *has transcendence degree* ≥ 1 *over* $\mathbb{Q}$*.*

Parts 3 and 4 of Theorem 24 are consequences of Schneider-Lang criterion [106], which deals with meromorphic functions satisfying differential equations, while parts 1 and 2 require a criterion which involves no differential equations. Such criteria have been given by Schneider [179,180], Lang [106] and Ramachandra [165,166] (see also [199] and [198]).

Theorem 24 includes also Theorem 20 apart from the case $b \neq 0$ in part 1 of that statement. However there are extensions of Theorem 24 which include results on Weierstraß zeta function (and also on Weierstraß sigma function in connection with elliptic integrals of the third kind). See [201,113,115–117,202,172,118].

Here is a corollary of Theorem 24.

Theorem 25 *Let* E *be an elliptic curve having algebraic invariants* g_2*,* g_3 *with complex multiplication. Let*

$$M = \begin{pmatrix} u_1 & u_2 & u_3 \\ & & \\ v_1 & v_2 & v_3 \end{pmatrix}$$

be a 2×3 *matrix whose entries are elliptic logarithms of algebraic numbers:* u_i *and* v_i *are in* $\mathcal{L}_E$*. Assume the three columns are linearly independent over* $\mathrm{End}(E)$ *and the two rows are also linearly independent over* $\mathrm{End}(E)$*. Then the matrix* M *has rank* 2*.*

In the case where the curve has no complex multiplication, a similar statement holds for 2×5 matrices. Also in the non CM case, one deduces from Theorem 24 that such 3×4 matrices $\big(u_{ij}\big)$ (where $\wp(u_{ij})$ are algebraic numbers) have rank ≥ 2.

There are further lower bounds going further than 2 for the rank of matrices of larger sizes but we shall not discuss this question here. We just mention the fact that higher dimensional considerations are relevant to a problem discussed by B. Mazur on the density of rational points on varieties [209].

4 Linear independence of numbers related with elliptic functions

From Schneider's Theorem 20 part 1, one deduces the linear independence over the field of algebraic numbers of the three numbers 1, ω and η, when ω is a nonzero period of a Weierstraß elliptic function (with algebraic invariants g_2 and g_3) and $\eta = \eta(\omega)$ is the associated quasiperiod of the corresponding Weierstraß zeta function. However the Gel'fond-Schneider method in one variable alone does not yield strong results of linear independence. Baker's method is better suited for this purpose.

4.1 Linear independence of periods and quasiperiods

Baker's method of proof for his Theorem 9 on linear independence of logarithms of algebraic numbers has been used as early as 1969 and 1970 by A. Baker himself [11,9] when he proved the transcendence of linear combinations with algebraic coefficients of the numbers ω_1, ω_2, η_1 and η_2 associated with an elliptic curve having algebraic invariants g_2 and g_3. His method is effective: it provides quantitative Diophantine estimates [10]

In 1971 J. Coates [42] proved the transcendence of linear combinations with algebraic coefficients of $\omega_1,\ \omega_2,\ \eta_1,\ \eta_2$ and $2i\pi$. Further, he proved in [41,43–45] that in the non-CM case, the three numbers ω_1, ω_2 and $2i\pi$ are $\overline{\mathbb{Q}}$-linearly independent.

From this point of view the final result has been reached by D.W. Masser in 1975 [122,123].

Theorem 26 (Masser, 1975)) *Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2 and g_3, denote by ζ the corresponding Weierstraß zeta function, let ω_1, ω_2 be a basis of the period lattice of $\wp$ and let η_1, η_2 be the associated quasiperiods of ζ. Then the six numbers 1, ω_1, ω_2, η_1, η_2 and $2i\pi$ span a $\overline{\mathbb{Q}}$-vector space of dimension 6 in the non CM case, 4 in the CM case:*

$$\dim_{\overline{\mathbb{Q}}}\{1, \omega_1, \omega_2, \eta_1, \eta_2, 2i\pi\} = 2 + 2\dim_{\overline{\mathbb{Q}}}\{\omega_1, \omega_2\}.$$

The fact that the dimension is 4 in the CM case means that there are two independent linear relations among these 6 numbers. One of them is $\omega_2 = \tau\omega_1$ with $\tau \in \overline{\mathbb{Q}}$. The second one (see [123]; see also [33]) can be written

$$C^2\tau\eta_2 - AC\eta_1 + \gamma\omega_1 = 0$$

where $A + BX + CX^2$ is the minimal polynomial of τ over $\mathbb{Z}$ and γ is an element in $\mathbb{Q}(g_2, g_3, \tau)$.

