Block Gauss Elimination Followed by a Classical Iterative Method for the Solution of Linear Systems

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Abstract

In the last four decades many papers have appeared in which the application of an iterative method for the solution of a linear system is preceded by a step of the Gauss elimination process in the hope that this will increase the rates of convergence of the iterative method. This combination of methods has been proven successful especially when the matrix A of the system is an $M-$matrix. The purpose of this paper is to extend the idea of one to more Gauss elimination steps, consider other classes of matrices A, e.g., $p-$cyclic consistently ordered, and generalize and improve some of the results known so far.

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1 Introduction and Preliminaries

For the numerical solution of the linear system

$$Ax = b, \quad A \in \mathbb{C}^{n,n}, \ b \in \mathbb{C}^n, \tag{1.1}$$

where $A \ (= (a_{ij}), \ i,j = 1(1)n)$, we apply a number of Gauss elimination steps followed by an iterative method. The idea of applying one elimination step preceding an iterative method has been given by Juncosa and Mulliken [10]. Based on this idea and on some similar ones (see, e.g., [18], [19] and [4]), Milaszewicz [15], [16], [17] improved the then known results. Similar works, in a different direction, by Gunawardena, Jain and Snyder [6], Kohno et al [11] and recently by Li and Sun [12] and Hadjidimos, Noutsos and Tzoumas [8] appeared.

To perform the elimination most of the researchers used left preconditioners on the linear system (1.1) and then applied a Jacobi or a Gauss-Seidel type iterative method to the preconditioned system. Specifically, Milaszewicz [17] considered, essentially, the preconditioner

$$P_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -a_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & 0 & \dots & 1 \end{bmatrix} \tag{1.2}$$

to eliminate the elements of the first column below the diagonal of the nonsingular $M-$matrix A. The preconditioner of Gunawardena, Jain and Snyder [6] eliminates the elements of the first upper diagonal of the (non)singular $M-$matrix A. Kohno et al [11], Li and Sun [12] and Hadjidimos, Noutsos and Tzoumas [8] introduced parameters in the above preconditioners to accelerate the convergence of the subsequent iterative method. It is noted that in [12] and [8] regular, weak regular and $M-$splittings (see, e.g., [23]) were considered to make comparisons between the spectral radii of the various iteration matrices involved.

In all the previous works one step of Gauss elimination was applied followed by a "point" iterative method. In the present work we apply more than one elimination steps followed by a "block" iterative method. Specifically, in Section 2, a block partitioning of the nonsingular $M-$matrix A is given, where

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1p} \\ A_{21} & A_{22} & \dots & A_{2p} \\ \vdots & \vdots & \ddots & \\ A_{p1} & A_{p2} & \dots & A_{pp} \end{bmatrix}, \tag{1.3}$$

and $A_{ij} \in \mathbb{C}^{n_i,n_j}, \ i,j = 1(1)p, \ \sum_{i=1}^{p} n_i = n$, and $\det(A_{ii}) \neq 0, \ i = 1(1)p$. To (1.1) we apply a block preconditioner P that eliminates the first n_1 columns of A below its diagonal. In Section 3, A is block $p-$cyclic consistently ordered and it is proved that applying P to A is equivalent to a block cyclic repartitioning from the $p-$cyclic to the $(p-1)-$cyclic case. So, problems that researchers, like Markham, Neumann and Plemmons [14], Pierce [20], Pierce, Hadjidimos and Plemmons [21], Eiermann, Niethammer and Ruttan [3], Galanis and Hadjidimos [5] and Hadjidimos and Plemmons [9] dealt with, reappear. In Section 4, the case of a singular A is discussed. Finally, in Section 5, some numerical examples are presented.

2 Nonsingular $M-$matrices

2.1 Basic theory

Consider the partitioning (1.3) for the nonsingular $M-$matrix A. It is known that the diagonal blocks $A_{ii} \in I\!\!R^{n_i,n_i}$, $i = 1(1)p$, of A are nonsingular $M-$matrices while the off-diagonal blocks $A_{ij} \in I\!\!R^{n_i,n_j}$, $i \neq j = 1(1)p$, are nonpositive matrices ($A_{ij} \leq 0$, $i \neq j = 1(1)p$) (see [2]). (Note: It is reminded that a matrix $A \in I\!\!R^{n,n}$ is a $Z-$matrix if all its off-diagonal elements are nonpositive. A $Z-$matrix A is an $M-$ matrix if $A = sI - B$, where $s \geq \rho(B)$, $s > 0$ and $B \geq 0$ (see, e.g., [2]), with $\rho(.)$ denoting spectral radius.) In [2], it is said that in the triangular decomposition of a nonsingular $M-$matrix A $(= LU)$, L, U are lower and upper triangular matrices, have positive diagonal elements and are nonsingular $M-$matrices. We may assume that L has unit diagonal elements.

For our results the following lemma by Funderlic and Plemmons [4] is used.

Lemma 2.1 *Let $A = (a_{ij}) \in I\!\!R^{n,n}$, $n \geq 2$, be a nonsingular $M-$matrix, and let*

$$L_1^{-1} = \begin{bmatrix} 1 & 0 & & \\ \frac{-a_{21}}{a_{11}} & 1 & & \\ \vdots & \vdots & \ddots & \\ \frac{-a_{n1}}{a_{11}} & 0 & \dots & 1 \end{bmatrix}.$$

*Then, the matrices $\widetilde{A} = {L_1}^{-1}A$ and $\widetilde{A}_1$, which is obtained from $\widetilde{A}$ by deleting its first row and column, are nonsingular $M-$matrices. (**Note:** If A is irreducible then so is $\widetilde{A}_1$ and if, in addition, A is singular then so are $\widetilde{A}$ and $\widetilde{A}_1$.)*

Based on Lemma 2.1 we prove our first result.

Theorem 2.1 *Let $A \in I\!\!R^{n,n}$ be a nonsingular $M-$matrix partitioned as in (1.3). Then n_1 successive applications of the Gauss elimination process on A are equivalent to premultiplying A by the (preconditioning) matrix*

$$P = \begin{bmatrix} L_{11}^{-1} & O_{12} & \dots & O_{1p} \\ -A_{21}A_{11}^{-1} & I_{22} & \dots & O_{2p} \\ \vdots & \vdots & \ddots & \\ -A_{p1}A_{11}^{-1} & O_{p2} & \dots & I_{pp} \end{bmatrix} = Q + S, \tag{2.1}$$

$$Q = \operatorname{diag}(L_{11}^{-1}, I_{22}, \dots, I_{pp}) \geq 0, \quad I_{ii} \in I\!\!R^{n_i,n_i}, \ i = 2(1)p, \tag{2.2}$$

$$S = \begin{bmatrix} O_{11} & O_{12} & \dots & O_{1p} \\ -A_{21}A_{11}^{-1} & O_{22} & \dots & O_{2p} \\ \vdots & \vdots & \ddots & \\ -A_{p1}A_{11}^{-1} & O_{p2} & \dots & O_{pp} \end{bmatrix} \geq 0, \tag{2.3}$$

with L_{11} being the lower triangular matrix in the LU triangular decomposition of A_{11}. Moreover, $\widetilde{A} = PA$ and the matrix $\widetilde{A}_1$, obtained from $\widetilde{A}$ by deleting its first n_1 rows and columns, are also nonsingular $M-$matrices. (**Note:** *If A is irreducible then so is $\widetilde{A}_1$ while if A is, in addition, singular then so are $\widetilde{A}$ and $\widetilde{A}_1$.)*

