On a series of Gorenstein cyclic quotient singularities admitting a unique projective crepant resolution

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Abstract. Let G be a finite subgroup of $\mathrm{SL}(r, \mathbb{C})$. In dimensions $r = 2$ and $r = 3$, McKay correspondence provides a natural bijection between the set of irreducible representations of G and a cohomology-ring basis of the overlying space of a projective, crepant desingularization of $\mathbb{C}^r/G$. For $r = 2$ this desingularization is unique and is known to be determined by the Hilbert scheme of the G-orbits. Similar statements (including a method of distinguishing just *one* among all possible smooth minimal models of $\mathbb{C}^3/G$), are very probably true for all G's $\subset \mathrm{SL}(3, \mathbb{C})$ too, and recent Hilbert-scheme-techniques due to Ito, Nakamura and Reid, are expected to lead to a new fascinating uniform theory. For dimensions $r \geq 4$, however, to apply analogous techniques one needs extra modifications. In addition, minimal models of $\mathbb{C}^r/G$ are smooth only under special circumstances. $\mathbb{C}^4/$(involution), for instance, cannot have any smooth minimal model. On the other hand, all abelian quotient spaces which are c.i.'s can always be fully resolved by torus-equivariant, crepant, projective morphisms. Hence, from the very beginning, the question whether a given Gorenstein quotient space $\mathbb{C}^r/G$, $r \geq 4$, admits special desingularizations of this kind, seems to be absolutely crucial.

In the present paper, after a brief introduction to the existence-problem of such desingularizations (for abelian G's) from the point of view of toric geometry, we prove that the Gorenstein cyclic quotient singularities of type

$$\frac{1}{l}\,(1, \ldots, 1, l - (r-1))$$

with $l \geq r \geq 2$, have a *unique* torus-equivariant projective, crepant, partial resolution, which is "full" iff either $l \equiv 0 \bmod (r-1)$ or $l \equiv 1 \bmod (r-1)$. As it turns out, if one of these two conditions is fulfilled, then the exceptional locus of the full desingularization consists of $\lfloor \frac{l}{r-1} \rfloor$ prime divisors, $\lfloor \frac{l}{r-1} \rfloor - 1$ of which are isomorphic to the total spaces of $\mathbb{P}^1_{\mathbb{C}}$-bundles over $\mathbb{P}^{r-2}_{\mathbb{C}}$. Moreover, it is shown that intersection numbers are computable explicitly and that the resolution morphism can be viewed as a composite of successive (normalized) blow-ups. Obviously, the monoparametrized singularity-series of the above type contains (as its "first member") the well-known Gorenstein singularity defined by the origin of the affine cone which lies over the r-tuple Veronese embedding of $\mathbb{P}^{r-1}_{\mathbb{C}}$.

1. Introduction

(a) Let $f : Y \to X$ be a birational morphism between two normal, $\mathbb{Q}$-Gorenstein complex varieties X and Y of index j. Denote by $\omega_X = \mathcal{O}(K_X)$ and $\omega_Y = \mathcal{O}(K_Y)$ the dualizing sheaves, and by K_X and K_Y representatives of canonical divisors of X and Y, respectively. f is called *crepant* if $\omega_X^{[j]} \cong f_* \left(\omega_Y^{\otimes j}\right)$, or, in other words, if the *discrepancy* $jK_Y - f^*(jK_X)$ vanishes. The "prototype" for a crepant morphism is the proper birational map which desingularizes the usual double-point-locus

$$X = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 \;\middle|\; z_1^2 + z_2^2 + z_3^2 = 0 \right\}$$

by blowing up $\mathbf{0} \in X \subset \mathbb{C}^3$. Crepant birational morphisms were mainly used in the past two decades in algebraic geometry to reduce the singularities of complex 3-folds (and, sometimes, n-folds) to terminal (or even $\mathbb{Q}$-factorial terminal) singularities, and to treat of *minimal models* in high dimensions. For X being the underlying space of a Gorenstein quotient singularity, they are "by definition" related to McKay-type correspondences.

(b) Let r be an integer ≥ 2, G a finite subgroup of $\mathrm{GL}(r, \mathbb{C})$ containing no pseudoreflections and acting linearly on $\mathbb{C}^r$, and $p : \mathbb{C}^r \to \mathbb{C}^r/G$ the corresponding quotient map. The underlying space $\mathbb{C}^r/G$ of the (germ of the) quotient singularity $(\mathbb{C}^r/G, [\mathbf{0}])$, with $[\mathbf{0}] := p(\mathbf{0})$, is canonically equipped with the structure of a normal, Cohen-Macaulay complex variety (or complex-analytic space).