In [123], D.W. Masser also gives quantitative estimates (measures of linear independence). In 1976, R. Franklin and D.W. Masser [130] produce an extension involving a logarithm of an algebraic number.

Further results can be found in papers by P. Bundschuh [36], S. Lang [110] (see also his surveys [107,108]), D.W. Masser [133,131], M. Anderson [4] and in the joint paper [5] by M. Anderson and D.W. Masser.

4.2 Elliptic analog of Baker's Theorem

The elliptic analog of Baker's Theorem 9 on linear independence of logarithms was proved by D.W. Masser in 1974 [122,123] in the CM case.

His proof yields also quantitative estimates (measures of linear independence of elliptic logarithms of algebraic points on an elliptic curve). Such estimates have a number of applications: this was shown by A.O. Gel'fond for usual logarithms of algebraic numbers [80] and further consequences of such lower bounds in the case of elliptic curves for solving Diophantine equations (integer points on elliptic curves) have been derived by S. Lang [111].

Lower bounds for linear combinations of elliptic logarithms in the CM case have been obtained by several mathematicians including J. Coates [42], D.W. Masser [124,128,129], J. Coates and S. Lang [46], M. Anderson [4]. The work of Yu Kunrui [218] yields similar estimates, but his method is not Baker-Masser's one: instead of using a generalization of Gel'fond's solution to Hilbert's seventh problem, Yu Kunrui uses a generalization in several variables of Schneider's solution to the same problem. Again, this method is restricted to the CM case.

The question of linear independence of elliptic logarithms in the non CM case has been settled only in 1980 by D. Bertrand and D.W. Masser [28,29]. They found a new proof of Baker's Theorem 9 using functions of several variables and they succeeded to extend this argument to the situation of elliptic functions, either with or without complex multiplication. The criterion they use is the one that Schneider established in 1949 [178] for his proof of the transcendence of Beta values. This criterion (revisited by S. Lang in [106]) deals with Cartesian products. From the several variables point of view, this is a rather degenerate situation; much deeper results are available, including Bombieri's solution in 1970 of Nagata's Conjecture [106,202], which involves Hörmander L^2-estimates for analytic functions of several variables. But so far these deeper results do not give further transcendence results in our context.

Theorem 27 (D.W. Masser 1974 for the CM case, D. Bertrand and D.W. Masser 1980 for the non CM case) *Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2, g_3 and field of endomorphisms k. Let $u_1, \dots, u_n$ be k-linearly independent complex numbers. Assume, for $1 \leq i \leq n$, that either $u_i \in \Omega$ or else $\wp(u_i) \in \overline{\mathbb{Q}}$. Then the numbers $1, u_1, \dots, u_n$ are linearly independent over the field $\overline{\mathbb{Q}}$.*

This means that *for an elliptic curve E which is defined over $\overline{\mathbb{Q}}$, if $u_1,\dots,u_n$ are elements in $\mathcal{L}_E$ which are linearly independent over the field of endomorphisms of E, then the numbers $1,u_1,\dots,u_n$ are linearly independent over $\overline{\mathbb{Q}}$.*

The method of Bertrand-Masser yields only weak Diophantine estimates (measures of linear independence of logarithms).

4.3 Further results of linear independence

Theorem 26 deals only with periods and quasiperiods associated with one lattice, Theorem 27 deals only with elliptic logarithms of algebraic points on one elliptic curve. A far reaching generalization of both results has been achieved by G. Wüstholz in 1987 [215–217] when he succeeded to extend Baker's Theorem to Abelian varieties and integrals, and, more generally, to commutative algebraic groups. If we restrict his general result to products of a commutative linear group, of copies of elliptic curves as well as of extensions of elliptic curves by the additive or the multiplicative group, the resulting statement settles the questions of linear independence of logarithms of algebraic numbers, of elliptic logarithms of algebraic points, including periods, quasiperiods, elliptic integrals of the first, second or third kind. This is a main step towards an answer to the questions of M. Kontsevich and D. Zagier on periods [104].

Wüstholz's method can be extended to yield measures of linear independence of logarithms of algebraic points on an algebraic group. The first effective such lower bound were given in 1989 in [162, 163]. As a special case, they provide the first measures of linear independence for elliptic logarithms which is also valid in the non CM case. More generally, they give effective lower bounds for any nonvanishing linear combination of logarithms of algebraic points on algebraic groups (including usual logarithms, elliptic logarithms, elliptic integrals of any kind).

Refinements have been obtained by N. Hirata Kohno [88–91], S. David [54], N. Hirata Kohno and S. David [56], M. Ably [2,3] and É. Gaudron [78,77,79] who uses the work of J-B. Bost [30] (slope inequalities) involving Arakelov's Theory. For instance, thanks to the recent work of David and Hirata-Kohno on one hand, of Gaudron on the other, one knows that the above mentioned nonvanishing linear combinations of logarithms of algebraic points are not Liouville numbers.