Proof: We apply Gauss elimination to the first n_1 columns of A. By Lemma 2.1 each elimination step, $k = 1(1)n_1$, yields a matrix $\widetilde{A}^{(k)}$ ($\widetilde{A}^{(0)} = A$) which is a nonsingular $M-$matrix and whose the bottom right corner submatrix $\widetilde{A}_{n_1+1-k} \in I\!R^{n-k,n-k}$, $k = 1(1)n_1$, is also a nonsingular $M-$matrix. So, $\widetilde{A} := \widetilde{A}^{(n_1)}$ and $\widetilde{A}_1$ are nonsingular $M-$matrices. (Note: By the same lemma, if A is irreducible then so will be $\widetilde{A}_1$ and if A is, in addition, singular so will be $\widetilde{A}$ and $\widetilde{A}_1$.) To express the above process in matrix form, note that in the k^{th} elimination step, $k = 1(1)n_1$, we multiply the $\widetilde{A}^{(k-1)}$ on the left by a lower triangular matrix with units on the diagonal and only nonzero elements in its k^{th} column. The product P of all these n_1 matrices will be a lower triangular matrix with units on the diagonal and only nonzero elements on the first n_1 columns. So, P, in a form consistent with A, will be

$$P = \begin{bmatrix} P_{11} & O_{12} & \dots & O_{1p} \\ P_{21} & I_{22} & \dots & O_{2p} \\ \vdots & \vdots & \ddots & \\ P_{p1} & O_{p2} & \dots & I_{pp} \end{bmatrix}, \tag{2.4}$$

where I_{jj}, $j = 2(1)p$, is the unit matrix of order n_j. To determine the block elements of P, we use $PA = \widetilde{A}$, and note that $\widetilde{A}_{ij} = P_{i1}A_{1j} + A_{ij}$, $i, j = 1(1)p$. Since $\widetilde{A}_{i1} = O_{i1}$, $i = 2(1)p$, then $P_{i1} = -A_{i1}A_{11}^{-1}$, $i = 2(1)p$. Observe that the first n_1 diagonal elements of $\widetilde{A}$ are the pivots of the elimination process. Since, after the elimination, $\widetilde{A}_{11}$ has zeros below its diagonal we conclude that $\widetilde{A}_{11}$ is the matrix U_{11} yielded after the elimination is applied to A_{11}. Thus, if $L_{11}U_{11}$ is the LU factorization of A_{11}, $P_{11} = L_{11}^{-1}$ and the proof is complete. •

Note: The nonnegativity of Q and S in (2.2) and (2.3) is based on the fact that the inverse of the nonsingular $M-$ matrices L_{11} and A_{11} are nonnegative matrices.

Corollary 2.1 *The application of P, of Theorem (2.1), to A of (1.3) results the matrix $\widetilde{A}$ whose block elements are given by the following expressions:*

$$\widetilde{A}_{ij} = \begin{cases} L_{11}^{-1}A_{1j}, & j = 1(1)p, \\ O_{i1}, & i = 2(1)p, \\ A_{ij} - A_{i1}A_{11}^{-1}A_{1j}, & i, j = 2(1)p. \end{cases} \tag{2.5}$$

Using the preconditioner P we premultiply system (1.1) to obtain equivalently

$$\widetilde{A}x = \widetilde{b} \; (= Pb), \tag{2.6}$$

where $x^T = [x_1 \; x_2 \; \dots \; x_p]$, $\widetilde{b}^T = [\widetilde{b}_1 \; \widetilde{b}_2 \; \dots \; \widetilde{b}_p]$, $x_i, \widetilde{b}_i \in I\!R^{n_i}$, $i = 1(1)p$, and $\widetilde{b}_1 = L_{11}^{-1}b_1$, $\widetilde{b}_i = b_i - A_{i1}A_{11}^{-1}b_1$, $i = 2(1)p$. In this work, for the solution of (2.6) we consider a classical block iterative method applied to (2.6) or a classical block iterative method applied to

$$\widetilde{A}_1[x_2^T \; x_3^T \; \dots \; x_p^T]^T = [\widetilde{b}_2^T \; \widetilde{b}_3^T \; \dots \; \widetilde{b}_p^T]^T, \tag{2.7}$$

followed by a back substitution applied to

$$U_{11}x_1 \;=\; \widetilde{b}_1. \tag{2.8}$$

2.2 Jacobi and Gauss-Seidel type iterative methods

To solve (2.6) using a classical iterative method we consider various splittings of $\widetilde{A}$. For this we define the following matrices in a way analogous to the point case in [8]:

$$SU = \widehat{L} + \widehat{D} + \widehat{U}, \tag{2.9}$$

where

$$\widehat{D} = \mathrm{diag}(O_{11}, A_{21}A_{11}^{-1}A_{12}, \cdots, A_{p1}A_{11}^{-1}A_{1p})\ (\geq 0), \tag{2.10}$$

$$\widehat{L} = \begin{bmatrix} O_{11} & O_{12} & \dots & O_{1p} \\ O_{21} & O_{22} & \dots & O_{2p} \\ O_{31} & A_{31}A_{11}^{-1}A_{12} & \dots & O_{3p} \\ \vdots & \vdots & \ddots & \vdots \\ O_{p1} & A_{p1}A_{11}^{-1}A_{12} & \dots & O_{pp} \end{bmatrix} (\geq 0), \tag{2.11}$$

$$\widehat{U} = \begin{bmatrix} O_{11} & O_{12} & O_{13} & \dots & O_{1p} \\ O_{21} & O_{22} & A_{21}A_{11}^{-1}A_{13} & \dots & A_{21}A_{11}^{-1}A_{1p} \\ O_{31} & O_{32} & O_{33} & \dots & A_{31}A_{11}^{-1}A_{1p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_{p1} & O_{p2} & O_{p3} & \dots & O_{pp} \end{bmatrix} (\geq 0). \tag{2.12}$$

Let

$$A = D - L - U, \tag{2.13}$$

$$D = \mathrm{diag}(A_{11}, A_{22}, \dots, A_{pp}),$$

$$L = \begin{bmatrix} O_{11} & O_{12} & \dots & O_{1p} \\ -A_{21} & O_{22} & \dots & O_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{p1} & -A_{p2} & \dots & O_{pp} \end{bmatrix}, \quad U = \begin{bmatrix} O_{11} & -A_{12} & \dots & -A_{1p} \\ O_{21} & O_{22} & \dots & -A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ O_{p1} & O_{p2} & \dots & O_{pp} \end{bmatrix},$$

then having in mind (2.9) and (2.13), we consider the following splittings of $\widetilde{A}$:

$$\widetilde{A} = (Q+S)(D-L-U) = \begin{cases} QD - (PL - SD + \widehat{L} + \widehat{D} + QU + \widehat{U}), \\ (QD - \widehat{D}) - (PL - SD + \widehat{L} + QU + \widehat{U}). \end{cases} \tag{2.14}$$

The block Jacobi and Gauss-Seidel as well as the block Jacobi and Gauss-Seidel type iteration matrices associated with the two splittings in (2.14) are:

$$B = D^{-1}(L+U)\ (\text{for } A) \tag{2.15}$$

$$B' = (QD)^{-1}(PL - SD + \widehat{L} + \widehat{D} + QU + \widehat{U}) \tag{2.16}$$

$$B'' = (QD - \widehat{D})^{-1}(PL - SD + \widehat{L} + QU + \widehat{U}) \tag{2.17}$$

$$H = (D - L)^{-1}U \text{ (for } A) \tag{2.18}$$

$$H' = (P(D - L) - \widehat{L})^{-1}(\widehat{D} + QU + \widehat{U}) \tag{2.19}$$

$$H'' = (P(D - L) - \widehat{L} - \widehat{D})^{-1}(QU + \widehat{U}). \tag{2.20}$$

Theorem 2.2 *Under the notation and the definitions so far, suppose that A in (1.3) is a nonsingular $M-$matrix and let $\rho(B) > 0$. Let B_1', B_1'', H_1, H_1', H_1'' denote the $(n - n_1) \times (n - n_1)$ bottom right corner submatrices of B', B'', H, H', H'', respectively. Then the following relationships hold:*

$$\rho(B_1'') \equiv \rho(B'') \leq \rho(B') \equiv \rho(B_1') < 1, \tag{2.21}$$

$$\rho(H_1'') \equiv \rho(H'') \leq \rho(H') \equiv \rho(H_1') \leq \rho(H) \equiv \rho(H_1) < 1. \tag{2.22}$$

