More on Modifications and Improvements of Classical Iterative Schemes for $Z-$Matrices

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Abstract

In the last four decades many articles have been devoted to the modifications and improvements of classes of preconditioners for linear systems whose matrix coefficient is a $Z-$ or an $M-$ matrix in order to improve the convergence rates of the classical iterative schemes (Jacobi, Gauss-Seidel etc.). The present work is a contribution towards the generalization of the most common preconditioners used so far.

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1 Introduction and Preliminaries

Consider the linear system of algebraic equations

$$Ax = b, \tag{1.1}$$

where $A \in I\!\!R^{n,n}$ belongs to the class of $Z-$*matrices* (see [1]), that is its off-diagonal elements are nonpositive, $a_{ij} \leq 0,\ i,j = 1(1)n,\ j \neq i$, and $b \in I\!\!R^n$. In this work we restrict to *nonsingular* $M-$*matrices* (see [1] and also [12], [14]) that is to $Z-$matrices for which A^{-1} exists and $A^{-1} \geq 0$. Consider the usual *splitting* of A, namely

$$A = D - L - U, \tag{1.2}$$

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where D is a diagonal matrix and L and U are strictly lower and strictly upper triangular matrices, respectively. Since A is a nonsingular $M-$matrix, $a_{ii} > 0,\ i = 1(1)n$, where to simplify the notation we may assume that $a_{ii} = 1,\ i = 1(1)n$. Under the assumptions so far, the iterative methods of Jacobi and of Gauss-Seidel associated with (1.1) converge and by virtue of the Stein-Rosenberg theorem ([12], [14], [1]) they exclussively hold

$$\text{either} \quad \rho(H) = \rho(B) = 0 \quad \text{or} \quad 0 < \rho(H) < \rho(B) < 1, \tag{1.3}$$

with

$$B := L + U, \quad H := (I - L)^{-1}U, \tag{1.4}$$

the Jacobi and Gauss-Seidel iteration matrices, respectively, associated with $A = I - L - U$.

Many researchers have considered left preconditioners, applied to system (1.1) that made the associated Jacobi and Gauss-Seidel (type) methods converge asymptotically faster than the original ones. Such modifications or improvements based on prechosen preconditioners were considered by Milaszewicz [11] who, in turn, based his idea on previous ones (see, e.g., [10], [2], [7]), by Gunawardena et al [3], by Kohno et al [8] who extended the main idea in [3], and very recently by Li and Sun [9] who extended the class of matrices considered in [8]. The preconditioners considered in these works are presented below.

In a simpler form Milaszewicz [11] considered the preconditioner

$$P_1 \;\equiv\; I + S_1 \;=\; \begin{bmatrix} 1 & & & & & & \\ -a_{21} & 1 & & & & & \\ -a_{31} & & 1 & & & & \\ \vdots & & & \ddots & & & \\ -a_{i1} & & & & 1 & & \\ \vdots & & & & & \ddots & \\ -a_{n1} & & & & & & 1 \end{bmatrix}, \tag{1.5}$$

which is nothing but the elementary matrix used to eliminate the elements of the first column of A below the diagonal.

Gunawardena et al [3] considered as a preconditioner the matrix

$$P_2 \;\equiv\; I + S_2 \;=\; \begin{bmatrix} 1 & -a_{12} & & & & & \\ & 1 & -a_{23} & & & & \\ & & \ddots & \ddots & & & \\ & & & 1 & -a_{i,i+1} & & \\ & & & & \ddots & \ddots & \\ & & & & & 1 & -a_{n-1,n} \\ & & & & & & 1 \end{bmatrix} \tag{1.6}$$

whose effect on A is to eliminate the elements of the first upper diagonal.

Finally, Kohno et al [8] considered a modification of P_2, specifically

$$P_2(\alpha) \equiv I + S_2(\alpha) = \begin{bmatrix} 1 & -\alpha_1 a_{12} & & & & & & \\ & 1 & -\alpha_2 a_{23} & & & & & \\ & & \ddots & \ddots & & & & \\ & & & 1 & -\alpha_i a_{i,i+1} & & & \\ & & & & \ddots & \ddots & & \\ & & & & & 1 & -\alpha_{n-1} a_{n-1,n} \\ & & & & & & 1 \end{bmatrix}, \quad (1.7)$$

where $\alpha = [\alpha_1\ \alpha_2\ \ldots\ \alpha_{n-1}]^T \in I\!R^{n-1}$ has all its components nonnegative. Obviously, for $\alpha = 0 \in I\!R^{n-1}$, $P_2(\alpha) \equiv P_2(0) = I$, while for $\alpha = [1\ 1\ \ldots\ 1]^T \in I\!R^{n-1}$, $P_2(\alpha) \equiv P_2(1) = P_2$. The same preconditioner was considered in [9] for $\alpha_i \in [0,1]$, $i = 1(1)n-1$, with some extensions of the splitting (1.2) to more general Gauss-Seidel type matrices.

The outline of this work is as follows: In Section 2, we generalize Milaszewicz's method by using the idea in [8] and base our Jacobi and Gauss-Seidel iterative schemes on the ones in [11] and [7] (see also [10]), sufficient conditions on the $\alpha_i's$ are given that guarantee convergence and the "best", in some sense, set of the $\alpha_i's$ is found. In Section 3 analogous work to the one in Section 2 is done. In this way all previous results, including some of the ones in [9], are extended. In Section 4, numerical examples are presented in support of the theory developed. Finally, in Section 5, a number of points are made that may help in the direction of improving on the results of this work and a number of open questions are addressed.

2 Generalizing Milaszewicz's Iterative Schemes

The generalized preconditioner used in this case is of the form

$$P_1(\alpha) \equiv I + S_1(\alpha) = \begin{bmatrix} 1 & & & & & & \\ -\alpha_2 a_{21} & 1 & & & & & \\ -\alpha_3 a_{31} & & 1 & & & & \\ \vdots & & & \ddots & & & \\ -\alpha_i a_{i1} & & & & 1 & & \\ \vdots & & & & & \ddots & \\ -\alpha_n a_{n1} & & & & & & 1 \end{bmatrix}, \quad (2.1)$$

where $\alpha = [\alpha_2\ \alpha_3\ \ldots\ \alpha_n]^T \in I\!R^{n-1}$ with $\alpha_i,\ i = 2(1)n$, constants. The selection of $\alpha' s$ will be made from the $(n-1)-$dimensional nonnegative cone K_{n-1} so that none of the diagonal elements of the preconditioned matrix $P_1(\alpha)A$ vanishes as the point $(\alpha_2, \alpha_3, \ldots, \alpha_n)$ moves continuously away from $(1, 1, \ldots, 1)$.

The following sets of integers will be useful in our analysis.

$$N := \{1,\ 2,\ \ldots,\ n\},\ N_1 := N \backslash \{1\},\ N_2 := \{i \in N_1 :\ a_{i1} \neq 0\}. \quad (2.2)$$

Additional Assumption: In the matrices A we are considering we will assume that there is at least one pair of indices $i, j \in N_1$, such that $a_{i1}a_{1j} \neq 0$, for otherwise the original system could be reduced to one of $n-1$ equations with $n-1$ unknowns.

2.1 Convergence of the Jacobi and Gauss-Seidel Iterative Schemes

Applying $P_1(\alpha)$ on (1.1) we obtain the equivalent linear system

$$\widetilde{A}(\alpha)x = \widetilde{b}(\alpha), \quad \text{with} \quad \widetilde{A}(\alpha) = (I + S_1(\alpha))\, A,\ \widetilde{b}(\alpha) = (I + S_1(\alpha))\, b, \tag{2.3}$$

where, if needed, we will write

$$\widetilde{A}(\alpha) = \widetilde{D}(\alpha) - \widetilde{L}(\alpha) - \widetilde{U}(\alpha). \tag{2.4}$$

The elements $\widetilde{a}_{ij}(\alpha)$ of $\widetilde{A}(\alpha)$ are given by the relationships:

$$\widetilde{a}_{ij}(\alpha) = \begin{cases} a_{1j}, & i = 1,\ j \in N, \\ (1-\alpha_i)a_{i1}, & i \in N_1,\ j = 1, \\ a_{ij} - \alpha_i a_{i1} a_{1j}, & i, j \in N_1. \end{cases} \tag{2.5}$$

Requesting that $\widetilde{a}_{i1}(\alpha) = (1-\alpha_i)a_{i1} \leq 0,\ i \in N_1$, the nonpositivity of all the off-diagonal elements will be preserved and so will be the $Z-$character of $\widetilde{A}(\alpha)$. So, if $a_{i1} < 0$ then $\alpha_i \leq 1$. If $a_{i1} = 0$ then any value for α_i will do since the elements of the i^{th} row of A will remain unchanged To guarantee strict positivity for the diagonal elements we must have $\widetilde{a}_{ii}(\alpha) = 1 - \alpha_i a_{i1} a_{1i} > 0$, a condition covered by the previous one ($\alpha_i \leq 1$), since for a $Z-$matrix A the statement "A is a nonsingular $M-$matrix" is equivalent to the statement "all the principal minors of A are positive" (see Theorem 6.2.3, Condition (A_1) of [1]) implying $a_{i1}a_{1i} < 1,\ i \in N_1$.