• The singular locus $\mathrm{Sing}(\mathbb{C}^r/G)$ of $\mathbb{C}^r/G$ itself contains always $[\mathbf{0}]$, but for $r \geq 3$, it is possible to possess also other strata of $\mathbb{C}^r$ of codimension ≥ 2 passing through $[\mathbf{0}]$ (cf. 5.1 below).

• As it was proved by Watanabe [81], $\mathbb{C}^r/G$ is Gorenstein iff $G \subset \mathrm{SL}(r, \mathbb{C})$.

• If $r = 2$, $G \subset \mathrm{GL}(2, \mathbb{C})$, the quotient space $\mathbb{C}^2/G$ admits a unique *minimal* desingularization

$$f : \widehat{X} \to X = \mathbb{C}^2/G \tag{1.1}$$

("minimal" in the sence that the exceptional locus of f does not contain any curve with self-intersection number -1, or equivalently, that there exists, up to isomorphism, a unique morphism $h : \widetilde{X} \to \widehat{X}$ with $g = f \circ h$, for any desingularization $g : \widetilde{X} \to X$ of X). The description of the prime divisors (rational curves) consisting the exceptional locus of the above f, as well as that of the way of how these divisors intersect each other (tree configurations), is due to Hirzebruch [33] (for cyclic acting groups) and Brieskorn [8] (for all the other finite subgroups G of $\mathrm{GL}(2, \mathbb{C})$).

• The minimal desingularization (1.1) is crepant if and only if $G \subset \mathrm{SL}(2, \mathbb{C})$. In this special case, the (Gorenstein) quotient spaces $\mathbb{C}^2/G$ are embeddable as *A-D-E* hypersurfaces in $\mathbb{C}^3$ (Klein [43], Du Val [14], [15]) and are nothing but the *rational double points* treated in the classical theory of the simple hypersurface singularities. Table 1 contains all possible finite subgroups of $\mathrm{SL}(2, \mathbb{C})$.

Nr.	groups G	Dynkin's notation	$\lvert G \rvert$	$\#\left\{\begin{array}{c}\text{conjugacy}\\ \text{classes of } G\end{array}\right\}$
1.	cyclic groups $\mathbf{C}_k$ of order $k \geq 2$	A_{k-1}	k	k
2.	binary dihedral groups $\mathbf{D}_{k-2}$ with $k \geq 4$	D_k	$4(k-2)$	$k+1$
3.	binary tetrahedral group $\mathbf{T}$	E_6	24	7
4.	binary octahedral group $\mathbf{O}$	E_7	48	8
5.	binary icosahedral group $\mathbf{I}$	E_8	120	9

Table 1.

More precisely, taking into account the above group classification, one obtains for the quotient spaces $\mathbb{C}^2/G$, $G \subset \mathrm{SL}(2,\mathbb{C})$, and (1.1):

Theorem 1.1. *The quotient spaces* $\mathbb{C}^2/G = \text{Max-Spec}\left(\mathbb{C}[\mathfrak{x}_1,\mathfrak{x}_2]^G\right)$, *for* G *a finite subgroup of* $\mathrm{SL}(2,\mathbb{C})$, *are minimally embedded as hypersurfaces*

$$\left\{(z_1,z_2,z_3) \in \mathbb{C}^3 \;\mid h(z_1,z_2,z_3) = 0\right\}$$

in $\mathbb{C}^3$, *i.e.* $\mathbb{C}[\mathfrak{x}_1,\mathfrak{x}_2]^G \cong \mathbb{C}[z_1,z_2,z_3] \;/\; (h(z_1,z_2,z_3))$. (The normal form for the ideal generator in each individual case is mentioned in the third column of table 2.)

Theorem 1.2. *Every quotient singularity* $(\mathbb{C}^2/G,[\mathbf{0}])$, *with* G *a finite subgroup of* $\mathrm{SL}(2,\mathbb{C})$, *admits a unique minimal (= crepant) resolution*

$$f : \left(\widehat{\mathbb{C}^2/G},\mathbf{E}\right) \to (\mathbb{C}^2/G,[\mathbf{0}])$$

(up to isomorphism) with exceptional divisor $\mathbf{E}$ *consisting of a configuration of rational smooth curves with self-intersection number* -2. *The intersection form of* $\mathbf{E}$ *is negative definite and therefore the Dynkin diagrams of type A-D-E are the dual graphs of the irreducible components of* $\mathbf{E}$. (See table 2; each "$\bullet$" intimates a smooth rational curve and each edge a transversal intersection at one point. For details, see e.g. Lamotke [46] and Slodowy [72]).