Also, there exists a vector $y \in \mathbb{R}^n$, with $y \geq 0$, such that

$$B'y \leq By. \tag{2.23}$$

Moreover, the spectral radii of the iteration matrices involved satisfy the relationships below:

$$\rho(H'') \leq \rho(B''), \quad \rho(H') \leq \rho(B'), \quad 0 < \rho(H) < \rho(B) < 1. \tag{2.24}$$

If, in addition, A is irreducible, then all the inequalities in (2.21)-(2.24) are strict. Also, in (2.23), $y > 0$, implying that

$$\rho(B_1') \;\equiv\; \rho(B') \;<\; \rho(B). \tag{2.25}$$

Proof: By Theorem 2.1, $\widetilde{A}$ and $\widetilde{A}_1$ are $M-$matrices and if A is irreducible then so is $\widetilde{A}_1$. Based on properties of $M-$matrices ([23], [2]) and on the fact that $Q,\ S,\ \widehat{L},\ \widehat{U} \geq 0$, we conclude that $D,\ QD,\ QD - \widehat{D},\ D - L,\ P(D - L) - \widehat{L}$ and $P(D - L) - \widehat{L} - \widehat{D}$ are nonsingular $M-$matrices. Also, the second matrix factors in (2.15)-(2.20) are nonnegative. Therefore the splittings from which the iterative matrices $B,\ B',\ B'',\ H,\ H'$ and H'' are produced are $M-$splittings [22] hence all these matrices are convergent. From this point onwards the proof of the present theorem duplicates that in [8], where all the parameters $\alpha_i,\ i = 2(1)n$, in it, are equal to 1. The difference is that instead of "point" we deal with "block" iteration matrices. So, as in [8], use of the Perron-Frobenius theory for nonnegative matrices [23] and of regular splittings, weak regular splittings [2] and $M-$splittings [22] is made. The complete proof can be found in [1], is very long and so is not possible to be given here. •

The results of the "block" case of Theorem 2.2 can be compared with the ones of the corresponding "point" case in [17]. In the statement below we show that ours are better.

Theorem 2.3 *Under the notation and the definitions used in Theorem 2.2, suppose that A is a nonsingular $M-$matrix. Let $B'^{(k)}$, $B''^{(k)}$, $H'^{(k)}$ and $H''^{(k)}$, $k = 1(1)n_1$, denote the "point" iteration matrices (Jacobi and Gauss-Seidel type) associated with the matrix $\widetilde{A}^{(k)}$ ($\widetilde{A}^{(0)} = A$) of Theorems 2.1 and 2.2 after the k^{th} elimination step $k = 1(1)n_1$. Let also $B^{(0)}$, $H^{(0)}$ be the point Jacobi and Gauss-Seidel iteration matrices associated with A. Then, there will hold*

$$\rho(H'') \ \leq \ \rho(H''^{(n_1)}) \ \leq \ \rho(H''^{(1)}) \ \leq \ \rho(H^{(0)}) \ (< 1). \tag{2.26}$$

If, in addition, A is irreducible, then there will also hold

$$\rho(B'') \ < \ \rho(B''^{(n_1)}) \ < \ \rho(B''^{(1)}) \ < \ \rho(B^{(0)}) \ (< 1), \tag{2.27}$$

and all the inequalities in (2.26) will be strict.

Proof: Let us apply the preconditioners P_1, in (1.2), and P, in (2.1), to (1.1). The application of P_1 is equivalent to the first Gauss elimination step of Lemma 2.1. So, according to the notation of Theorem 2.1, $P_1 A \equiv \widetilde{A}^{(1)}$. For the corresponding matrices $B'^{(1)}$, $B''^{(1)}$, $H'^{(1)}$ and $H''^{(1)}$ (the point Jacobi and Gauss-Seidel type matrices) relationships analogous to those in (2.21)-(2.25) hold ([17] and [8]). Specifically, we note that $\rho(H''^{(1)}) \leq \rho(H^{(0)}) < 1$ and that if A is irreducible the leftmost inequality will be strict and there will also hold that $\rho(B''^{(1)}) < \rho(B^{(0)}) < 1$. So, after each elimination step the produced matrix $\widetilde{A}^{(k)}$ will have point Jacobi and Gauss-Seidel iteration matrices $B''^{(k)}$ and $H''^{(k)}$, $k = 1(1)n_1$, whose spectral radii will be connected with those of the previous step in the same way the corresponding spectral radii after the first elimination step are connected with those of the initial point Jacobi, $B^{(0)}$, and Gauss-Seidel, $H^{(0)}$, iteration matrices. Therefore, induction shows that

$$\rho(H''^{(n_1)}) \ \leq \ \rho(H''^{(1)}) \ (< 1). \tag{2.28}$$

If A is irreducible inequality (2.28) is strict and the following relationships will also hold

$$\rho(B''^{(n_1)}) \ < \ \rho(B''^{(1)}) \ (< 1). \tag{2.29}$$

Observe that after the n_1^{th} elimination step we apply "point" Jacobi and Gauss-Seidel iterative methods to $\widetilde{A}^{(n_1)}$, which is nothing but the matrix $\widetilde{A}$ of Theorem 2.1. If we apply the corresponding "block" iterative methods to $\widetilde{A}$, as in Theorem 2.2, then because $\widetilde{A}$ is a nonsingular $M-$matrix, all four "point" and "block" iterative methods correspond to $M-$splittings of the type $M - N$. It is checked that in these splittings the N matrices of the "point" Jacobi and Gauss-Seidel methods are greater than or equal to the corresponding ones of the "block" methods. Hence the "block" methods converge at least as fast as the "point" ones. The convergence will be strictly faster if A is irreducible, which gives us the leftmost inequalities in (2.26) and (2.27). This and the results in (2.28) and (2.29) conclude the proof. •

2.3 SOR type iterative methods

Based on (2.10)-(2.13) we consider the following SOR type splittings for A and $\widetilde{A}$:

$$\widetilde{A} \;=\; \begin{cases} \frac{1}{\omega}P(D-\omega L) - \frac{1}{\omega}P\left((1-\omega)D+\omega U\right) \\ \frac{1}{\omega}\Big(QD-\omega(PL-SD+\widehat{L})\Big) - \frac{1}{\omega}\Big((1-\omega)QD+\omega(\widehat{D}+QU+\widehat{U})\Big) \\ \frac{1}{\omega}\Big((QD-\widehat{D})-\omega(PL-SD+\widehat{L})\Big) - \frac{1}{\omega}\Big((1-\omega)(QD-\widehat{D})+\omega(QU+\widehat{U})\Big) \end{cases}, \tag{2.30}$$

where $\widehat{D}$, $\widehat{L}$, $\widehat{U}$ are given in (2.10), (2.11), (2.12), respectively. In view of (2.30) the block SOR and SOR type iteration matrices associated with A and $\widetilde{A}$ are:

$$\begin{aligned} \mathcal{L}_\omega &= (D-\omega L)^{-1}\left((1-\omega)D+\omega U\right) \ \ (\text{for} A \text{ and } \widetilde{A}) \\ \mathcal{L}'_\omega &= \Big(QD-\omega(PL-SD+\widehat{L})\Big)^{-1}\Big((1-\omega)QD+\omega(\widehat{D}+QU+\widehat{U})\Big) \\ \mathcal{L}''_\omega &= \Big((QD-\widehat{D})-\omega(PL-SD+\widehat{L})\Big)^{-1}\Big((1-\omega)(QD-\widehat{D})+\omega(QU+\widehat{U})\Big). \end{aligned} \tag{2.31}$$

Below we give a statement due to Milaszewicz ([15], [17]) and part of Theorem 3.5 of Marek and Szyld [13] which will be used in the proof of our main statement of this section.