In view of the discussion just made we restrict to $\alpha_i \in [0,1],\ i \in N_2$. The cases where some or all of the $\alpha_i' s,\ i \in N_2$, can be greater than 1 will be examined next.

Case I: $\alpha_i \in [0,1],\ i \in N_2$. Defining the matrices

$$D_\alpha := \text{diag}(0,\ \alpha_2 a_{21} a_{12},\ \ldots,\ \alpha_n a_{n1} a_{1n}) \ \text{ and } \ S_1(\alpha)U := L_\alpha + D_\alpha + U_\alpha, \tag{2.6}$$

where L_α, U_α the strictly lower and strictly upper triangular components of $S_1(\alpha)U$, then, from (2.5), the restrictions on the elements of A and on the $\alpha_i' s$, and the preceding discussion, we have that the three matrices on the right hand side of (2.4) are given by

$$\widetilde{D}(\alpha) = I - D_\alpha,\ \widetilde{L}(\alpha) = (I + S_1(\alpha))\, L - S_1(\alpha) + L_\alpha,\ \widetilde{U}(\alpha) = U + U_\alpha. \tag{2.7}$$

The diagonal elements of $\widetilde{D}(\alpha)$ are positive while those of $\widetilde{L}(\alpha)$ and $\widetilde{U}(\alpha)$ are nonnegative.

For the needs of one of our main statements the following splittings will be considered:

$$\widetilde{A}(\alpha) = \begin{cases} M(\alpha) - N(\alpha) &= (I + S_1(\alpha)) - (I + S_1(\alpha))\,(L+U), \\ M'(\alpha) - N'(\alpha) &= I - \left((I + S_1(\alpha))\, L - S_1(\alpha) + L_\alpha + D_\alpha + U + U_\alpha\right), \\ M''(\alpha) - N''(\alpha) &= (I - D_\alpha) - \left((I + S_1(\alpha))\, L - S_1(\alpha) + L_\alpha + U + U_\alpha\right). \end{cases} \tag{2.8}$$

Below we define the Jacobi and the Jacobi type iteration matrices associated with the above splittings as well as the corresponding Gauss-Seidel and Gauss-Seidel type ones:

$$\begin{aligned}
&B(\alpha) \equiv B := M^{-1}(\alpha)N(\alpha) = L + U,\\
&B'(\alpha) := M'^{-1}(\alpha)N'(\alpha) = (I + S_1(\alpha))\, L - S_1(\alpha) + D_\alpha + L_\alpha + U + U_\alpha,\\
&\widetilde{B}(\alpha) \equiv B''(\alpha) := M''^{-1}(\alpha)N''(\alpha) = (I - D_\alpha)^{-1}\left((I + S_1(\alpha))\, L - S_1(\alpha) + L_\alpha + U + U_\alpha\right),
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
&H(\alpha) \equiv H := (I - L)^{-1}U,\\
&H'(\alpha) := \left((I + S_1(\alpha))\,(I - L) - L_\alpha\right)^{-1}(D_\alpha + U + U_\alpha),\\
&\widetilde{H}(\alpha) \equiv H''(\alpha) := \left((I + S_1(\alpha))\,(I - L) - D_\alpha - L_\alpha\right)^{-1}(U + U_\alpha).
\end{aligned} \tag{2.10}$$

Theorem 2.1 *a) Under the assumptions and the notation so far, for any $\alpha \in K_{n-1}$ such that $\alpha_i \in [0,1],\ i \in N_2$, there hold:*
There exists $y \in I\!R^n$, with $y \geq 0$, such that

$$B'(\alpha)y \leq By, \tag{2.11}$$

$$\rho\left(\widetilde{B}(\alpha)\right) \leq \rho\left(B'(\alpha)\right) < 1, \tag{2.12}$$

$$\rho\left(\widetilde{H}(\alpha)\right) \leq \rho\left(H'(\alpha)\right) \leq \rho(H) < 1, \tag{2.13}$$

$$\rho\left(\widetilde{H}(\alpha)\right) \leq \rho\left(\widetilde{B}(\alpha)\right), \quad \rho\left(H'(\alpha)\right) \leq \rho\left(B'(\alpha)\right), \quad \rho(H) < \rho(B) < 1. \tag{2.14}$$

(Note: Equalities in (2.14) hold if and only if $\rho(B) = 0$.)
b) Suppose that A is irreducible. Then,
i) For $\alpha_i \in [0,1),\ i \in N_2$, provided that $\alpha \neq 0$, the matrices $\widetilde{B}(\alpha)$, $B'(\alpha)$ and B are irreducible and all the inequalities in (2.12)-(2.14) are strict. Moreover, there holds

$$\rho\left(B'(\alpha)\right) \leq \rho(B). \tag{2.15}$$

ii) For $\alpha_i = 1, i \in N_2$, the $(n-1)\times(n-1)$ matrices $B'_1(1)$ and $\widetilde{B}_1(1)$ of the bottom right corner of $B'(1)$ and $\widetilde{B}(1)$ are irreducible and all the inequalities in (2.12)-(2.15) are strict.

Proof: a) (2.11): To prove (2.11) we need the explicit expressions of the nonnegative elements of the two Jacobi and Jacobi type iteration matrices involved. Below we give the elements for all three matrices in the splittings (2.9):

$$\begin{cases} b_{ii} = 0, & i \in N,\\ b_{ij} = -a_{ij}, & i,j \in N,\ j \neq i, \end{cases} \tag{2.16}$$

$$\begin{cases}
b'_{ii}(\alpha) = \alpha_i a_{i1} a_{1i} = \alpha_i b_{i1} b_{1i}, & i \in N_2,\\
b'_{ii}(\alpha) = 0, & i \in N \backslash N_2,\\
b'_{ij}(\alpha) = -a_{ij} = b_{ij}, & i \in N\backslash N_2,\ j \in N_1,\ j \neq i,\\
b'_{i1}(\alpha) = (\alpha_i - 1)a_{i1} = (1-\alpha_i)b_{i1}, & i \in N_2,\\
b'_{ij}(\alpha) = \alpha_i a_{i1} a_{1j} - a_{ij} = \alpha_i b_{i1} b_{1j} + b_{ij}, & i \in N_2,\ j \in N_1,\ j \neq i,
\end{cases} \tag{2.17}$$

and

$$\begin{cases} \widetilde{b}_{ii}(\alpha) = 0, & i \in N, \\ \widetilde{b}_{ij}(\alpha) = -a_{ij} = b_{ij}, & i \in N \backslash N_2,\ j \in N_1,\ j \neq i, \\ \widetilde{b}_{i1}(\alpha) = \frac{(\alpha_i - 1)a_{i1}}{1-\alpha_i a_{i1} a_{1i}} = \frac{(1-\alpha_i)b_{i1}}{1-\alpha_i b_{i1} b_{1i}}, & i \in N_2, \\ \widetilde{b}_{ij}(\alpha) = \frac{\alpha_i a_{i1} a_{1j} - a_{ij}}{1-\alpha_i a_{i1} a_{1i}} = \frac{\alpha_i b_{i1} b_{1j} + b_{ij}}{1-\alpha_i b_{i1} b_{1i}}, & i \in N_2,\ j \in N_1,\ j \neq i. \end{cases} \tag{2.18}$$

For the nonnegative Jacobi iteration matrix B there exists a nonnegative vector y such that $By = \rho(B)y$. Equating the i^{th} rows, for $i \in N_2$, of the two vectors and replacing the elements b_{ij} of B in terms of the elements $b'_{ij}(\alpha)$ of $B'(\alpha)$ using (2.16) and (2.17) we successively obtain

$$\begin{aligned} \rho(B)y_i &= \textstyle\sum_{j=1, j\neq i}^{n} b_{ij} y_j = b_{i1} y_1 + \sum_{j=2, j\neq i}^{n} b_{ij} y_j \\ &= (b'_{i1}(\alpha) + \alpha_i b_{i1}) y_1 + \textstyle\sum_{j=2, j\neq i}^{n} (b'_{ij}(\alpha) - \alpha_i b_{i1} b_{1j}) y_j + b'_{ii}(\alpha) y_i - b'_{ii}(\alpha) y_i \\ &= \textstyle\sum_{j=1}^{n} b'_{ij}(\alpha) y_j - \alpha_i b_{i1} \sum_{j=2, j\neq i}^{n} b_{1j} y_j - \alpha_i b_{i1} b_{1i} y_i + \alpha_i b_{i1} y_1 \\ &= \textstyle\sum_{j=1}^{n} b'_{ij}(\alpha) y_j - \alpha_i b_{i1} \sum_{j=2}^{n} b_{1j} y_j + \alpha_i b_{i1} y_1. \end{aligned} \tag{2.19}$$