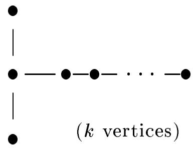
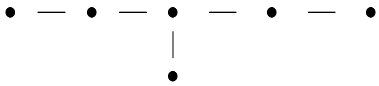
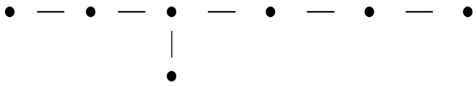
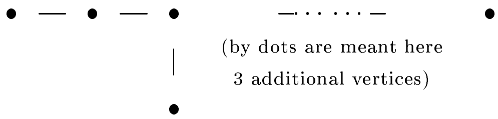
Nr.	G	$h(z_1, z_2, z_3)$	Dynkin diagram of the minimal resolution
1.	$\mathbf{C}_k$	$z_1^2 + z_2^2 + z_3^k$	
2.	$\mathbf{D}_{k-2}$	$z_1^2 + z_2^2\, z_3 + z_3^{k-1}$	
3.	$\mathbf{T}$	$z_1^2 + z_2^3 + z_3^4$	
4.	$\mathbf{O}$	$z_1^2 + z_2^2 + z_2\, z_3^3$	
5.	$\mathbf{I}$	$z_1^2 + z_2^3 + z_3^5$	

Table 2.

• Denoting by $\mathfrak{D}$ a simply-laced Dynkin diagram belonging to the fourth column of Table 2, there is an "extended" Dynkin diagram $\mathfrak{D}^{\text{ext}}$ obtained by adding one vertex $\mathbf{v}_0$ to $\mathfrak{D}$ and by connecting the vertices, say $\mathbf{v}_1, \ldots, \mathbf{v}_\nu$, of $\mathfrak{D}$ with $\mathbf{v}_0$ in the following way: Let R be the root system corresponding to $\mathfrak{D}$. The vertices $\mathbf{v}_1, \ldots, \mathbf{v}_\nu$ give rise to simple roots $\mathfrak{r}_1, \ldots, \mathfrak{r}_\nu$ with respect to the system R_+ of the positive roots. Furthermore, $\mathbf{v}_0$ corresponds to $\mathfrak{r}_0 = -\theta$, where θ is the "longest" root (i.e., θ is positive and $\theta + \mathfrak{r}_i$ non-positive for all i, $i \in \{1, .., \nu\}$). In $\mathfrak{D}^{\text{ext}}$ one connects $\mathbf{v}_0$ and $\mathbf{v}_i$ by an *edge* if and only if $(\mathfrak{r}_0, \mathfrak{r}_i) \neq 0$. In the late seventies, McKay [49] observed a remarkable connection between the representation theory of the finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ and the extended Dynkin diagrams $\mathfrak{D}^{\text{ext}}$. To formulate it explicitly, let us point out that each vertex $\mathbf{v}_i$ of $\mathfrak{D}^{\text{ext}}$ is accompanied by a label $q_i \in \mathbb{N}$, with $q_0 = 1$ and $\sum_{i=0}^{\nu} \mathfrak{r}_i\, q_i = 0$, or equivalently, $\theta = \sum_{i=1}^{\nu} \mathfrak{r}_i\, q_i$.

Theorem 1.3 (Classical McKay Correspondence). *Let G be a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$ and $\mathfrak{D}$ its Dynkin diagram. Then there is an one-to-one correspondence*

$$\left\{\text{vertices of } \mathfrak{D}^{\text{ext}}\right\} \ni \mathbf{v}_i \longleftrightarrow \boldsymbol{\rho}_i \in \left\{ \begin{array}{c} \text{equivalence classes of} \\ \text{irreducible representations of } G \end{array} \right\}$$

so that $\boldsymbol{\rho}_i$ has dimension q_i. Moreover, for any two dimensional representation $\boldsymbol{\rho}'$ of G in $\mathbb{C}^2$, there exist isomorphisms for all i, $i \in \{0, 1, ..., \nu\}$,

$$\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}' \xrightarrow{\cong} \left(\bigoplus_{j \text{ incident to } i} \boldsymbol{\rho}_j \right) .$$

• Having this theorem as starting-point, Gonzalez-Sprinberg, Verdier [25], and later Knörrer [45], constructed a purely geometric, correspondence of "McKay-type" between the set of irreducible representations of G, and the *cohomology ring* of $\widehat{X} = \widehat{\mathbb{C}^2/G}$ via "tautological sheaves". Recently, Ito, Nakamura [40], and Reid [63], introduced new techniques for the study of McKay correspondence involving Hilbert schemes of G-orbits.