Lemma 2.2 *Let $V,T \in I\!R^{n,n}$, $V,T \geq 0$, for which $\rho(V) < \rho(V+T)$ holds. Then, $\rho(V+tT)$ strictly increases with $t \in [0,\infty)$ and is unbounded. Moreover, if $\rho(V) < 1$ there exists a unique $t = t_1 > 0$ such that $\rho(V+t_1T) = 1$. Also $\rho\left((I-V)^{-1}T\right) = \frac{1}{t_1}$.*

Corollary 2.2 *If $V+T$ is irreducible Lemma 2.2 holds by the Perron-Frobenius theory of nonnegative matrices.*

Lemma 2.3 *Let $A^{-1} \geq 0$. Let $A = M_1 - N_1 = M_2 - N_2$ be two weak splittings with $T_1 = M_1^{-1}N_1$, $T_2 = M_2^{-1}N_2$ having property "d", and $\rho(T_1) < 1$, $\rho(T_2) < 1$. Let $z > 0$ such that $T_2 z = \rho(T_2)z$. If $N_2 z \geq N_1 z$ then $\rho(T_1) \leq \rho(T_2)$.*

Based on the splittings (2.30) and the matrices in (2.31) we will prove the statement below.

Theorem 2.4 *Under the assumptions of Theorem 2.2, for the block SOR and SOR type matrices defined previously and $\forall\ \omega \in (0,1]$ there hold:*

$$\text{a) } \rho(\mathcal{L}_\omega) < 1, \quad \text{b) } \rho(\mathcal{L}'_\omega) < 1, \quad \text{c) } \rho(\mathcal{L}''_\omega) < 1, \tag{2.32}$$

and

$$\rho(\mathcal{L}''_\omega) \ \leq \ \rho(\mathcal{L}'_\omega). \tag{2.33}$$

Also, for ω_1, ω_2, such that $0 < \omega_1 < \omega_2 \leq 1$, there hold:

$$\text{a) } \rho(\mathcal{L}_{\omega_2}) \ \leq \ \rho(\mathcal{L}_{\omega_1}), \quad \text{b) } \rho(\mathcal{L}'_{\omega_2}) \ \leq \ \rho(\mathcal{L}'_{\omega_1}), \quad \text{c) } \rho(\mathcal{L}''_{\omega_2}) \ \leq \ \rho(\mathcal{L}''_{\omega_1}). \tag{2.34}$$

Moreover, there exists a vector $z \in I\!R^n$, $z \geq 0$, such that

$$\mathcal{L}'_\omega z \leq \mathcal{L}_\omega z, \quad 0 < \omega \leq 1. \tag{2.35}$$

If, in addition, A is irreducible and $\mathcal{L}'_{\omega,1}$, $\mathcal{L}''_{\omega,1}$ denote the SOR type matrices corresponding to the last two splittings in (2.30) and are associated with the matrix $\widetilde{A}_1$ of Theorem 2.1, then $\widetilde{A}_1$ is irreducible and the corresponding relationships in (2.33)-(2.34) are strict. If, $B = D^{-1}(L+U)$ is irreducible the vector z in (2.35) is positive and it is also implied that

$$\rho(\mathcal{L}'_\omega) \;\leq\; \rho(\mathcal{L}_\omega), \quad 0 < \omega \leq 1. \tag{2.36}$$

(**Note:** *For $\omega = 1$ (Gauss-Seidel case) some of the above assertions are proved in Theorem 2.2 and will not be proved here although continuity arguments can cover this case as well.)*

Proof: (2.32): In view of the assumptions, the nonsingular $M-$matrix character of A and $\widetilde{A}$ and the fact $0 < \omega \leq 1$ we observe the following:
a) The iteration matrix $\mathcal{L}_\omega = (D-\omega L)^{-1}((1-\omega)D + \omega U)$ is derived from the first splitting in (2.30) and is also derived from the splitting $\frac{1}{\omega}(D-\omega L) - \frac{1}{\omega}((1-\omega)D+\omega U)$. In the latter, the matrix $D - \omega L$ is a nonsingular $M-$matrix because D is a block nonsingular $M-$matrix, the block strictly lower triangular matrix $-\omega L$ is nonpositive and $\frac{1}{\omega} > 0$. Also $\frac{1}{\omega}((1-\omega)I + \omega D^{-1}U) \geq 0$. Therefore the splitting from which $\mathcal{L}_\omega$ is obtained is an $M-$splitting and thus it is convergent implying (a) of (2.32).
b) In the same way it can be proved that the splitting $M' - N'$ for $\mathcal{L}'_\omega$ is an $M-$splitting and therefore convergent. Indeed, $M' = \frac{1}{\omega}(QD)^{-1}(M_1 - \omega\widehat{L})$, where

$$M_1 \;=\; QD - \omega(PL - SD) \;=\; \begin{bmatrix} U_{11} & O_{12} & O_{13} & \dots & O_{1p} \\ O_{21} & A_{22} & O_{23} & \dots & O_{2p} \\ O_{31} & \omega A_{32} & A_{33} & \dots & O_{3p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_{p1} & \omega A_{p2} & \omega A_{p3} & \dots & A_{pp} \end{bmatrix}.$$

M_1 is a block lower triangular $Z-$matrix whose diagonal blocks are nonsingular $M-$matrices and so it is a nonsingular $M-$matrix itself since its inverse is nonnegative. For the same reason $M_1 - \omega\widehat{L}$ is a nonsingular $M-$matrix. Since $QD = \text{diag}(U_{11},\ A_{22},\ \dots,\ A_{pp})$ it can be checked by direct calculation that M' is a nonsingular $M-$matrix. On the other hand N' is nonnegative and so the splitting yielding $\mathcal{L}'_\omega$ is an $M-$splitting.
c) The proof goes along the same lines as in the previous one except that $QD - \widehat{D}$ and $QU + \widehat{U}$ play the roles of QD and $QU + \widehat{D} + \widehat{U}$, respectively.
(2.33): The last two splittings for $\widetilde{A}$ are $M-$splittings and as can be checked from the second splitting $M'' - N''$ it is $M'' = (QD - \widehat{D})^{-1} = \left(I - (QD)^{-1}\widehat{D}\right)^{-1}(QD)^{-1} =$
$\left(I + (QD)^{-1}\widehat{D} + \cdots + ((QD)^{-1}\widehat{D})^{p-1}\right)(QD)^{-1} \geq (QD)^{-1} = M'$ and according to Woźnicki [24], (2.33) holds true.
(2.34): As was seen all three splittings from which the three SOR type matrices are produced are $M-$splittings of the form $M_{\omega_i} - N_{\omega_i}$, $i = 1, 2$. It is checked that in view of $0 < \omega_1 < \omega_2 \leq 1$ for each one of them there holds $N_{\omega_2} \leq N_{\omega_1}$. Consequently, (2.34) are valid.
(2.35): Use of Lemma 2.2 will be made. Let $V = \omega D^{-1}L \geq 0$ and $T = (1-\omega)I + \omega D^{-1}U \geq 0$. Since $\rho(V) = 0$ and $\rho(V+T) = 1 - \omega + \omega\rho(B) > 0$, because $0 < \omega \leq 1$ and $\rho(B) > 0$,

the assumptions of the lemma are satisfied. Therefore, there exists a $t_1 > 0$ such that $\rho(V+t_1T) = 1$. Note that $t_1 > 1$ because $0 < \rho(B) < 1$, $0 < \rho(V+T) = 1-\omega\,(1-\rho(B)) < 1$. Since $V + t_1T \geq 0$ there exists a vector $z \geq 0$ (eigenvector) such that

$$\left(\omega D^{-1}L + t_1\left((1-\omega)I + \omega D^{-1}U\right)\right) z \;=\; z, \tag{2.37}$$

from which

$$\mathcal{L}_\omega z = (I-\omega D^{-1}L)^{-1}\left((1-\omega)I + \omega D^{-1}U\right) z = \frac{1}{t_1} z. \tag{2.38}$$