Using the fact that $\rho(B)y_1 = \sum_{j=2}^{n} b_{1j} y_j$ and replacing in (2.19) we have that

$$\rho(B) y_i = \sum_{j=1}^{n} b'_{ij}(\alpha) y_j + \alpha_i b_{i1} \left(\frac{1}{\rho(B)} - 1 \right) \sum_{j=2}^{n} b_{1j} y_j. \tag{2.20}$$

Since the second term on the sum in (2.20) is nonnegative we have that

$$\sum_{j=1}^{n} b'_{ij}(\alpha) y_j \leq \sum_{j=1}^{n} b_{ij} y_j \tag{2.21}$$

from which (2.11) follows.

a) (2.12): For a $Z-$matrix A the statement "A is a nonsingular $M-$matrix" is equivalent to the statement "there exists a positive vector $y(>0) \in I\!R^n$, that is $y_i > 0,\ i \in N$, such that $Ay > 0$" (see Theorem 6.2.3, Condition (I_{27}) of [1]). But $P_1(\alpha) = I + S_1(\alpha) \geq 0$, implies $\widetilde{A}(\alpha)y = P_1(\alpha)Ay > 0$. Consequently, $\widetilde{A}(\alpha)$, which **is** a $Z-$matrix, is a nonsingular $M-$matrix. So, the last two splittings in (2.8) are regular splittings because $M'^{-1}(\alpha) = I^{-1} = I \geq 0$, $N'(\alpha) \geq 0$ and $M''^{-1}(\alpha) = (I - D_\alpha)^{-1} \geq 0$, $N''(\alpha) \geq 0$ and so they are convergent. Since $M''^{-1}(\alpha) \geq M'^{-1}(\alpha)$, it is implied (see [13]) that the left inequality in (2.12) is true.

a) (2.13): To prove the inequalities (2.13) we use regular splittings of the matrix $\widetilde{A}(\alpha)$. Specifically, consider the following splittings that define the iteration matrices in (2.10):

$$\widetilde{A}(\alpha) = \begin{cases} M(\alpha) - N(\alpha) &= (I + S_1(\alpha))(I - L) - (I + S_1(\alpha))U, \\ M'(\alpha) - N'(\alpha) &= ((I + S_1(\alpha))(I - L) - L_\alpha) - (D_\alpha + U + U_\alpha), \\ M''(\alpha) - N''(\alpha) &= ((I + S_1(\alpha))(I - L) - L_\alpha - D_\alpha) - (U + U_\alpha), \end{cases} \tag{2.22}$$

where we have used the same symbols for the two matrices of each splitting as in the case of (2.8). The matrix $M(\alpha) = (I + S_1(\alpha))(I - L)$ of the first splitting is a lower triangular one with units on the diagonal, elements of the first column $(1 - \alpha_i)a_{i1},\ i \in N_1$, and remaining elements of the strictly lower triangular part $a_{ij},\ i \in N_1\backslash\{2\}, j \in N_1, j < i$. So all the off-diagonal elements of $M(\alpha)$ are nonpositive and therefore $M(\alpha)$ is a nonsingular $Z-$matrix. On the other hand, we have

$$\begin{aligned} M^{-1}(\alpha) &= (I-L)^{-1}\,(I+S_1(\alpha))^{-1} \\ &= (I+L+L^2+\cdots+L^{n-2}+L^{n-1})\,(I-S_1(\alpha)) \\ &= I+(I+L+L^2+\cdots+L^{n-2})(L-S_1(\alpha)) \end{aligned} \tag{2.23}$$

since $L^{n-1}S_1(\alpha)=0$. But since $L \geq 0$ and also $L-S_1(\alpha) \geq 0$ it is implied that $M^{-1}(\alpha) \geq 0$. Consequently $M(\alpha)$ is a nonsingular $M-$matrix. Also, $(I+S_1(\alpha))\,U \geq 0$, so the first splitting in (2.22) is a regular one. Similarly, $M'(\alpha)$ of the second splitting is a nonsingular $Z-$matrix. Since $M'(\alpha)$ can be written as $M'(\alpha) = M(\alpha) - L_\alpha = M(\alpha)(I - M^{-1}(\alpha)L_\alpha)$, we can set $\overline{L} = M^{-1}(\alpha)L_\alpha \geq 0$ and have

$$M'^{-1}(\alpha) = (I-\overline{L})^{-1}M^{-1}(\alpha) = \left(I+\overline{L}+\overline{L}^2+\cdots+\overline{L}^{n-1}\right)M^{-1}(\alpha) \geq 0. \tag{2.24}$$

Consequently $M'(\alpha)$ is a nonsingular $M-$matrix. But $N'(\alpha) = D_\alpha + U + U_\alpha \geq 0$ and so the second splitting in (2.22) is also a regular one. Finally, the last splitting is a regular one since $\widetilde{A}(\alpha)$ is a nonsingular $M-$matrix and so is $M''(\alpha)$ since the latter is derived from the former by setting some off-diagonal elements equal to zero (see Theorem 3.12 of [12]) and $N''(\alpha) = U + U_\alpha \geq 0$. The inequalities in (2.13) are readily established because we can simply notice that $N(\alpha) = L_\alpha + D_\alpha + U + U_\alpha \geq N'(\alpha) = D_\alpha + U + U_\alpha \geq N''(\alpha) = U + U_\alpha$.
a) (2.14): Since A is a nonsingular $M-$matrix, the rightmost inequality is a straightforward implication of the Stein-Rosenberg Theorem as was mentioned before. The other two inequalities in (2.14) are implied directly by the facts that $\widetilde{A}(\alpha)$ is a nonsingular $M-$matrix, and the last two pairs of splittings in (2.8) and (2.22), from which the four matrices involved, $\widetilde{H}(\alpha),\ \widetilde{B}(\alpha),\ H'(\alpha),\ B'(\alpha)$, are produced, are regular ones with $(I+S_1(\alpha))\,L - S_1(\alpha) + L_\alpha + U + U_\alpha \geq U + U_\alpha$ and $(I+S_1(\alpha))\,L - S_1(\alpha) + L_\alpha + D_\alpha + U + U_\alpha \geq D_\alpha + U + U_\alpha$.
b) For $\alpha_i \in [0,1)$, $\widetilde{A}(\alpha)$ is obviously irreducible because it inherits the nonzero structure of the irreducible matrix A.
bi) (2.12)-(2.15): By virtue of the irreducible character of the corresponding matrices involved, the theorems used previously are also applied to prove the strict inequalities in (2.12)-(2.14), while (2.15) is proved in Theorem 2.2 of [11].
bii) We consider the block partitionings

$$A = \left[\begin{array}{c|c} 1 & a_h^T \\ \hline a_v & A_1 \end{array}\right], \quad P_1(1) = \left[\begin{array}{c|c} 1 & 0_{n-1}^T \\ \hline -a_v & I_1 \end{array}\right], \quad \widetilde{A}(1) = \left[\begin{array}{c|c} 1 & a_h^T \\ \hline 0_{n-1} & \widetilde{A}_1(1) \end{array}\right]. \tag{2.25}$$

Then the associated block Jacobi and Gauss-Seidel iteration matrices will be

$$B = \left[\begin{array}{c|c} 0 & -a_h^T \\ \hline -a_v & B_1 \end{array}\right], \quad B'(1) = \left[\begin{array}{c|c} 0 & -a_h^T \\ \hline 0_{n-1} & B_1'(1) \end{array}\right], \quad \widetilde{B}(1) = \left[\begin{array}{c|c} 0 & -a_h^T \\ \hline 0_{n-1} & \widetilde{B}_1(1) \end{array}\right], \tag{2.26}$$

and

$$H = \left[\begin{array}{c|c} 0 & -a_h^T \\ \hline 0_{n-1} & H_1 \end{array}\right], \quad H'(1) = \left[\begin{array}{c|c} 0 & -a_h^T \\ \hline 0_{n-1} & H_1'(1) \end{array}\right], \quad \widetilde{H}(1) = \left[\begin{array}{c|c} 0 & -a_h^T \\ \hline 0_{n-1} & \widetilde{H}_1(1) \end{array}\right], \tag{2.27}$$

respectively.
bii) (2.12)-(2.15): By studying the structure of the matrices B_1, $B_1'(1)$, $\widetilde{B}_1(1)$, H_1, $H_1'(1)$ and $\widetilde{H}_1(1)$ we can find out that the associated irreducibility properties will hold for these matrices. So, the theorems used previously are also applied in each case to prove the strict inequalities (2.12)-(2.14) while (2.15) is proved in Theorem 2.2 of [11]. □

If all or some of the $\alpha_i's$, $i \in N_2$, can take values greater than 1, then some of the elements of the Jacobi iteration matrix $\widetilde{B}(\alpha)$ can be negative and so the theory used in Case I can not be directly applied. This forces us to restrict the class of the invertible $M-$matrices A to the class of strictly diagonally dominant (SDD) and the class of irreducibly diagonally dominant (IDD) matrices. At this point some further notation and terminology are introduced.