In the p-adic case there is a paper of G. Rémond and F. Urfels [167] dealing with two elliptic logarithms. The general case of n logarithms would also deserve to be dealt with.

Further applications to elliptic curves of the Baker-Masser-Wüstholz method have been derived by D.W. Masser and G.Wüstholz [142,143].

A survey on questions related to the isogeny Theorem is [154]. Other surveys dealing with the questions of *small points*, Bogomolov conjecture and the André-Oort conjecture are [48,55]. We do not cover these aspects of the theory in the present paper. Other related topics which would deserve more attention are the theory of height and theta functions as well as ultrametric questions.

Extensions of the above mentioned results to Abelian varieties have been considered by D.W. Masser [124–129,132,134–136], S. Lang [109], J. Coates and S. Lang [46], D. Bertrand and Yu.V. Flicker [26], Yu.V. Flicker [76], D. Bertrand [23,24]. For instance J. Wolfart and G. Wüstholz [213] have shown that the only linear dependence relations with algebraic coefficients between the values $B(a,b)$ of Euler Beta function at points $(a,b)\in\mathbb{Q}^2$ are those which follow from the Deligne-Koblitz-Ogus relations (see further references in [212]). They deduce also the transcendence of the values at algebraic points of hypergeometric functions with rational parameters.

5 Algebraic independence of numbers related with elliptic functions

5.1 Small transcendence degree

We keep the notations and assumptions of section 3.2.

The following extension of Theorem 24 to a result of algebraic independence containing Gel'fond's 1949 results (see [80]) is a consequence of the works of many a mathematician, including A.O. Gel'fond [80], A.A. Šmelev [186,187], R. Tijdeman, W.D. Brownawell [31], W.D. Brownawell and K.K. Kubota [33], D.W. Masser and G. Wüstholz [139]. Further references are given in [204,205].

Theorem 28
1. *Assume* $(d_1 + d_2)m \geq 2(m + d_1 + 2d_2)$. *Then the field* K_1 *has transcendence degree* ≥ 2 *over* $\mathbb{Q}$.
2. *Assume* $(d_1 + d_2)m \geq m + 2(d_1 + 2d_2)$. *Then* K_2 *has transcendence degree* ≥ 2 *over* $\mathbb{Q}$.
3. *Assume* $(d_1 + d_2)m \geq 2m + d_1 + 2d_2$. *Then* K_3 *has transcendence degree* ≥ 2 *over* $\mathbb{Q}$.
4. *Assume* $(d_1 + d_2)m > m + d_1 + 2d_2$. *Then* K_4 *has transcendence degree* ≥ 2 *over* $\mathbb{Q}$.

Quantitative estimates (measures of algebraic independence) exist (R. Tubbs [190], E.M. Jabbouri [92], Yu.V. Nesterenko [145–147]).

Further related results are due to R. Tubbs [192,191,193,194,189,190], É.Reyssat [174], M. Toyoda and T. Yasuda [188]. See also the measure of algebraic independence given by M. Ably in [1] and by S.O. Shestakov [181].

A survey on results related with small transcendence degree is given in [205] (see also Chapter 13 of [153]).

Again, like for Theorem 24, there is an extension of Theorem 28 which includes results on Weierstraß zeta function. Also results on functions of several variables are known, as well as results related to Abelian functions [206].

5.2 Algebraic independence of periods and quasiperiods

Deep results have been achieved by G.V. Čudnovs′kiĭ starting in the 1970's [49,50,52,51,53]. He succeeded to prove sharp results of algebraic independence (large transcendence degree) for values of the exponential function, generalizing Gel'fond's Theorem 13 (previous results in this direction were extremely limited). Also he proved strong results of algebraic independence (small transcendence degree) related with elliptic functions. We first describe the latter. Among his other contributions are results dealing with G-functions (see [53]; see also Y. André's work [6,7]).

One of G.V. Čudnovs′kiĭ 's most spectacular contributions [49,52,53] was obtained in 1976:

Theorem 29 (G.V. Čudnovs′kiĭ, 1976) *Let* $\wp$ *be a Weierstraß elliptic function with invariants* g_2, g_3. *Let* (ω_1, ω_2) *be a basis of the lattice period of* $\wp$ *and* $\eta_1 = \eta(\omega_1)$, $\eta_2 = \eta(\omega_2)$ *the associated quasiperiods of the associated Weierstraß zeta function. Then two at least of the numbers* $g_2,\ g_3,\ \omega_1,\ \omega_2,\ \eta_1,\ \eta_2$ *are algebraically independent.*

In the case where g_2 and g_3 are algebraic the algebraic independence of two at least of the four numbers $\omega_1,\ \omega_2,\ \eta_1,\ \eta_2$ is also a consequence of the next result.