From (2.37) we can readily obtain

$$(\omega PL + t_1((1-\omega)PD + \omega PU))\, z = PDz$$

or

$$(\omega PL + t_1(1-\omega)QD + t_1(1-\omega)SD + \omega t_1 QU + \omega t_1 SU)\, z = (QD+SD)z$$

or equivalently after some manipulation

$$\begin{array}{rcl}\left(QD - \omega(PL - SD + \widehat{L})\right) z &=& t_1\left((1-\omega)QD + \omega(QU + \widehat{D} + \widehat{U})\right) z \;+ \\ (t_1-1)\left((1-\omega)SD + \omega\widehat{L}\right) z &\geq& t_1\left((1-\omega)QD + \omega(QU + \widehat{D} + \widehat{U})\right) z,\end{array} \tag{2.39}$$

Because $(t_1-1)\left((1-\omega)SD + \omega\widehat{L}\right) z \geq 0$. Combining (2.38) and (2.39), (2.35) is proved. If A is irreducible then so will be $\widetilde{A}_1$. Therefore, $\widetilde{A}_1^{-1} > 0$ and the $M-$splittings, from which $\mathcal{L}'_\omega$, $\mathcal{L}'_{\omega_1}$, $\mathcal{L}'_{\omega_2}$ and $\mathcal{L}''_\omega$, $\mathcal{L}''_{\omega_1}$, $\mathcal{L}''_{\omega_2}$ are yielded, give strict inequalities in (2.33) and (2.34).
(2.36): If B (≥ 0) is irreducible so will be $\mathcal{L}_\omega$, $\omega \in (0,1)$, because after some algebra it is

$$\begin{array}{rcl}\mathcal{L}_\omega &=& (1-\omega)I + (1-\omega)\omega D^{-1}L + \omega D^{-1}U + \text{ nonnegative terms} \\ &=& (1-\omega)I + (1-\omega)\omega D^{-1}(L+U) + \omega^2 D^{-1}U + \text{nonnegative terms } \geq 0,\end{array} \tag{2.40}$$

and the rightmost matrix expression is irreducible since $B = D^{-1}(L+U)$ is. Hence the eigenvector w corresponding to the spectral radius of $\mathcal{L}_\omega$ will be positive. The first splitting in (2.30) is a weak (nonnegative) convergent one for the nonsingular $M-$matrix $\widetilde{A}$ and at the same time it is an $M-$ splitting for the matrix A. Let $\widetilde{A} = M_A - N_A = M_{\widetilde{A}} - N_{\widetilde{A}}$ denote the two splittings that give the iteration matrices $\mathcal{L}_\omega$ and $\mathcal{L}'_\omega$. We have that $N_A - N_{\widetilde{A}} = \frac{1}{\omega}\left(P(1-\omega)D + \omega PU\right) - \frac{1}{\omega}\left((1-\omega)QD + \omega(\widehat{D} + QU + \widehat{U})\right) = \frac{1}{\omega}\left((1-\omega)SD + \omega\widehat{L}\right) \geq 0$ and therefore $(N_A - N_{\widetilde{A}})w \geq 0$. So, according to Lemma 2.3, (2.36) is valid. •

3 Nonsingular p-cyclic Consistently Ordered Matrices

We consider the nonsingular matrix $A \in \mathbb{C}^{n,n}$ of the special block form

$$A \;=\; \begin{bmatrix} A_{11} & O_{12} & O_{13} & \dots & A_{1p} \\ A_{21} & A_{22} & O_{23} & \dots & O_{2p} \\ O_{31} & A_{32} & A_{33} & \dots & O_{3p} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O_{p1} & O_{p2} & \dots & A_{p,p-1} & A_{pp} \end{bmatrix}, \tag{3.1}$$

with $p \geq 3$ and A_{ii}, $i = 1(1)p$, nonsingular matrices. The matrix A is block $p-$cyclic consistently ordered [23] and its block Jacobi matrix will be

$$B = \begin{bmatrix} O_{11} & O_{12} & O_{13} & \dots & -A_{11}^{-1}A_{1p} \\ -A_{22}^{-1}A_{21} & O_{22} & O_{23} & \dots & O_{2p} \\ O_{31} & -A_{33}^{-1}A_{32} & O_{33} & \dots & O_{3p} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O_{p1} & O_{p2} & \dots & -A_{pp}^{-1}A_{p,p-1} & O_{pp} \end{bmatrix}. \tag{3.2}$$

The matrix B is then weakly cyclic of index p and consistently ordered [23]. Applying the preconditioner P of (2.1) we obtain according to (2.5) that

$$\widetilde{A} = \begin{bmatrix} U_{11} & O_{12} & O_{13} & \dots & L_{11}^{-1}A_{1p} \\ O_{21} & A_{22} & O_{23} & \dots & -A_{21}A_{11}^{-1}A_{1p} \\ O_{31} & A_{32} & A_{33} & \dots & O_{3p} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O_{p1} & O_{p2} & \dots & A_{p,p-1} & A_{pp} \end{bmatrix}.$$

The block Jacobi matrix B'' associated with $\widetilde{A}$ (see (2.17)) will be

$$B'' = \begin{bmatrix} O_{11} & O_{12} & O_{13} & \dots & -A_{11}^{-1}A_{1p} \\ O_{21} & O_{22} & O_{23} & \dots & A_{22}^{-1}A_{21}A_{11}^{-1}A_{1p} \\ O_{31} & -A_{33}^{-1}A_{32} & O_{33} & \dots & O_{3p} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O_{p1} & O_{p2} & \dots & -A_{pp}^{-1}A_{p,p-1} & O_{pp} \end{bmatrix}. \tag{3.3}$$

The matrix B''_1, yielded from B'' as $\widetilde{A}_1$ is yielded from $\widetilde{A}$ will be block weakly cyclic of index $p-1$ and consistently ordered and $\widetilde{A}_1$ will be block $(p-1)-$cyclic consistently ordered.

Lemma 2.1.12 of Young [25] given below will be used.

Lemma 3.1 *Let $E \in \mathbb{C}^{n,m}$, $F \in \mathbb{C}^{m,n}$ and $\lambda \in \mathbb{C}\backslash\{0\}$. Then $\lambda \in \sigma(EF)$ if and only if $\lambda \in \sigma(FE)$, with $\sigma(.)$ denoting eigenvalue spectrum.*

Direct computations show that the powers B^p and $B_1''^{p-1}$ are diagonal matrices. Using Lemma 3.1 we find that their eigenvalue spectra are connected via the following relationship

$$\sigma(B_1''^{p-1})\backslash\{0\} \equiv \sigma(B^p)\backslash\{0\}. \tag{3.4}$$

However, (3.4) and especially the expressions for B, B'' and B''_1 strongly remind us of the problem of the best block $p-$cyclic SOR repartitioning first considered and studied by Markham, Neumann and Plemmons [14]. Indeed, it can be seen that if we repartition the matrix A into the following block $(p-1)-$cyclic consistently ordered form

$$A = \left[\begin{array}{cc|c|c|c} A_{11} & O_{12} & O_{13} & \dots & A_{1p} \\ A_{21} & A_{22} & O_{23} & \dots & O_{2p} \\ \hline O_{31} & A_{32} & A_{33} & \dots & O_{3p} \\ \hline \vdots & \vdots & \ddots & \ddots & \vdots \\ \hline O_{p1} & O_{p2} & \dots & A_{p,p-1} & A_{pp} \end{array}\right] \tag{3.5}$$

we can find out that the block Jacobi matrix, J, associated with A of (3.5) is identically the same with the matrix B'' associated with A without any repartitioning, since

$$J = \left[\begin{array}{cc|c|c|c} O_{11} & O_{12} & O_{13} & \dots & -A_{11}^{-1}A_{1p} \\ O_{21} & O_{22} & O_{23} & \dots & A_{22}^{-1}A_{21}A_{11}^{-1}A_{1p} \\ \hline O_{31} & -A_{33}^{-1}A_{32} & O_{33} & \dots & O_{3p} \\ \hline \vdots & \vdots & \ddots & \ddots & \vdots \\ \hline O_{p1} & O_{p2} & \dots & -A_{pp}^{-1}A_{p,p-1} & O_{pp} \end{array}\right].$$

Consequently, $J \equiv B''$ and therefore

$$\sigma(J^{p-1})\backslash\{0\} \equiv \sigma(B''^{p-1})\backslash\{0\} \equiv \sigma(B_1''^{p-1})\backslash\{0\} \equiv \sigma(B^p)\backslash\{0\}. \tag{3.6}$$

Based on the result just obtained we can readily prove the following statement.