For the splitting (1.2), the *companion matrix* of $A \in \mathbb{C}^{n,n}$, denoted by $\mathcal{M}(A)$, is the matrix

$$\mathcal{M}(A) = |D| - |L| - |U|, \tag{2.28}$$

where $|.|$ is the matrix whose elements are the moduli of the elements of the given matrix. $A \in \mathbb{C}^{n,n}$ is said to be an $H-matrix$ if and only if its companion matrix is an $M-$matrix. Also, for reasons which will become clear soon, we define the quantities below

$$\begin{array}{l} d_i = |a_{ii}|, \ i \in N, \quad l_1 = 0, \quad l_i = \sum_{j=1}^{i-1} |a_{ij}|, \ i \in N_1, \\ u_i = \sum_{j=i+1}^{n} |a_{ij}|, \ i \in N \backslash \{n\}, \quad u_n = 0, \end{array} \tag{2.29}$$

and assume that

$$|l_i| + |u_i| > 0, \ i \in N, \tag{2.30}$$

so that in each row there is at least one off-diagonal element different from zero; for otherwise we would practically have a linear system of $n-1$ equations with $n-1$ unknowns to solve.

As in [8], we set

$$p_i = a_{i1}a_{1i}, \quad q_i = a_{i1}\sum_{j=1}^{i-1} a_{1j}, \quad r_i = a_{i1}\sum_{j=i+1}^{n} a_{1j}, \tag{2.31}$$

so

$$p_i + q_i + r_i = a_{i1}\sum_{j=1}^{n} a_{1j} = a_{i1}(1 - l_1 - u_1) \le 0, \ i \in N_2. \tag{2.32}$$

Note: Strict inequality can be assumed in (2.32) because there always exists an i for which $1 - l_i - u_i > 0$ and a similarity permutation transformation interchanging the first and the i^{th} rows and columns does not destroy any of the characteristics of the original matrix.

For the Jacobi and the Gauss-Seidel iteration matrices, $\widetilde{B}(\alpha)$ and $\widetilde{H}(\alpha)$, to converge sufficient conditions are (see, e.g., [5] and [6]),

$$\rho\left(\widetilde{B}(\alpha)\right) \ \le \ \max_{i \in N} \frac{\widetilde{l}_i(\alpha) + \widetilde{u}_i(\alpha)}{\widetilde{d}_i(\alpha)} < 1, \quad \rho\left(\widetilde{H}(\alpha)\right) \ \le \ \max_{i \in N} \frac{\widetilde{u}_i(\alpha)}{\widetilde{d}_i(\alpha) - \widetilde{l}_i(\alpha)} < 1, \tag{2.33}$$

which will be used whenever needed.

Case II: $\alpha_i > 1,\ i \in N_2$. In the sequel we examine the case where the $\alpha_i' s$ are allowed to take values greater than 1 without destroying the positivity of the diagonal elements $\tilde{a}_{ii}(\alpha)$ and preserving at the same time for each row of the companion matrix $\mathcal{M}(\tilde{A}(\alpha))$ of $\tilde{A}(\alpha)$, at least, the inequalities that the corresponding row of A satisfies.

For $\tilde{a}_{ii}(\alpha) > 0,\ i \in N_2$, to hold there must be $\alpha_i a_{i1} a_{1i} < 1$ because for $i \in N \backslash N_2$, $\tilde{a}_{ii}(\alpha) = a_{ii} = 1 > 0$. Since A is an $M-$matrix, and therefore $a_{i1}a_{1i} < 1$, $\tilde{a}_{ii}(\alpha) > 0$ implies

$$\alpha_i \in \begin{cases} \left(1, \frac{1}{a_{i1}a_{1i}}\right) & \text{if } a_{i1}a_{1i} \neq 0, \\ & \qquad\qquad i \in N_2. \\ (1, \infty) & \text{if } a_{i1}a_{1i} = 0, \end{cases} \tag{2.34}$$

On the other hand, if (2.34) hold then

$$\tilde{a}_{ij}(\alpha) \begin{cases} > 0 & \text{if } i = j \in N, \\ \geq 0 & \text{if } i \in N_1,\ j = 1, \\ \leq 0 & \text{if } i \in N_1,\ j \in N \backslash \{1, i\}. \end{cases} \tag{2.35}$$

In the present case it is

$$\begin{array}{ll} \tilde{d}_i(\alpha) = |\tilde{a}_{ii}(\alpha)| = & \tilde{a}_{ii}(\alpha) = 1 - \alpha_i a_{i1} a_{1i} = 1 - \alpha_i p_i, \\ \tilde{l}_i(\alpha) = \sum_{j=1}^{i-1} |\tilde{a}_{ij}(\alpha)| = & \tilde{a}_{i1}(\alpha) - \sum_{j=2}^{i-1} \tilde{a}_{ij}(\alpha) = a_{i1} - \alpha_i a_{i1} + \alpha_i a_{i1} \sum_{j=2}^{i-1} a_{1j} \\ & - \sum_{j=2}^{i-1} a_{ij} = 2a_{i1} + l_i - 2\alpha_i a_{i1} + \alpha_i q_i, \\ \tilde{u}_i(\alpha) = \sum_{j=i+1}^{n} |\tilde{a}_{ij}(\alpha)| = & - \sum_{j=i+1}^{n} \tilde{a}_{ij}(\alpha) = \alpha_i a_{i1} \sum_{j=i+1}^{n} a_{1j} - \sum_{j=i+1}^{n} a_{ij} = \alpha_i r_i + u_i, \end{array} \tag{2.36}$$

hence

$$\begin{array}{rl} \tilde{d}_i(\alpha) - \tilde{l}_i(\alpha) - \tilde{u}_i(\alpha) = & 1 - \alpha_i p_i - (2a_{i1} + l_i - 2\alpha_i a_{i1} + \alpha_i q_i) - (\alpha_i r_i + u_i) = \\ & 1 - l_i - u_i - 2a_{i1} + \alpha_i a_{i1}(1 + l_1 + u_1). \end{array} \tag{2.37}$$

Requiring to always have $\tilde{d}_i(\alpha) > \tilde{l}_i(\alpha) + \tilde{u}_i(\alpha)$ we obtain $\alpha_i(-a_{i1})(1+l_1+u_1) < 1-l_i-u_i-2a_{i1}$. For $i \in N_2$, since $1 + l_1 + u_1 > 0$, there must be

$$\alpha_i < \frac{1 - l_i - u_i - 2a_{i1}}{(-a_{i1})(1 + l_1 + u_1)}, \tag{2.38}$$

provided that the expression in (2.38) is greater than 1. However,

$$\frac{1 - l_i - u_i - 2a_{i1}}{(-a_{i1})(1 + l_1 + u_1)} \geq \frac{-2a_{i1}}{(-a_{i1})(1 + l_1 + u_1)} = \frac{2}{1 + l_1 + u_1} > 1. \tag{2.39}$$

If $a_{1i} = 0$ then the upper bound for the value of α_i is not ∞ as (2.34) indicates but the expression given in (2.38). If, on the other hand, $a_{1i} \neq 0$ then the upper bound for the value of α_i should be the smallest out of $\frac{1}{a_{i1}a_{1i}}$ in (2.34) and the one in (2.38) above. But

$$\frac{1}{a_{i1}a_{1i}} - \frac{1 - l_i - u_i - 2a_{i1}}{(-a_{i1})(1 + l_1 + u_1)} = \frac{(1 - a_{i1}a_{1i}) + (l_1 + u_1 + a_{1i}) + (-a_{1i})(l_i + u_i + a_{i1})}{a_{i1}a_{1i}(1 + l_1 + u_1)} > 0,$$

since all the terms in the numerator are nonnegative with the first one being strictly positive. So, from (2.38)

$$\alpha_i \in I_i := \left(1, \frac{1 - l_i - u_i - 2a_{i1}}{(-a_{i1})(1 + l_1 + u_1)}\right), \; i \in N_2. \tag{2.40}$$

Note: In [8] it is said that values of $\alpha_i \geq 1,\ i \in N_2$, exist that make the corresponding $\widehat{u}_i(\alpha_i)'s$ be zero. However, this can not happen, as can be readily proved, except for only α_{n-1} in which case $\alpha_{n-1} = 1$.