Theorem 30 (G.V. Čudnovs′kiĭ, 1976) *Assume* g_2 *and* g_3 *are algebraic. Let* ω *be a non-zero period of* $\wp$, *set* $\eta = \eta(\omega)$ *and let* u *be a complex number which is not a period such that* u *and* ω *are* $\mathbb{Q}$-*linearly independent:* $u \notin \mathbb{Q}\omega \cup \Omega$. *Assume* $\wp(u) \in \overline{\mathbb{Q}}$. *Then the two numbers*

$$\zeta(u) - \frac{\eta}{\omega}u, \quad \frac{\eta}{\omega}$$

are algebraically independent.

From either Theorem 29 or Theorem 30 one deduces:

Corollary 31 *Let* ω *be a nonzero period of* $\wp$ *and* $\eta = \eta(\omega)$. *If* g_2 *and* g_3 *are algebraic, then the two numbers* π/ω *and* η/ω *are algebraically independent.*

The following consequence of Corollary 31 shows that in the CM case, Čudnovsʹkiĭ's results are sharp:

Corollary 32 *Assume g_2 and g_3 are algebraic and the elliptic curve has complex multiplication. Let ω be a nonzero period of $\wp$. Then the two numbers ω and π are algebraically independent.*

As a consequence of formulae (4) and (5), one deduces:

Corollary 33 *The numbers π and $\Gamma(1/4)$ are algebraically independent. Also the numbers π and $\Gamma(1/3)$ are algebraically independent.*

In connexion with these result let us quote a conjecture of S. Lang in 1971 [107].

Conjecture 34 If $j(\tau)$ is algebraic with $j'(\tau)\neq 0$, then $j'(\tau)$ is transcendental.

Since

$$j'(\tau)=18\frac{\omega_1^2}{2i\pi}\cdot\frac{g_2}{g_3}j(\tau),$$

Conjecture 34 amounts to the transcendence of ω^2/π. Hence Corollary 32 implies that Conjecture 34 is true at least in the CM case (see [20]):

Corollary 35 *If $\tau\in\mathcal{H}$ is quadratic and $j'(\tau)\neq 0$, then π and $j'(\tau)$ are algebraic independent.*

A quantitative refinement (measure of algebraic independence) of Corollary 32 due to G. Philibert [156] turns out to be useful in connexion with Nesterenko's work in 1996 (further references on this topic are given in [208]).

A transcendence measures for $\Gamma(1/4)$ has been obtained by P. Philippon [160,161] and S. Bruiltet [35]:

Theorem 36 *For $P\in\mathbb{Z}[X,Y]$ with degree d and height H,*

$$\log|P(\pi,\Gamma(1/4)|>-10^{326}\bigl((\log H+d\log(d+1)\bigr)d^2\bigl(\log(d+1)\bigr)^2.$$

Corollary 37 *The number $\Gamma(1/4)$ is not a Liouville number:*

$$\left|\Gamma(1/4)-\frac{p}{q}\right|>\frac{1}{q^{10^{330}}}\cdot$$

Further references related to Čudnovsʹkiĭ's results are papers by D. Bertrand [17] and E. Reyssat's [170,173] (see also the Bourbaki lecture [200] and the book of E.B. Burger and R. Tubbs [38]).

We conclude this section by the following open problem, which simultaneously generalizes Theorems 29 and 30 of G.V. Čudnovsʹkiĭ.

Conjecture 38 Let $\wp$ be a Weierstraß elliptic function with invariants g_2, g_3, let ω be a non-zero period of $\wp$, set $\eta=\eta(\omega)$ and let $u\in\mathbb{C}\setminus\{\mathbb{Q}\omega\cup\Omega\}$. Then two at least of the five numbers

$$g_2,\quad g_3,\quad \wp(u),\quad \zeta(u)-\frac{\eta}{\omega}u,\quad \frac{\eta}{\omega}$$

are algebraically independent.

Čudnovsʹkiĭ's method has been extended by K.G. Vasil'ev [195] and P. Grinspan [84], who proved that two at least of the three numbers π, $\Gamma(1/5)$ and $\Gamma(2/5)$ are algebraically independent. Their proof involves the Jacobian of the Fermat curve $X^5+Y^5=Z^5$, which is an Abelian variety of dimension 2. See also Pellarin's paper [155].