Theorem 3.1 *Under the notation of the previous Section let $A \in \mathbb{C}^{n,n}$ be a nonsingular block $p-$cyclic consistently ordered matrix of the form (3.1) with $\rho(B) < 1$. Then for the spectral radii of the block Jacobi and Gauss-Seidel iteration matrices B B'', B_1'', H, H'' and H_1'' the following relationships hold:*

$$\rho(B_1'') \equiv \rho(B'') = \rho^{\frac{p}{p-1}}(B) < \rho(B) < 1 \tag{3.7}$$

and

$$\rho(H_1'') \equiv \rho(H'') = \rho(H) = \rho^p(B) < 1. \tag{3.8}$$

Proof: The proof of (3.7) is an immediate consequence of (3.6). The validity of (3.8) is because the spectrum of the Gauss-Seidel matrix of a $p-$cyclic consistently ordered matrix is the union of 0 and of the p^{th} powers of the eigenvalues of the corresponding Jacobi matrix. •

We conclude this section by stating a theorem concerning the best of the optimal block SOR methods associated with a block $p-$cyclic consistently ordered matrix A of the form (3.1) and its preconditioned one $\widetilde{A}$ (or $\widetilde{A}_1$) when the $\sigma(B^p)$ is real. Its proof consists of a very delicate and careful modification of an extract from a general statement given in Theorem 2.1 and Table 1 of [5] and is not possible to be given here since it will be a tedious and long repetition of the argumentation in [5]. So, only an outline of the modification is presented.

Theorem 3.2 *Let B be the block Jacobi matrix (3.2) associated with the block $p-$cyclic consistently ordered matrix A in (3.1), $p \geq 3$, and let $\sigma(B^p) \subset [-\alpha^p, \beta^p]$ with $-\alpha^p$, $\beta^p \in \sigma(B^p)$, where $0 \leq \beta < 1$ and $0 \leq \alpha < \infty$. Consider $\widetilde{A}$ partitioned in a block $(p-1)-$cyclic form consistent with the partitioning of A in (3.5). Denote by ω_k and $\rho(\mathcal{L}_{\omega_k})$, $k = p, p-1$, the real optimal SOR factor and the optimal spectral radius associated with A ($k = p$) and $\widetilde{A}$ or $\widetilde{A}_1$ ($k = p-1$), respectively, and by $\rho(\mathcal{L}_\omega(A))$, $\rho\left(\mathcal{L}_\omega(\widetilde{A})\right)$ the spectral radii of the SOR matrices for A and $\widetilde{A}$. Then there hold:*

(i) If $\frac{\alpha}{\beta} \in \left[0, \left(\frac{p-3}{p-1}\right)^{\frac{p-1}{p}}\right]$, then

$$\rho(\mathcal{L}_{\omega_{p-1}}) < \rho(\mathcal{L}_{\omega_p}) < 1. \tag{3.9}$$

(ii) If $\left(\frac{p-3}{p-1}\right)^{\frac{p-1}{p}} < \frac{\alpha}{\beta} < \frac{p-2}{p}$*, then for any* $\beta \in (0,1)$ *there exists a unique value of* α*,* $\alpha_{p-1,p} := \alpha(\beta) \in \left(\left(\frac{p-3}{p-1}\right)^{\frac{p-1}{p}}\beta, \frac{p-2}{p}\beta\right)$ *which is given by the expression* $\alpha_{p-1,p} = \left(\frac{2\rho^{\frac{1}{p-1}} - (1+\rho)\beta^{\frac{p}{p-1}}}{1-\rho}\right)^{\frac{p-1}{p}}$*, where* ρ *is the unique root in* $(0,1)$ *of the equation*

$$\beta^p\left(p-1+\rho\right)^p - p^p\rho = 0. \tag{3.10}$$

The root ρ *is such that*

$$\rho(\mathcal{L}_{\omega_{p-1}}) \;<\; \rho(\mathcal{L}_{\omega_p}) \;<\; 1 \quad \text{for} \quad \left(\frac{p-3}{p-1}\right)^{\frac{p-1}{p}}\beta < \alpha < \alpha_{p-1,p}, \tag{3.11}$$

$$\rho(\mathcal{L}_{\omega_{p-1}}) \;=\; \rho(\mathcal{L}_{\omega_p}) \;<\; 1 \quad \text{for} \quad \alpha = \alpha_{p-1,p}, \tag{3.12}$$

$$\rho(\mathcal{L}_{\omega_p}) \;<\; \rho(\mathcal{L}_{\omega_{p-1}}) \;<\; 1 \quad \text{for} \quad \alpha_{p-1,p} < \alpha < \left(\frac{p-2}{p}\right)\beta. \tag{3.13}$$

(iii) If $\frac{p-2}{p} \le \frac{\alpha}{\beta} < 1$*, then*

$$\rho(\mathcal{L}_{\omega_p}) \;<\; \rho(\mathcal{L}_{\omega_{p-1}}) \;<\; 1. \tag{3.14}$$

(iv) If $\frac{\alpha}{\beta} = 1$*, then*

$$\rho(H_1'') = \rho(H'') = \rho(H) = \rho^p(B) < 1, \tag{3.15}$$

that is both optimal SOR matrices are identical with their corresponding Gauss-Seidel ones and both converge equally well.
(v) If $1 < \frac{\alpha}{\beta} \le \frac{p}{p-2}$*, then*

$$\rho(\mathcal{L}_{\omega_p}) \;<\; \rho(\mathcal{L}_{\omega_{p-1}}) \;<\; 1. \tag{3.16}$$

(vi) If $\frac{p}{p-2}\beta < \alpha \le \frac{p}{p-2}$*, then for any such* α *there exists a unique value of* β*,* $\beta_{p-1,p} := \beta(\alpha) \in \left(\left(\frac{p-3}{p-1}\right)^{\frac{p-1}{p}}\alpha, \frac{p-2}{p}\alpha\right]$ *which is given by the expression* $\beta_{p-1,p} = \beta(\alpha) \in \left(\frac{2\rho^{\frac{1}{p-1}} - (1-\rho)\alpha^{\frac{p}{p-1}}}{1+\rho}\right)^{\frac{p-1}{p}}$*, where* ρ *is the unique root in* $(0,1)$ *of the equation*

$$\alpha^p\left(p-1-\rho\right)^p - p^p\rho = 0. \tag{3.17}$$

The root ρ *is such that*

$$\rho(\mathcal{L}_{\omega_{p-1}}) \;<\rho(\mathcal{L}_{\omega_p}) \;<\; 1 \quad \text{for} \quad \left(\frac{p-3}{p-1}\right)^{\frac{p-1}{p}}\alpha < \beta < \beta_{p-1,p}, \tag{3.18}$$

$$\rho(\mathcal{L}_{\omega_{p-1}}) \;=\; \rho(\mathcal{L}_{\omega_p}) \;<\; 1 \quad \text{for} \quad \beta = \beta_{p-1,p} \tag{3.19}$$

$$\rho(\mathcal{L}_{\omega_p}) \;<\; \rho(\mathcal{L}_{\omega_{p-1}}) \;<\; 1 \quad \text{for} \quad \beta_{p-1,p} < \beta < \frac{p-2}{p}\alpha. \tag{3.20}$$