After the analysis just given it is implied that the companion matrix $\mathcal{M}(\widetilde{A}(\alpha))$ of $\widetilde{A}(\alpha)$ for all $\alpha_i \in I_i,\ i \in N_2$, is a $Z-$matrix with the same nonzero pattern as the original matrix A. Also, it has row sums positive even in cases where A had corresponding sums zero. Consequently, $\mathcal{M}(\widetilde{A}(\alpha))$ is a nonsingular $M-$matrix. Also, if A is irreducible, so is $\mathcal{M}(\widetilde{A}(\alpha))$.

In this present Case II we observe that in view of (2.35) the off-diagonal elements of $\widetilde{A}(\alpha)$ are **not** all nonpositive. This simply suggests that a direct conclusion regarding the convergence of the iteration matrices considered in Case I can **not** be drawn. So, we turn our attention to the companion matrix $\mathcal{M}(\widetilde{A}(\alpha))$ of $\widetilde{A}(\alpha)$ and its associated Jacobi, Jacobi type, Gauss-Seidel and Gauss-Seidel type iteration matrices associated with it. First, we consider the main splitting of $\mathcal{M}(\widetilde{A}(\alpha))$ as in (2.4) for $\widetilde{A}(\alpha)$, which, because $\widetilde{D}(\alpha),\ \widetilde{U}(\alpha) \geq 0$, is

$$\mathcal{M}(\widetilde{A}(\alpha)) = \widetilde{D}(\alpha) - |\widetilde{L}(\alpha)| - \widetilde{U}(\alpha), \tag{2.41}$$

We consider the Jacobi iteration matrix associated with $\mathcal{M}(\widetilde{A}(\alpha))$ given by

$$|\widetilde{B}(\alpha)| := \widetilde{D}(\alpha)^{-1}\left(|\widetilde{L}(\alpha)| + \widetilde{U}(\alpha)\right),$$

whose elements are all nonnegative and are given by the expressions

$$|\widetilde{b}_{ij}(\alpha)| = \begin{cases} 0 & \text{if} \quad i = j \in N, \\ -a_{ij} = b_{ij} & \text{if} \quad i \in N \backslash N_2,\ j \in N_1,\ j \neq i, \\ \frac{a_{ij} - \alpha_i a_{ij}}{1 - \alpha_i a_{i1} a_{1i}} = \frac{(\alpha_i - 1) b_{ij}}{1 - \alpha_i b_{i1} b_{1i}} & \text{if} \quad i \in N_2,\ j = 1, \\ \frac{-a_{ij} + \alpha_i a_{i1} a_{1j}}{1 - \alpha_i a_{i1} a_{1i}} = \frac{b_{ij} + \alpha_i b_{i1} b_{1j}}{1 - \alpha_i b_{i1} b_{1i}} & \text{if} \quad i \in N_2,\ j \in N_1,\ j \neq i. \end{cases} \tag{2.42}$$

Here we give a lemma which will be used in some of the proofs in the sequel.

Lemma 2.1 *The sign of the derivative of a function of z of the form $\frac{az+b}{cz+d}$, a, b, c, d constants with $|c| + |d| \neq 0$, is the same as that of the quantity $ad - bc$. Specifically,*

$$\text{sign}\left(\frac{\partial}{\partial z}\left(\frac{az + b}{cz + d}\right)\right) = \text{sign}(ad - bc). \tag{2.43}$$

Proof: We readily get $\frac{\partial}{\partial z}\left(\frac{az+b}{cz+d}\right) = \frac{ad-bc}{(cz+d)^2}$ from which (2.43) follows directly. □

Based on the lemma just given we can find how the elements of the Jacobi iteration matrix associated with $\mathcal{M}(\widetilde{A}(\alpha))$ vary as any of the $\alpha_i,\ i \in N_2$, increases from 1 onwards. Applying Lemma 2.1 directly to the elements of $|\widetilde{B}(\alpha)|$ given in (2.42) we have

$$\begin{array}{c} \text{sign}\left(\frac{\partial|\widetilde{b}_{i1}(\alpha)|}{\partial\alpha_i}\right) = \text{sign}\left(b_{i1}(1-b_{i1}b_{1i})\right) = 1,\ i \in N_2, \\ \text{sign}\left(\frac{\partial|\widetilde{b}_{ij}(\alpha)|}{\partial\alpha_i}\right) = \text{sign}\left(b_{i1}(b_{1j}+b_{1i}b_{ij})\right) = 1,\ i \in N_2,\ j \in N_1,\ j \neq i. \end{array} \tag{2.44}$$

From (2.44) we have some results which are stated in the statements below. For their proof the reader is referred to [4].

Theorem 2.2 *a) For any two distinct $\alpha, \alpha' \in I\!R^{n-1}$, with components $\alpha_i, \alpha_i' \in I_i,\ i \in N_2$, such that $\alpha_i \leq \alpha_i'$, with $[1\ 1\ \cdots\ 1]^T \leq \alpha = [\alpha_2\ \alpha_3\ \cdots\ \alpha_n]^T \leq \alpha' = [\alpha_2'\ \alpha_3'\ \cdots\ \alpha_n']^T$, and where $\alpha_i = \alpha_i' = 1,\ i \in N_1 \backslash N_2$, we have*

$$\rho\left(\widetilde{B}(1)\right) \equiv \rho\left(|\widetilde{B}(1)|\right) \leq \rho\left(|\widetilde{B}(\alpha)|\right) \leq \rho\left(|\widetilde{B}(\alpha')|\right) < 1. \tag{2.45}$$

b) Moreover, except for some very special cases (see [4]), increasing a certain $\alpha_i \in I_i,\ i \in N_2$, for which $a_{1i} < 0$, from 1 onwards, with all the other α_i's remaining fixed, there exists a value of the α_i in question, denoted by $\widehat{\alpha}_i$, strictly to the right of the interval I_i and less than $\frac{1}{a_{i1}a_{1i}}$, such that

$$\rho\left(\widetilde{B}([\alpha_2\ \alpha_3\ \cdots\ \widehat{\alpha}_i\ \cdots\ \alpha_n]^T)\right) = 1. \tag{2.46}$$

c) If A is irreducible and α and α' are as in part (a), with $\alpha_i, \alpha_i' \in I_i,\ i \in N_2$, then the inequalities in (2.45) are strict.

Theorem 2.3 *Let $H\left(\mathcal{M}(\widetilde{A}(\alpha))\right)$ denote the Gauss-Seidel iteration matrix associated with $\mathcal{M}(\widetilde{A}(\alpha))$. Then*
a) For any $\alpha \in I\!R^{n-1}$, with components $\alpha_i \in I_i,\ i \in N_2$, there hold

$$\rho\left(H\left(\mathcal{M}(\widetilde{A}(\alpha))\right)\right) \leq \rho\left(|\widetilde{B}(\alpha)|\right) < 1. \tag{2.47}$$

b) For any two distinct $\alpha, \alpha' \in I\!R^{n-1}$, with components $\alpha_i, \alpha_i' \in I_i,\ i \in N_2$, such that $\alpha_i \leq \alpha_i'$, with $[1\ 1\ \cdots\ 1]^T \leq \alpha = [\alpha_2\ \alpha_3\ \cdots\ \alpha_n]^T \leq \alpha' = [\alpha_2'\ \alpha_3'\ \cdots\ \alpha_n']^T$, and where $\alpha_i = \alpha_i' = 1,\ i \in N_1 \backslash N_2$, we have

$$\rho(\widetilde{H}(1)) \equiv \rho\left(H\left(\mathcal{M}(\widetilde{A}(1))\right)\right) \leq \rho\left(H\left(\mathcal{M}(\widetilde{A}(\alpha))\right)\right) \leq \rho\left(H\left(\mathcal{M}(\widetilde{A}(\alpha'))\right)\right) < 1. \tag{2.48}$$

c) Moreover, except for some very special cases as in Theorem 2.2, increasing a certain $\alpha_i \in I_i,\ i \in N_2$, for which $a_{1i} < 0$, from 1 onwards, with all the other α_i's remaining fixed, there exists a value of the α_i in question, denoted by $\widehat{\alpha}_i$, strictly to the right of the interval I_i and strictly less than $\frac{1}{a_{i1}a_{1i}}$, such that

$$\rho\left(H\left(\mathcal{M}(\widetilde{A}([\alpha_2\ \alpha_3\ \cdots\ \widehat{\alpha}_i\ \cdots\ \alpha_n]^T))\right)\right) = 1. \tag{2.49}$$

d) If A is irreducible and α and α' are as in part (b) with $\alpha_i, \alpha_i' \in I_i \backslash \{1\},\ i \in N_2$, then the inequalities in (2.47) and (2.48) are strict.