5.3 Large transcendence degree

Another important (and earlier) contribution of G.V. Čudnovs′kiĭ is that in 1974 he succeeded to apply Gel'fond's method in order to prove results on large transcendence degree (see references in [53,200]). He first proved that three at least of the numbers

$$\alpha^{\beta}, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d-1}} \tag{10}$$

are algebraically independent if α is a nonzero algebraic number, $\log\alpha$ a nonzero logarithm of α and β an algebraic number of degree $d \geq 7$. The same year, by a much more difficult proof, he succeded to prove that there exist at least n algebraically independent numbers in the set (10), provided that $d \geq 2^n - 1$. This was a remarkable achievement since no such result providing a lower bound for the transcendence degree was known. Later, thanks to the work of several mathematicians, especially P. Philippon, the exponential lower bound for d was reduced to a polynomial bound, until G. Diaz [57] obtained the best know results so far: the transcendence degree is at least $[(d+1)/2]$. The case $d = 5$ had been settled by G.V. Čudnovs′kiĭ who also obtained elliptic analogs (see [50,52,53,200,170]).

Also G.V. Čudnovs′kiĭ [51] succeeded in 1980 to prove the Lindemann-Weierstraß Theorem 5 by means of his extension of the Gel'fond-Schneider's method to large transcendence degree. This method enabled P. Philippon [157–159] and G. Wüstholz [214] in 1982 to prove the elliptic analog of Lindemann Weierstraß Theorem on the algebraic independence of $e^{\alpha_1}, \dots, e^{\alpha_n}$ in the CM case:

Theorem 39 *Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2, g_3 and complex multiplication. Let $\alpha_1, \dots, \alpha_m$ be algebraic numbers which are linearly independent over the field of endomorphisms of E. Then the numbers $\wp(\alpha_1), \dots, \wp(\alpha_n)$ are algebraically independent.*

The same conclusion should hold also in the non-CM case – so far only the algebraic independence of at least $n/2$ of these numbers is known.

Further results on large transcendence degree are due to D.W. Masser and G. Wüstholz [140,141] W.D. Brownawell [32], W.D. Brownawell and R. Tubbs [34] (for a survey of this topic, see [205]; see also [153] Chap. 14).

5.4 Modular functions

Ramanujan introduced the following functions

$$P(q) = 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}, \quad Q(q) = 1 + 240\sum_{n=1}^{\infty} \frac{n^3q^n}{1-q^n}, \quad R(q) = 1 - 504\sum_{n=1}^{\infty} \frac{n^5q^n}{1-q^n}\cdot$$

They are special cases of Eisenstein series. Recall the Bernoulli numbers:

$$\frac{z}{e^z-1} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} B_k \frac{z^{2k}}{(2k)!},$$

$$B_1 = 1/6, \quad B_2 = 1/30, \quad B_3 = 1/42.$$

For $k \geq 2$ the Eisenstein series of weight k is (compare with (3))

$$E_{2k}(z) = 1 + (-1)^k \frac{4k}{B_k} \sum_{n=1}^{\infty} \frac{n^{2k-1}z^n}{1-z^n}\cdot$$

The connection with Ramanujan's notation is

$$P(z) = E_2(z), \quad Q(z) = E_4(z), \quad R(z) = E_6(z).$$

The discriminant Δ and the modular invariant J are related to these functions by

$$\Delta = 12^{-3}(Q^3 - R^2) = q\prod_{n=1}^{\infty}(1-q^n)^{24} \quad \text{and} \quad J = Q^3/\Delta.$$

Let q be a complex number, $0 < |q| < 1$. There exists τ in the upper half plane $\mathcal{H}$ such that $q = e^{2i\pi\tau}$. Select any twelfth root of $\Delta(q)$ and set $\omega = 2\pi\Delta(q)^{1/12}$. The invariants g_2 and g_3 of the Weierstraß $\wp$ function attached to the lattice $(\mathbb{Z} + \mathbb{Z}\tau)\omega$ satisfy $g_2^3 - 27g_3^2 = 1$ and

$$P(q) = 3\frac{\omega}{\pi}\cdot\eta\pi, \quad Q(q) = \frac{3}{4}\left(\frac{\omega}{\pi}\right)^4 g_2, \quad R(q) = \frac{27}{8}\left(\frac{\omega}{\pi}\right)^6 g_3.$$