(vii) If $\frac{p}{p-2} < \alpha \leq \left(\frac{p-1}{p-3}\right)^{\frac{p-1}{p}}$, *then*

$$\rho(\mathcal{L}_{\omega_{p-1}}) \ < \ 1 \ \text{ and } \ \rho\left(\mathcal{L}_{\omega}(A)\right) \ > \ 1, \ \ \forall\, \omega \in I\!R. \tag{3.21}$$

(viii) If $(\frac{p-1}{p-3})^{\frac{p-1}{p}} < \alpha$, *then*

$$\rho\left(\mathcal{L}_{\omega}(\widetilde{A})\right) \ > \ 1 \ \text{ and } \ \rho\left(\mathcal{L}_{\omega_p}(A)\right) \ > \ 1, \ \ \forall\, \omega \in I\!R. \tag{3.22}$$

Proof: It is known that in a $p-$cyclic consistently ordered case, like the one we are working on, when $\alpha > \frac{p}{p-2}$ there is no $\omega \in I\!R$ for which the associated SOR method converges. Having in mind the note just made then by means of the formulas given in Theorem 2.1 and Table 1 of [5] (of the Best Cyclic Repartitioning for Optimal SOR) one can determine the value of $k = 2(1)p$ that gives the best (repartitioned) optimal SOR. Having determined the specific value of k one can find the optimal SOR parameter by means of the formulas (2.20)-(2.22) of [5]. In our case, however, we must restrict to the values of $k = p-1$ and p. So, we have to appropriately exploit the results in [5]. For this we observe that because of (3.6), it will be $\alpha_{p-1}^{p-1} = \alpha^p$ and $\beta_{p-1}^{p-1} = \beta^p$, where $\alpha_{p-1}, \beta_{p-1}$ play the roles of α, β for the block $(p-1)-$cyclic consistently ordered matrix $\widetilde{A}$ or $\widetilde{A}_1$. However, for these matrices convergence of the associated block SOR will take place for values of $\alpha_{p-1} \in \left[0, \frac{p-1}{p-3}\right]$ while for the original $p-$cyclic SOR the corresponding interval for α will be $\left[0, \frac{p}{p-2}\right]$. The latter interval for α is contained in the former one for α which is then $\left[0, \left(\frac{p-1}{p-3}\right)^{\frac{p-1}{p}}\right]$ since as can be proved the function $\left(\frac{x-2}{x}\right)^x$, $x \in [2, \infty)$, strictly increases with x. So, the $(p-1)-$cyclic case for $\widetilde{A}$ can be advantageous over the $p-$cyclic case for A. From this point on the statement of our theorem describes more specifically the various situations that can arise and gives for each one of them the (optimal) convergence results that can be obtained. •

4 Comments and Discussion on the Singular Case

I) Let $A \in I\!R^{n,n}$, in (1.3), be a **singular** irreducible $M-$matrix. For such an A, each principal submatrix, except itself, is a nonsingular $M-$matrix ([2]). By Lemma 2.1 and Theorem 2.1, $\widetilde{A}_1$ is irreducible (unless $p = 2$ and $n_1 = n-1$, when $\widetilde{A}_1 = O \in I\!R^{1,1}$) and both $\widetilde{A}$ and $\widetilde{A}_1$ are singular $M-$matrices. All the splittings in Sections 2.2 and 2.3 that were $M-$splittings still are and so all the corresponding iteration matrices there, e.g., B, H, $\mathcal{L}_\omega$, $\omega \in (0,1]$, are well defined and have spectral radii 1. As is known [2] for the (semi)converge of a linear first order iterative scheme, with iteration matrix satisfying $\rho(T) = 1$, for any initial guess $x_0 \in I\!R^n$, provided that $b \in \text{range}(A)$, $\sigma(T)$ must satisfy the three conditions below (see [2]). In such a case a factor $\gamma(T)$, which is equal to the modulus of the second largest in modulus eigenvalue of T, plays the role of the spectral radius.
i) $\rho(T) = 1$.

ii) If $\lambda \in \sigma(T)$ with $|\lambda| = 1$, then $\lambda = 1$.
iii) index$(I - T) = 1$, that is in the Jordan canonical form of T all eigenvalues of modulus 1 are associated with 1×1 Jordan blocks.
All iteration matrices arising from $M-$splittings satisfy condition (i) and in view of the irreducibility of A, and of $\widetilde{A}_1$, the iteration matrices associated with them satisfy condition (iii). However, condition (ii) cannot be always satisfied. E.g., for A being also $p-$cyclic consistently ordered its Jacobi matrix has, besides 1, as eigenvalues the numbers $\exp\left(i\frac{2\pi k}{p}\right)$, $k = 1(1)p-1$, of modulus 1. So, stronger conditions are necessary and a search for them is being made. We also note that even if all three conditions (i)-(iii) are met for the iteration matrices of Theorems 2.2 and 2.4 it is not clear whether their semiconvergence factors $\gamma(.)$ will satisfy relationships analogous to those in (2.21)-(2.24) and in (2.32)-(2.36). This remains an open problem.

II) The case where some of the results obtained are carried over to the singular case is when $A \in \mathbb{C}^{n,n}$ is block $p-$cyclic consistently ordered of the form (3.1). Suppose then that $B = I - D^{-1}A$ satisfies condition (iii). This condition, index$(I - B) = 1$, implies index$(I - \mathcal{L}_\omega) = 1$, $\forall\ \omega \neq \{0, \frac{p}{p-1}\}$ (see Theorem 3.1 of [7]). Hence by virtue of Theorem 3.1 of [9], Theorem 3.2 holds, provided that in its assumptions $\sigma(B^p) \subset [-\alpha^p, \beta^p] \cup \{1\}$ will replace $\sigma(B^p) \subset [-\alpha^p, \beta^p]$ and therefore Theorem 3.2 will be valid except that $\gamma(.)'s$ will replace the corresponding $\rho(.)'s$.

A direct application to the previous results in the present section is the determination of the stationary probability distribution vector in the Markov Chain Analysis where the coefficient matrix A is a singular $M-$matrix with zero column (or row) sums.
If A is irreducible (ergodic chain) Theorems 2.2 and 2.4 hold in the way explained above.
If A is, in addition, of the form (3.1) (periodic chain of period p) and $B = I - D^{-1}A$ is irreducible, then Theorem 3.2 holds as was explained. Since $\rho(B) = 1$, α in Theorem 3.2 must satisfy $\alpha < 1$. This restriction modifies slightly the conclusions of the theorem. Specifically: The three cases (vi), (vii) and (viii) are expressed as one under the main assumption $\frac{p}{p-2} < \frac{\alpha}{\beta} \leq \infty$, where " $= \infty$" means $\beta = 0$ and $\alpha > 0$. Then, $\beta_{p-1,p} \in \left[0, \frac{p-2}{p}\alpha\right]$, with $\beta_{p-1,p} = 0$ corresponding to $\frac{\alpha}{\beta} = \infty$. The conclusions are those of case (vi).