2.2 "Best" Convergent Jacobi and Gauss-Seidel Iterative Schemes

Out of the convergent Jacobi and Gauss-Seidel iterative schemes derived in the previous Section 2.1 and for the nonsingular $M-$matrices A considered in Cases I and II one may wonder which of the permissible $\alpha \in I\!R^{n-1}$ gives the fastest iterative schemes. To find such an optimal α seems to be a very difficult problem.

In Case II, by Theorems 2.2 and 2.3 we can conclude that for the matrices A dealt with there if $\alpha_i \in I_i,\ i \in N_2$, the "best" Jacobi and Gauss-Seidel type schemes, which are associated with the companion matrix $\mathcal{M}(\widetilde{A}(\alpha))$ of A, are the ones with $\alpha = [1\ 1\ \ldots\ 1]^T \in I\!R^{n-1}$.

In Case I, for the general class of nonsingular $M-$matrices considered there a similar conclusion can not be drawn as was mentioned before. However, if we restrict to the classes of SDD or IDD matrices we may apply Lemma 2.1 for either the associated Jacobi or the Gauss-Seidel iteration matrix and try to minimize the upper bounds of their spectral radii.

For the Jacobi method: The elements $\widetilde{a}_{ij}(\alpha),\ i \in N_2, j \in N$, are given in (2.5), where the diagonal ones are positive and the off-diagonal ones are nonnegative, therefore

$$\begin{array}{ll} \widetilde{d}_i(\alpha) = |\widetilde{a}_{ii}(\alpha)| = & \widetilde{a}_{ii}(\alpha) = 1 - \alpha_i a_{i1} a_{1i} = 1 - \alpha_i p_i, \\ \widetilde{l}_i(\alpha) = \sum_{j=1}^{i-1} |\widetilde{a}_{ij}(\alpha)| = & -\sum_{j=1}^{i-1} \widetilde{a}_{ij}(\alpha) = \alpha_i a_{i1} \sum_{j=1}^{i-1} a_{1j} - \sum_{j=1}^{i-1} a_{ij} = \alpha_i q_i + l_i, \\ \widetilde{u}_i(\alpha) = \sum_{j=i+1}^{n} |\widetilde{a}_{ij}(\alpha)| = & -\sum_{j=i+1}^{n} \widetilde{a}_{ij}(\alpha) = \alpha_i a_{i1} \sum_{j=i+1}^{n} a_{1j} - \sum_{j=i+1}^{n} a_{ij} = \alpha_i r_i + u_i. \end{array} \tag{2.50}$$

Applying Lemma 2.1 to minimize the expression $\frac{\widetilde{u}_i(\alpha)+\widetilde{l}_i(\alpha)}{\widetilde{d}_i(\alpha)}$, in (2.33), in view of (2.50), where $a = q_i + r_i$, $b = l_i + u_i$, $c = -p_i$ and $d = 1$ we have $(ad-bc)_i = a_{i1}(1-l_1-u_1)-(1-l_i-u_i) < 0$. Therefore the ratio in question is a strictly decreasing function of α_i meaning that the "best" value for all $\alpha_i's,\ i \in N_2$, in the sense explained, is 1. Consequently, for $\alpha = [1\ 1\ \cdots\ 1]^T \in I\!R^{n-1}$ one obtains the "best" Jacobi method.

For the Gauss-Seidel method: For the application of Lemma 2.1 one has to minimize $\frac{\widetilde{u}_i(\alpha)}{\widetilde{d}_i(\alpha)-\widetilde{l}_i(\alpha)}$, in (2.33). Now, $a = r_i$, $b = u_i$, $c = -(p_i + q_i)$ and $d = 1 - l_i$. Therefore, $(ad-bc)_i = a_{i1}(1-l_i)\left(\frac{u_i}{1-l_i} - u_1\right)$. Obviously, for $i \in N_2$, $\text{sign}\,((ad-bc)_i) = \text{sign}\left(u_1 - \frac{u_i}{1-l_i}\right)$, which can be $+1,\ -1$ or 0, while if $n \in N_2$, $\text{sign}\,((ad-bc)_n) = +1$, since $u_n = 0$. Consequently, depending on the sign of the expression $(ad-bc)_i$ for each $i \in N_2$, the ratio we are interested in minimizing can be a strictly increasing or strictly decreasing function of the corresponding α_i or can be constant. So, the "best" value for a certain $\alpha_i,\ i \in N_2$, can be either 0 or 1 and the values of $\alpha_i's$ obtained in this way give the "best" Gauss-Seidel method.

3 Generalizing Kohno et al's Iterative Schemes

The generalized preconditioner we use in this section is that in (1.7). Our analysis follows the steps of Section 2 and extends the existing theory in various directions. First, we consider the Jacobi and Jacobi type methods associated with the preconditioned matrix A, $\widehat{A}(\alpha) = P_2(\alpha)A$, where $\alpha = [\alpha_1\ \alpha_2\ \cdots\ \alpha_{n-1}]^T \in I\!R^{n-1}$, $\alpha_i,\ i = 1(1)n-1$, extending the main theory in [3]. Next, we consider the Gauss-Seidel and Gauss-Seidel type iteration methods associated with $\widehat{A}(\alpha)$, extending and completing the theory developed in [3], [8] and [9]. Then, comparisons of the corresponding Jacobi and Gauss-Seidel type methods are made, regarding the spectral radii of their iteration matrices in case of convergence, and finally, an attempt to find the "best" of the schemes considered is presented. The sets of integers to be used and the *additional assumption* made in the previous section are redefined as:

$$N := \{1, 2, \cdots, n\},\ N_1 := N\backslash\{n\},\ N_2 := \{i \in N_1 :\ a_{i,i+1} \neq 0\}. \tag{3.1}$$

Additional Assumption: We assume that there exists a pair of indices $i \in N_2$ and $j \in N_1$ such that $a_{i,i+1}a_{i+1,j} \neq 0$, so that at least one element of $\widehat{A}(\alpha)$ is different from that of A.

3.1 Jacobi and Gauss-Seidel Type Iterative Schemes

Applying the new preconditioner $P_2(\alpha)$ on (1.1) we obtain a linear system which looks precisely like the one in (2.3) with the elements $\widehat{a}_{ij}(\alpha)$ of $\widehat{A}(\alpha)$ being given by

$$\widehat{a}_{ij}(\alpha) = \begin{cases} a_{ij} - \alpha_i a_{i,i+1}a_{i+1,j} & i \in N_1,\ j \in N_1\backslash\{i+1\}, \\ (1-\alpha_i)a_{i,i+1}, & i \in N_1,\ j = i+1, \\ a_{nj}, & i = n,\ j \in N. \end{cases} \tag{3.2}$$

Case I: $\alpha_i \in [0,1],\ i \in N_2$. Defining the matrices

$$\widehat{D}_\alpha := \mathrm{diag}(\ \alpha_1 a_{12}a_{21},\ \ldots,\ \alpha_{n-1}a_{n-1,n}a_{n,n-1},\ 0) \ \text{ and }\ S_2(\alpha)L := \widehat{L}_\alpha + \widehat{D}_\alpha, \tag{3.3}$$

where $\widehat{D}_\alpha$ and $\widehat{L}_\alpha$ the diagonal and the strictly lower triangular components of $S_2(\alpha)L$, then from (3.2) and the preceding discussion, we have that the three matrices on the right hand side of the corresponding to (2.4) relationships are given by

$$\widehat{D}(\alpha) = I - \widehat{D}_\alpha,\ \widehat{L}(\alpha) = L + \widehat{L}_\alpha,\ \widehat{U}(\alpha) = (I + S_2(\alpha))\,U - S_2(\alpha). \tag{3.4}$$