According to formulae (4) and (5), here are a few special values (see for instance [210]). For $\tau = i$, $q = e^{-2\pi}$,

$$P(e^{-2\pi}) = \frac{3}{\pi}, \quad Q(e^{-2\pi}) = 3\left(\frac{\omega_1}{\pi}\right)^4, \quad R(e^{-2\pi}) = 0 \quad \text{and} \quad \Delta(e^{-2\pi}) = \frac{1}{2^6}\left(\frac{\omega_1}{\pi}\right)^{12}, \tag{11}$$

with

$$\omega_1 = \frac{\Gamma(1/4)^2}{\sqrt{8\pi}} = 2.6220575542\ldots$$

For $\tau = \varrho$, $q = -e^{-\pi\sqrt{3}}$,

$$P(-e^{-\pi\sqrt{3}}) = \frac{2\sqrt{3}}{\pi},\ Q(-e^{-\pi\sqrt{3}}) = 0,\ R(-e^{-\pi\sqrt{3}}) = \frac{27}{2}\left(\frac{\omega_1}{\pi}\right)^6,\ \Delta(-e^{-\pi\sqrt{3}}) = -\frac{27}{256}\left(\frac{\omega_1}{\pi}\right)^{12}, \tag{12}$$

with

$$\omega_1 = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} = 2.428650648\ldots$$

5.5 Mahler-Manin problem on $J(q)$

After Schneider's Theorem (Corollary 21) on the transcendence of the values of the modular function $j(\tau)$, the first results on Eisenstein series (cf. § 5.6) go back to D. Bertrand's 1977 paper [18]. See also his papers [17, 19, 21, 22] and his paper with M. Laurent on values of theta functions : [27].

The first transcendence proof using modular forms is due to a team from St Étienne (K. Barré-Sirieix, G. Diaz, F. Gramain and G. Philibert) — hence the nickname *théorème stéphanois* for the next result, from [15] (see also [81, 82] and Chap. 2 of [153]), which answers a conjecture of K. Mahler in the complex case and of Yu. V. Manin in the p-adic case (we state the result only in the complex case — the paper [15] deals with both cases).

Theorem 40 (K. Barré, G. Diaz, F. Gramain, G. Philibert, 1996) *Let $q \in \mathbb{C}$, $0 < |q| < 1$. If q is algebraic, then $J(q)$ is transcendental.*

The solution of Manin's problem has several consequences. It is a tool both for R. Greenberg in his study of zeroes of p-adic L functions, and for H. Hida, J. Tilouine and É. Urban in their solution of the main Conjecture for the Selmer group of the symmetric square of an elliptic curve with multiplicative reduction at p.

The proof of Theorem 40 involves upper bounds for the growth of the coefficients of the modular function $J(q)$. Such estimates have been produced first by K. Mahler [121]. A refined estimate, due to G. Philibert, for the coefficients $c_k(m)$ (which are nonnegative rational integers) in

$$\bigl(zJ(z)\bigr)^k = \sum_{m=0}^{\infty} c_k(m)z^m,$$

is
$$c_k(m) \le e^{4\sqrt{km}}.$$
As pointed out by D. Bertrand [25], the upper bound
$$|\tilde{c}_{Nk}(m)| \le C^N m^{12N}$$
($0 \le k \le N$, $N \ge 1$, $m \ge 1$, with an absolute constant C) for the coefficients in the Taylor development at the origin of $\Delta^{2N} J^k$:
$$\Delta(z)^{2N} J(z)^k = \sum_{m=1}^{\infty} \tilde{c}_{Nk}(m) z^m$$
is sufficient for the proof of Theorem 40 and is an easy consequence of a Theorem of Hecke together with the fact that Δ^2 and $\Delta^2 J$ are parabolic modular forms of weight 24.

Another auxiliary lemma used in the proof of Theorem 40 is an estimate for the degrees and height of $J(q^n)$ in terms of $J(q)$ (which is assumed to be algebraic) and $n \ge 1$. There exists a symmetric polynomial $\Phi_n \in \mathbb{Z}[X,Y]$, of degree
$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right)$$
in each variable, such that $\Phi_n\big(J(q), J(q^n)\big) = 0$. Again, K. Mahler [120, 121] was the first to investigate the coefficients of the polynomial $\Phi_n(X,Y)$: he proved that its length (sum of the absolute values of the coefficients) satisfies
$$L(\Phi_n) \le e^{cn^{3/2}}$$
with an absolute constant c. In the special case $n = 2^m$ he improved his result
$$L(\Phi_n) \le 2^{57n} n^{36n} \qquad \text{if } n = 2^m,$$
and claimed (see [120] p. 97) that if the sharper upper bound
$$L(\Phi_n) \le 2^{Cn} \qquad \text{if } n = 2^m,$$
with a positive absolute constant $C > 0$, were true, he could prove Theorem 40. However in 1984 P. Cohen [47] produced asymptotic estimates which show that Mahler's expectation was too optimistic:
$$\lim_{\substack{n=2^m \\ m\to\infty}} \frac{1}{n \log n} \log L(\Phi_n) = 9.$$
In fact she proved more precise results, without the condition $n = 2^m$, which imply for instance $\log L(\Phi_n) \sim 6\psi(n) \log n$ for $n \to \infty$.