5 Numerical Examples

Example I: The matrix below (without the partitioning is found in [6]) is obviously an irreducible $Z-$matrix and since $\rho(I - A) \approx .9807 < 1$, A is also a nonsingular $M-$matrix,

$$A = \left[\begin{array}{rr|r|rr} 1 & -.2 & -.1 & -.4 & -.2 \\ -.2 & 1 & -.3 & -.1 & -.6 \\ \hline -.3 & -.2 & 1 & -.1 & -.6 \\ \hline -.1 & -.1 & -.1 & 1 & -.01 \\ -.2 & -.3 & -.4 & -.3 & 1 \end{array}\right]. \qquad (5.1)$$

Therefore, Theorems 2.1-2.4 apply. Indeed, preserving the notation in these statements it is:

$$\begin{array}{rrr} \rho(B^{(0)}) = .9807 & \rho(H^{(0)}) = .9611 & \rho(\mathcal{L}^{(0)}_{.75}) = .9768 \\ \rho(B) = .9785 & \rho(H) = .9570 & \rho(\mathcal{L}_{.75}) = .9743 \\ \rho(B'^{(n_1)}) = .9778 & \rho(H'^{(n_1)}) = .9565 & \rho(\mathcal{L}'^{(n_1)}_{.75}) = .9749 \\ \rho(B''^{(n_1)}) = .9770 & \rho(H''^{(n_1)}) = .9531 & \rho(\mathcal{L}''^{(n_1)}_{.75}) = .9734 \\ \rho(B') = .9684 & \rho(H') = .9505 & \rho(\mathcal{L}'_{.75}) = .9676 \\ \rho(B'') = .9577 & \rho(H'') = .9172 & \rho(\mathcal{L}''_{.75}) = .9502 \end{array}, \tag{5.2}$$

where the matrices in the first two rows refer to the "point" and "block" iteration matrices associated with A, the matrices of the third and fourth rows refer to the "point" iteration matrices of $\widetilde{A}$ and the matrices of the last two rows refer to the "block" ones of $\widetilde{A}$. It is checked that all the relationships (strict inequalities) of Theorems 2.1-2.4 are verified.

Example II: We consider the following singular block $3-$cyclic consistently ordered matrix

$$A = \begin{bmatrix} I_3 & O & -C \\ -C & I_3 & O \\ O & -C & I_3 \end{bmatrix}, \text{ with } C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha\beta & \alpha-\beta+\alpha\beta & 1-\alpha+\beta \end{bmatrix}, \tag{5.3}$$

$I_3 \in I\!R^{3,3}$ the identity matrix and $-\alpha \leq 0 \leq \beta < 1,\ \alpha \neq \beta$. It is $B = \text{diag}(I_3, I_3, I_3) - A$ hence $B^3 = \text{diag}(C^3, C^3, C^3)$, with $\sigma(C^3) = \{1, -\alpha^3, \beta^3\}$, $-\alpha^3 \leq 0 \leq \beta^3 < 1,\ \alpha \neq \beta$. Therefore

$$\sigma(B) = \left\{ \exp\left(i\frac{2k\pi}{3}\right), \beta \exp\left(i\frac{2k\pi}{3}\right), -\alpha \exp\left(i\frac{2k\pi}{3}\right) \right\},\ k = 0, 1, 2,$$

the eigenvalue 1 is simple and index$(I - B) = 1$. Consequently, all the assumptions of Theorem 3.2, in the singular case, are satisfied. So, depending on the value of the ratio $\frac{\alpha}{\beta}$ the best of the two optimal SORs associated with A and $\widetilde{A}_1$, if they exist, will be associated with either the original block $3-$cyclic or, after the preconditioning takes place, with the block $2-$cyclic one. It is noted that for the optimal $3-$cyclic SOR to exist, $\alpha < 3$ must hold, while the $2-$cyclic one exists for all $\alpha \in [0, \infty)$!

References

[1] M. Alanelli. *Elementary Block Elimination Preconditioners for the Numerical Solution of Nonsingular and Singular Linear Systems.* Master Dissertation. Department of Mathematics, University of Crete, GR-714 09 Heraklion, Greece, 2001 (in Greek).

[2] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences. Classics in Applied Mathematics.* SIAM, Philadelphia, 1994.

[3] M. Eiermann, W. Niethammer, and A. Ruttan. *Optimal Successive Overrelaxation Iterative Methods for p−Cyclic Matrices. Numer. Math.*, 57:593–606, 1990.

[4] R.E. Funderlic and R.J. Plemmons. *LU Decomposition of M-Matrices by Elimination Without Pivoting. Linear Algebra Appl.* 41 (1981), 99–110.

[5] S. Galanis and A. Hadjidimos. *Best Cyclic Repartitioning for Optimal Overrelaxation Convergence. SIAM J. Matrix Anal. Appl.* 13 (No. 1) (1992), 102-120.

[6] A.D. Gunawardena, S.K. Jain and L. Snyder. *Modified Iterative Methods for Consistent Linear Systems. Linear Algebra Appl.* 154–156 (1991), 123–143.

[7] A. Hadjidimos. *On the Optimization of the Classical Iterative Schemes for the Solution of Complex Singular Linear Systems. SIAM J. Alg. Disc. Meth.*, 6:555–566, 1985.

[8] A. Hadjidimos, D. Noutsos and M. Tzoumas. *More on Modifications and Improvements of Classical Iterative Schemes for $Z-$Matrices.* Preprint no. 01-1/2001, Department of Mathematics, University of Crete, GR-714 09 Heraklion, Greece, 2001.

[9] A. Hadjidimos and R.J. Plemmons. *Optimal $p-$Cyclic SOR. Numer. Math.,* 67:475–490, 1994.

[10] M.L. Juncosa and T.W. Mulliken. *On the Increase of Convergence Rates of Relaxation Procedures for Elliptic Partial Differential Equations. J. Assoc. Comput. Mach.* 7 (1960), 29–36.

[11] T. Kohno, H. Kotakemori, H. Niki and M. Usui. *Improving the Gauss-Seidel Method for Z-Matrices. Linear Algebra Appl.* 267 (1997), 113–123.

[12] W. Li and W. Sun. *Modified Gauss-Seidel type methods and Jacobi type methods for Z-matrices. Linear Algebra Appl.* 317 (2000), 227–240.

[13] I. Marek and D.B. Szyld. *Comparison Theorems for Weak Splittings of Bounded Operators. Numer. Math.* 58 (1990), 387-397.

[14] T.L. Markham, M. Neumann and R.J. Plemmons. *Convergence of a Direct-Iterative Method for Large-Scale Least Squares Problem. Linear Algebra Appl.* 69 (1985), 155-167.

[15] J.P. Milaszewicz. *A Generalization of the Stein-Rosenberg Theorem to Banach Spaces. Numer. Math.* 34 (1980), 403-409.

[16] J.P. Milaszewicz. *On Modified Jacobi Linear Operators*, *Linear Algebra Appl.* 51 (1983), 127–136.

[17] J.P. Milaszewicz. *Improving Jacobi and Gauss-Seidel Iterations*, *Linear Algebra Appl.* 93 (1987), 161–170.

[18] F. Robert. *Algorithmes tronqués de découpe linéaire. RAIRO, Revue de l' AFCET,* (1972), 45-64.

[19] F. Robert. *Autour du théorème de Stein-Rosenberg. Numer. Math.* 27(1976), 133-141.

[20] D.J. Pierce. *Parallel Least Squares Computations and Related Material.* Ph.D. thesis, North Carolina State University, Raleigh, NC, (1987).

[21] D.J. Pierce, A. Hadjidimos and R.J. Plemmons. *Optimality Relationships for p-Cyclic SOR. Numer. Math.* 56 (1990), 635-643.

[22] H. Schneider. *Theorems on $M-$splittings of a Singular $M-$matrix Which Depend on Graph Structure. Linear Algebra Appl.* 58 (1984), 407-424.

[23] R.S. Varga. *Matrix Iterative Analysis.* Prentice-Hall, Englewood Cliffs, NJ, 1962. (Also: 2nd Edition, Revised and Expanded, Springer, Berlin, 2000.)

[24] Z. Woźnicki. *Nonnegative Splitting Theory, Japan Journal of Industrial and Applied Mathematics* 11 (1994), 289–342.

[25] D.M. Young. *Iterative Solution of Large Linear Systems.* Academic Press, New York, 1971.