$U - S_2(\alpha)$ is nonnegative, the diagonal elements of $\widehat{D}(\alpha)$ are positive while $\widehat{L}(\alpha)$ and $\widehat{U}(\alpha)$ are nonnegative. In the sequel the following splittings will be considered:

$$\widehat{A}(\alpha) \ = \begin{cases} M(\alpha) - N(\alpha) & = \ (I + S_2(\alpha)) - (I + S_2(\alpha))\,(L + U), \\ M'(\alpha) - N'(\alpha) & = \ I - \left(L + \widehat{L}_\alpha + \widehat{D}_\alpha + (I + S_2(\alpha))\,U - S_2(\alpha)\right), \\ M''(\alpha) - N''(\alpha) & = \ \left(I - \widehat{D}_\alpha\right) - \left(L + \widehat{L}_\alpha + (I + S_2(\alpha))\,U - S_2(\alpha)\right). \end{cases} \tag{3.5}$$

Below, we define the Jacobi and the Jacobi type iteration matrices associated with the above splittings as well as the corresponding Gauss-Seidel and Gauss-Seidel type ones:

$$\begin{aligned} &B := M^{-1}(\alpha)N(\alpha) = L + U, \\ &\widehat{B}'(\alpha) := M'^{-1}(\alpha)N'(\alpha) = \left(L + \widehat{L}_\alpha + \widehat{D}_\alpha + (I + S_2(\alpha))\,U - S_2(\alpha)\right), \\ &\widehat{B}''(\alpha) := M''^{-1}(\alpha)N''(\alpha) = \left(I - \widehat{D}_\alpha\right)^{-1}\left(L + \widehat{L}_\alpha + (I + S_2(\alpha))\,U - S_2(\alpha)\right) \end{aligned} \tag{3.6}$$

and

$$
\begin{aligned}
&H := (I-L)^{-1}U,\\
&\widehat{H}'(\alpha) := \left(I - L - \widehat{L}_\alpha\right)^{-1}\left(\widehat{D}_\alpha + (I+S_2(\alpha))\,U - S_2(\alpha)\right),\\
&\widehat{H}''(\alpha) := \left(I - \widehat{D}_\alpha - L - \widehat{L}_\alpha\right)^{-1}\left((I+S_2(\alpha))\,U - S_2(\alpha)\right).
\end{aligned}
\tag{3.7}
$$

Theorem 3.1 *a) Under the assumptions and the notation so far, for any $\alpha \in K_{n-1}$, where K_{n-1} is the $(n-1)-$dimensional nonnegative cone, such that $\alpha_i \in [0,1]$, $i \in N_2$, there hold: There exists $y \in \mathbb{R}^n$, with $y \geq 0$, such that*

$$\widehat{B}'(\alpha)y \leq By, \tag{3.8}$$

$$\rho\left(\widehat{B}''(\alpha)\right) \leq \rho\left(\widehat{B}'(\alpha)\right) < 1, \tag{3.9}$$

$$\rho\left(\widehat{H}''(\alpha)\right) \leq \rho\left(\widehat{H}'(\alpha)\right) \leq \rho(H) < 1, \tag{3.10}$$

$$\rho\left(\widehat{H}''(\alpha)\right) \leq \rho\left(\widehat{B}''(\alpha)\right),\quad \rho\left(\widehat{H}'(\alpha)\right) \leq \rho\left(\widehat{B}'(\alpha)\right),\quad \rho(H) < \rho(B) < 1. \tag{3.11}$$

(Note: Equalities in (3.11) hold if and only if $\rho(B) = 0$.)
b) Suppose that A is irreducible. Then,
i) For $\alpha_i \in [0,1)$, $i \in N_2$, provided that $\alpha \neq 0$, the matrices $\widehat{B}''(\alpha)$, $\widehat{B}'(\alpha)$ and B are irreducible and all the inequalities in (3.9)-(3.11) are strict. Moreover, there holds

$$\rho\left(\widehat{B}'(\alpha)\right) \leq \rho(B). \tag{3.12}$$

ii) For $\alpha_i = 1$, $i \in N_2$, the matrices $\widehat{B}''(1)$, $\widehat{B}'(1)$ and B are irreducible and all the inequalities in (3.9)-(3.12) are strict.

Proof: a) (3.8): The expressions of the nonnegative elements of the Jacobi type iteration matrix $\widehat{B}'(\alpha)$ are the following:

$$
\begin{cases}
\widehat{b}'_{ii}(\alpha) = \alpha_i a_{i,i+1}a_{i+1,i} = \alpha_i b_{i,i+1}b_{i+1,i}, & i \in N_2,\\
\widehat{b}'_{ii}(\alpha) = 0, & i \in N\backslash N_2,\\
\widehat{b}'_{ij}(\alpha) = -a_{ij} = b_{ij}, & i \in N\backslash N_2,\ j \in N_1,\ j \neq i,\\
\widehat{b}'_{i,i+1}(\alpha) = (\alpha_i - 1)a_{i,i+1} = (1-\alpha_i)b_{i,i+1}, & i \in N_2,\\
\widehat{b}'_{ij}(\alpha) = \alpha_i a_{i,i+1}a_{i+1,j} - a_{ij} = \alpha_i b_{i,i+1}b_{i+1,j} + b_{ij}, & i \in N_2,\ j \in N_1,\ j \neq i, i+1.
\end{cases}
\tag{3.13}
$$

It is observed that (3.13) are exactly the same expressions as those in (2.17) with the index $i+1$ replacing 1. So, the proof of the part (a) of the present theorem follows the same lines as that of Theorem 2.1 and also use of regular splittings is made. The only difference is in the middle inequality of (3.10) which holds by Theorem 3.1 of [9].
bi) For $\alpha_i \in [0,1)$, if A is irreducible then so is $\widehat{A}(\alpha)$, since the nonzero structure of A is inherited by $\widehat{A}(\alpha)$, as is seen from (3.13). The irreducibility of $\widehat{A}(\alpha)$ and the way of proof in Theorem 2.1 guarantee that the inequalities in (3.9)-(3.12) are strict.

bii) The irreducibility of $\widehat{A}(1)$ from that of A is proved as follows: From (3.2) it is implied that if an element $a_{ij},\ j \neq i+1$, of A is nonzero then so is $\widehat{\alpha}_{ij}(1)$ of $\widehat{A}(1)$. If $a_{ij} = 0$ then there will be a path in the graph $G(A)$ of A joining the node i with the node j. Let this path consist of the edjes joining the concecutive nodes $i,\ i_1,\ \cdots,\ i_p,\ i_q,\ i_r\ \cdots,\ i_s,\ j$. Then $a_{i,i_1} \cdots a_{i_p,i_q} a_{i_q,i_r} \cdots a_{i_s,j} \neq 0$. If none of the nodes of the path is the node $i+1$ then, because $\widehat{a}_{i,i_1}(1) \cdots \widehat{a}_{i_p,i_q}(1) \widehat{a}_{i_q,i_r}(1) \cdots \widehat{a}_{i_s,j}(1) \neq 0$, the same path will be present in the graph $G(\widehat{A}(1))$ of $\widehat{A}(1)$. If, however, one of the nodes is the node $q = i+1$, then because $a_{i_p,i_q} a_{i_q,i_r} \neq 0$, it will be $\widehat{a}_{i_p,i_r} \neq 0$, and therefore $\widehat{a}_{i,i_1}(1) \cdots \widehat{a}_{i_p,i_r}(1) \cdots \widehat{a}_{i_s,j}(1) \neq 0$, implying that in the graph of $\widehat{A}(1)$, there is still a path joining the nodes i and j. Consequently, $\widehat{A}(1)$ is irreducible. From this irreducibilty that of the three Jacobi and Jacobi type natrices readily follows and from the latter follows the strictness of the inequalities in (3.9)–(3.11). □

In cases where all of the $\alpha_i' s,\ i \in N_2$, can take values greater than 1, we observe that some of the elements of the Jacobi matrix $\widehat{B}(\alpha)$ can be negative and so to study those cases we adopt the analysis in Section 2. Analogously to (2.28)-(2.30) we set

$$\widehat{p}_i = a_{i,i+1} a_{i+1,i} > 0, \quad \widehat{q}_i = a_{i,i+1} \sum_{j=1}^{i-1} a_{i+1,j} \geq 0, \quad \widehat{r}_i = a_{i,i+1} \sum_{j=i+1}^{n} a_{i+1,j} \leq 0, \tag{3.14}$$

and so we have

$$\widehat{p}_i + \widehat{q}_i + \widehat{r}_i = a_{i,i+1} \sum_{j=1}^{n} a_{i+1,j} = a_{i,i+1}(1 - l_{i+1} - u_{i+1}) \leq 0,\ i \in N_2. \tag{3.15}$$