Further related results are given in [61] (G. Diaz and G. Philibert) for the j-function and [138] (D.W. Masser) for $\wp$-function.

The proof of [15] can be adapted to yield quantitative estimates [13, 14].

A Corollary to Theorem 40 on the transcendence of $J(q)$ is the following mixed analog of the four exponentials Conjecture 12:

Corollary 41 *Let $\log \alpha$ be a logarithm of a non-zero algebraic number. Let $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice with algebraic invariants g_2, g_3. Then the determinant*
$$\begin{vmatrix} \omega_1 & \log \alpha \\ \omega_2 & 2i\pi \end{vmatrix}$$
does not vanish.

The four exponentials conjecture for the product of an elliptic curve by the multiplicative group is the following more general open problem:

Conjecture 42 Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2, g_3. Let E be the corresponding elliptic curve, u_1 and u_2 be two elements in $\mathcal{L}_E$ and $\log\alpha_1$, $\log\alpha_2$ be two logarithms of algebraic numbers. Assume further that the two rows of the matrix

$$M = \begin{pmatrix} u_1 & \log\alpha_1 \\ u_2 & \log\alpha_2 \end{pmatrix}$$

are linearly independent over $\mathbb{Q}$. Then the determinant of M does not vanish.

Another special case of Conjecture 42, stronger than Corollary 41, is the next question of Yu. V. Manin:

Conjecture 43 (Yu.V. Manin) Let $\log\alpha_1$ and $\log\alpha_2$ be two non-zero logarithms of algebraic numbers and let $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice with algebraic invariants g_2 and g_3. Then

$$\frac{\omega_1}{\omega_2} \neq \frac{\log\alpha_1}{\log\alpha_2}.$$

In this direction let us quote some of the open problems raised by G. Diaz [59, 60].

Conjecture 44 (G. Diaz)
1. For any $z \in \mathbb{C}$ with $|z| = 1$ and $z \neq \pm 1$, the number $e^{2i\pi z}$ is transcendental.
2. If q is an algebraic number with $0 < |q| < 1$ such that $J(q) \in [0, 1728]$, then $q \in \mathbb{R}$.
3. The function J is injective on the set of algebraic numbers α with $0 < |\alpha| < 1$.

Remark (G. Diaz). The third part 3 of Conjecture 44 implies the two first ones, it follows from the four exponentials Conjecture 12, also it follows from the next Conjecture of D. Bertrand.

Conjecture 45 (D. Bertrand) If α_1 and α_2 are two multiplicatively independent algebraic numbers in the domain $\{z \in \mathbb{C}\,;\, 0 < |z| < 1\}$, then the two numbers $J(\alpha_1)$ and $J(\alpha_2)$ are algebraically independent.

This Conjecture 45 implies the special case of the four exponentials Conjecture 12, where two of the algebraic numbers are roots of unity and the two others have modulus $\neq 1$.

5.6 Nesterenko's Theorem

In 1976 [17], D. Bertrand pointed out that Schneider's Theorem 20 implies:
For any $q \in \mathbb{C}$ with $0 < |q| < 1$, one at least of the two numbers $Q(q),\ R(q)$ is transcendental.

Two years later [19], he noticed that G.V. Čudnovs′kiĭ's Theorem 29 yields:
For any $q \in \mathbb{C}$ with $0 < |q| < 1$, two at least of the numbers $P(q),\ Q(q),\ R(q)$ are algebraically independent.

The following result of Yu.V. Nesterenko [148, 149] (see also [208, 151, 210, 152] as well as Chap. 3 and 4 of [153]) goes one step further:

Theorem 46 (Nesterenko, 1996) *For any $q \in \mathbb{C}$ with $0 < |q| < 1$, three at least of the four numbers q, $P(q)$, $Q(q)$, $R(q)$ are algebraically independent.*

Among the tools used by Nesterenko in his proof is the following result due to K. Mahler [119] (see also Chap. 1 of [153]):

The functions P, Q, R are algebraically independent over $\mathbb{C}(q)$.