For the Jacobi and the Gauss-Seidel iteration matrices, $\widehat{B}(\alpha)$ and $\widehat{H}(\alpha)$, to converge to converge we consider the sufficient conditions

$$\rho\left(\widehat{B}(\alpha)\right) \ \leq\ \max_{i \in N} \frac{\widehat{l}_i(\alpha) + \widehat{u}_i(\alpha)}{\widehat{d}_i(\alpha)} < 1, \quad \rho\left(\widehat{H}(\alpha)\right) \ \leq\ \max_{i \in N} \frac{\widehat{u}_i(\alpha)}{\widehat{d}_i(\alpha) - \widehat{l}_i(\alpha)} < 1. \tag{3.16}$$

Case II: $\alpha_i > 1,\ i \in N_2$. An analogous analysis for the companion matrix $\mathcal{M}(\widehat{A}(\alpha))$ of $\widehat{A}(\alpha)$ as in Case II of Section 2 can be done here as well. After the analysis, for which the reader is referred to [4], one can arrive at the following intervals for the $\alpha_i' s$,

$$\alpha_i \in I_i := \left(1, \frac{1 - l_i - u_i - 2a_{i,i+1}}{(-a_{i,i+1})(1 + l_{i+1} + u_{i+1})}\right),\ i \in N_2. \tag{3.17}$$

In view of (3.17) statements analogous to Theorems 2.2 and 2.3 can be stated and proved for the matrices $B\left(\mathcal{M}(\widehat{A}(\alpha))\right)$ and $H\left(\mathcal{M}(\widehat{A}(\alpha))\right)$.

3.2 "Best" Convergent Jacobi and Gauss-Seidel Schemes

As in Section 2, successive applications of Lemma 2.1 to the associated ratios, for the Jacobi type and the Gauss-Seidel type methods, give as a final result that the "best" value for all $\alpha_i' s$ for both methods is 1.

4 Numerical Examples

For over 10,000 randomly generated nonsingular $M-$matrices for $n = 5$ to $n = 100$ we determined the spectral radii of the iteration matrices of Jacobi, Gauss-Seidel as well as those of the corresponding iteration matrices after the application of the Milaszewicz's preconditioner, the Gunawardena et al's one and, also, the successive application of the two preconditioners. Below we present four representative example matrices for which the spectral radii of the corresponding iteration matrices considered are given in the subsequent Table. In the Table J and GS denote Jacobi and Gauss-Seidel type methods while the indices M and G denote that Milaszewicz's preconditioner and Gunawardena et al's were used, respectively. $M-G$ means that the application of Milaszewicz's preconditioner was used first followed by Gunawardena et al's, while $G-M$ means the reverse situation.

$$A_1 = \begin{bmatrix} 1.00000 & -0.00580 & -0.19350 & -0.25471 & -0.03885 \\ -0.28424 & 1.00000 & -0.16748 & -0.21780 & -0.21577 \\ -0.24764 & -0.26973 & 1.00000 & -0.18723 & -0.08949 \\ -0.13880 & -0.01165 & -0.25120 & 1.00000 & -0.13236 \\ -0.25809 & -0.08162 & -0.13940 & -0.04890 & 1.00000 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1.00000 & -0.15359 & -0.24342 & -0.02303 & -0.03363 \\ -0.01756 & 1.00000 & -0.00630 & -0.14703 & -0.18174 \\ -0.01087 & -0.03714 & 1.00000 & -0.25258 & -0.17673 \\ -0.12507 & -0.01414 & -0.07603 & 1.00000 & -0.14130 \\ -0.00515 & -0.24496 & -0.23477 & -0.27707 & 1.00000 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1.00000 & -0.27149 & -0.20650 & -0.02972 & -0.12557 \\ -0.12416 & 1.00000 & -0.18328 & -0.07729 & -0.25528 \\ -0.31163 & -0.02827 & 1.00000 & -0.15184 & -0.39463 \\ -0.12292 & -0.00477 & -0.23299 & 1.00000 & -0.20115 \\ -0.37067 & -0.09086 & -0.20368 & -0.30835 & 1.00000 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1.00000 & -0.23661 & -0.37369 & -0.25833 & -0.05480 \\ -0.13602 & 1.00000 & -0.10578 & -0.38675 & -0.32750 \\ -0.12569 & -0.01525 & 1.00000 & -0.26597 & -0.17207 \\ -0.14603 & -0.18344 & -0.34914 & 1.00000 & -0.35613 \\ -0.15730 & -0.34795 & -0.09515 & -0.00397 & 1.00000 \end{bmatrix}.$$

Matrix	Method	Spectral Radius
A_1	J_M	0.553502
	J_G	0.584773
	J_{G-M}	0.524133
A_2	J_M	0.460575
	J_G	0.418960
	J_{M-G}	0.245048
A_3	GS_M	0.480367
	GS_G	0.497869
	GS_{M-G}	0.351696
A_4	GS_M	0.622791
	GS_G	0.568660
	GS_{M-G}	0.490150

From all the examples we run, we can make the following observations: When each of the two preconditioners is used alone in connection with the Jacobi method in approximately 50% of the cases Gunawardena et al's (G) preconditioner gave better results than Milaszewicz's (M). However, when they were used with the Gauss-Seidel method G preconditioner gave better results in approximately 70% of the cases. The successive application of the two preconditioners, as one might have expected from the theory developed, always gave better results than the application of either one of them. For the Jacobi method the application of the two preconditioners in the order $M-G$ gives better results than in the order $G-M$ in approximately 65% of the cases while for the Gauss-Seidel method and in more than 90% of the cases the $M-G$ preconditioner gives better results. The percentages given seem to increase in almost all the cases as the order of the matrix increases from 5 to 100.

5 Concluding Remarks and Discussion

As one may have noticed from our analysis many questions are raised directly or indirectly. For example: Can one prove a monotone behavior in either of the two methods used when any of the $\alpha_i' s$ strictly increases in $[0,1]$? Can the intervals of convergence for the $\alpha_i' s$ be extended further for the entire class of nonsingular $M-$matrices? Can "optimal" values for the $\alpha' s$ be obtained theoretically? Can one prove theoretically some of the "facts" that the numerical evidence provide? Can the two preconditioners be exploited further without increasing the computational complexity of the problem solved? Can these or similar preconditioners be used efficiently with other classes of matrices? These and many other issues have being studied and partial answers to some of them have already been given.

References

[1] A. Berman and R.J. Plemmons *Nonnegative Matrices in the Mathematical Sciences. Classics in Applied Mathematics.* SIAM, Philadelphia, 1994.

[2] R.E. Funderlic and R.J. Plemmons, *LU Decomposition of M-Matrices by Elimination Without Pivoting.* Linear Algebra Appl. 41 (1981), 99–110.

[3] A.D. Gunawardena, S.K. Jain and L. Snyder, *Modified Iterative Methods for Consistent Linear Systems.* Linear Algebra Appl. 154–156 (1991), 123–143.

[4] A. Hadjidimos, D. Noutsos and M. Tzoumas, *More on Modifications and Improvements of Classical Iterative Schemes for $Z-$Matrices.* Preprint no. 01-1/2001, Department of Mathematics, University of Crete, GR-714 09 Heraklion, Greece, 2001.

[5] K.R. James, *Convergence of Matrix Iterations Subject to Diagonal Dominance.* SIAM J. Numer. Anal. 10 (1973), 478–484.

[6] K.R. James and W. Riha, *Convergence Criteria for Successive Overrelaxation.* SIAM J. Numer. Anal. 12 (1975), 137–143.

[7] M.L. Juncosa and T.W. Mulliken, *On the Increase of Convergence Rates of Relaxation Procedures for Elliptic Partial Differential Equations.* J. Assoc. Comput. Mach. 7 (1960), 29–36.

[8] T. Kohno, H. Kotakemori, H. Niki and M. Usui, *Improving the Gauss-Seidel Method for Z-Matrices.* Linear Algebra Appl. 267 (1997), 113–123.

[9] W. Li and W. Sun, *Modified Gauss-Seidel type methods and Jacobi type methods for Z-matrices.* Linear Algebra Appl. 317 (2000), 227–240.

[10] J.P. Milaszewicz, *On Modified Jacobi Linear Operators*, Linear Algebra Appl. 51 (1983), 127–136.

[11] J.P. Milaszewicz, *Improving Jacobi and Gauss-Seidel Iterations*, Linear Algebra Appl. 93 (1987), 161–170.

[12] R.S. Varga, *Matrix Iterative Analysis.* Prentice-Hall, Englewood Cliffs, NJ, 1962. (Also: 2nd Edition, Revised and Expanded, Springer, Berlin, 2000.)

[13] Z. Woźnicki, *Nonnegative Splitting Theory*, Japan Journal of Industrial and Applied Mathematics 11 (1994), 289–342.

[14] D.M. Young, *Iterative Solution of Large Linear Systems.* Academic Press, New York, 1971.