Also he uses the fact (see again Chap. 1 of [153]) that they satisfy a system of differential equations for $D = q\, d/dq$:

$$12\frac{DP}{P} = P - \frac{Q}{P}, \quad 3\frac{DQ}{Q} = P - \frac{R}{Q}, \quad 2\frac{DR}{R} = P - \frac{Q^2}{R}\cdot$$

One of the main steps in his original proof [149] is his following zero estimate:

Theorem 47 (Nesterenko's zero estimate) *Let L_0 and L be positive integers, $A \in \mathbb{C}[z, X_1, X_2, X_3]$ a nonzero polynomial in four variables of degree $\le L_0$ in z and $\le L$ in each of the three other variables X_1, X_2, X_3. Then the multiplicity at the origine of the analytic function $A\bigl(z, P(z), Q(z), R(z)\bigr)$ is at most $2 \cdot 10^{45} L_0 L^3$.*

In the special case where $J(q)$ is algebraic, P. Philippon [161] produced a simpler proof where this zero estimate 47 is not used; in place of it he uses Philibert's measure of algebraic independence for ω/π and η/π (see [156] and § 5.2 above).

Using (11) one deduces from Theorem 46

Corollary 48 *The three numbers π, e^{π}, $\Gamma(1/4)$ are algebraically independent.*

while using (12) one deduces

Corollary 49 *The three numbers π, $e^{\pi\sqrt{3}}$, $\Gamma(1/3)$ are algebraically independent.*

Consequences of Corollary 48 are the transcendence of the numbers

$$\sigma_{\mathbb{Z}[i]}(1/2) = 2^{5/4}\pi^{1/2}e^{\pi/8}\Gamma(1/4)^{-2}$$

and (P. Bundschuh [37])

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}}\cdot$$

D. Duverney, K. and K. Nishioka, and I. Shiokawa [62,63,65,64,66,67] as well as D. Bertrand [25] derived from Nesterenko's Theorem 46 a number of interesting corollaries, including the following ones.

Corollary 50 *Rogers-Ramanujan continued fraction:*

$$RR(\alpha) = 1 + \cfrac{\alpha}{1 + \cfrac{\alpha^2}{1 + \cfrac{\alpha^3}{1 + \ddots}}}$$

is transcendental for any algebraic α with $0 < |\alpha| < 1$.

Corollary 51 *Let $(F_n)_{n\ge 0}$ be the Fibonacci sequence: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$. Then the number*

$$\sum_{n=1}^{\infty} \frac{1}{F_n^2}$$

is transcendental.

Jacobi Theta Series are defined by

$$\theta_2(q) = 2q^{1/4}\sum_{n\geq 0} q^{n(n+1)} = 2q^{1/4}\prod_{n=1}^{\infty}(1-q^{4n})(1+q^{2n}),$$

$$\theta_3(q) = \sum_{n\in\mathbb{Z}} q^{n^2} = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1})^2,$$

$$\theta_4(q) = \theta_3(-q) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} = \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1})^2.$$

Corollary 52 . *Let i, j and $k \in \{2,3,4\}$ with $i \neq j$. Let $q \in \mathbb{C}$ satisfy $0 < |q| < 1$. Then each of the two fields*

$$\mathbb{Q}\big(q, \theta_i(q), \theta_j(q), D\theta_k(q)\big) \quad \textit{and} \quad \mathbb{Q}\big(q, \theta_k(q), D\theta_k(q), D^2\theta_k(q)\big)$$

has transcendence degree ≥ 3 over $\mathbb{Q}$.

As an example, *for an algebraic number $q \in \mathbb{C}$ with $0 < |q| < 1$, the number*

$$\theta_3(q) = \sum_{n\in\mathbb{Z}} q^{n^2}$$

is transcendental. The number $\theta_3(q)$ was explicitly considered by Liouville in his 1844 memoir (see [153] p.30).

The proof of Yu.V. Nesterenko is effective and yields quantitative refinements (measures of algebraic independence): [150, 161, 83].

5.7 Further open problems

Among many open problems, we mention

- the algebraic independence of the three numbers π, $\Gamma(1/3)$, $\Gamma(1/4)$.
- the algebraic independence of at least three numbers among π, $\Gamma(1/5)$, $\Gamma(2/5)$, $e^{\pi\sqrt{5}}$.
- the algebraic independence of the four numbers e, π, e^{π} and $\Gamma(1/4)$.

The main conjectures in this domain are due to S. Schanuel, A. Grothendieck, Y. André [8] and C. Bertolin [16]. The proof of the algebraic independence of π and e^{π} requires elliptic and modular functions. One may expect that higher dimensional objects (Abelian varieties, motives) should be involved to go further. In this respect we conclude by alluding to the remarkable progress which have been achieved recently in finite characteristic (after the works by Jing Yu, G.W. Anderson and D. Thakur, L. Denis, W.D. Brownawell, J.F. Voloch, M. Papanikolas among others).

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