• In particular, for $r = 2$, the main result of Ito and Nakamura [38], [39], [40], [51], [52] can be roughly stated as follows:

Theorem 1.4. *Let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ with $l = |G|$. Then there is a unique irreducible component $\mathcal{H}^G(\mathbb{C}^2)$ of the G-fixed point set $\mathcal{H}^l\left[\mathbb{C}^2\right]^G$ of the Hilbert scheme $\mathcal{H}^l\left[\mathbb{C}^2\right]$ parametrizing all clusters of length l on $\mathbb{C}^2$, such that the induced proper birational morphism*

$$\mathcal{H}^G\left(\mathbb{C}^2\right) \longrightarrow \mathbb{C}^2/G = X$$

gives again the minimal resolution (1.1) *of X* (*up to isomorphism*). *Moreover, the original correspondence of* [49], [25], [45], *between the non-trivial representations of G and the exceptional prime divisors of* (1.1) *can be reinterpreted exclusively in terms of suitable ideals of $\mathcal{H}^G\left(\mathbb{C}^2\right)$.*

• More generally, for arbitrary r, if $f : \widehat{X} \to X$ denotes a projective crepant ("full") desingularization of $X = \mathbb{C}^r/G$, then the expected bijections are those of the following box:

$$\begin{array}{cccc}
\boxed{\mathbf{A}} & \left\{\begin{array}{c}\text{irreducible}\\ \text{representations}\\ \text{of the group } G\end{array}\right\} & \overset{1:1}{\longleftrightarrow} & \left\{\begin{array}{c}\text{a suitable basis}\\ \text{of the cohomology}\\ \text{ring } H^*\left(\widehat{X};\mathbb{Z}\right)\end{array}\right\} \\
 & \updownarrow & & \\
\boxed{\mathbf{B}} & \left\{\begin{array}{c}\text{conjugacy}\\ \text{classes of } G\end{array}\right\} & \overset{1:1}{\longleftrightarrow} & \left\{\begin{array}{c}\text{a suitable basis of the}\\ \text{homology ring } H_*\left(\widehat{X};\mathbb{Z}\right)\end{array}\right\}
\end{array}$$

which is now known as *Reid's slogan* :

$$\text{representation theory of } G \quad \text{“}=\text{”} \quad \text{homology of } \widehat{X}.$$

This conjecture is accompanied by the remark that the above bijections probably satisfy certain "compatibilities" (as for $r = 2$) with respect to the behaviour of the cup product, the image of the character table of G, the duality interrelation etc.

• Although on the level of "counting" dimensions of *rational* cohomology groups, or even on that of providing formal correspondences between the left and the right hand side, the required techniques are meanwhile well-understood (by toric methods [3] for G abelian, and by recent results of Batyrev [2] involving non-archimedian integrals, for arbitrary G's), there are still lots of open questions of how one might work with (co)homology groups the coefficients of which are taken from $\mathbb{Z}$.

• Reid's approach to this generalized McKay-type-conjecture is two-fold. The first idea (concerning correspondence **B**) relies on the application of the following *Ito-Reid theorem* [41] in order to construct a suitable collection of loci within $\widehat{X}$ (i.e., a collection of centers of monomial valuations on $\mathbb{C}(X)$) generating $H_*\left(\widehat{X};\mathbb{Z}\right)$.

Theorem 1.5. *Let G be a finite subgroup of $\mathrm{SL}(r,\mathbb{C})$ acting linearly on $\mathbb{C}^r$, $r \geq 2$, and $X = \mathbb{C}^r/G$. Then there is a canonical one-to-one correspondence between the junior conjugacy classes in G and the crepant discrete valuations of X.*

• The second idea (w.r.t. bijection **A**) is to consider the tautological sheaves $\mathfrak{F}_\rho$ assigned to the irreducible representations ρ of G. Reid [63] conjectures (and proves for several examples) that appropriate $\mathbb{Z}$-linear combinations of the Chern classes of $\mathfrak{F}_\rho$'s lead to a canonical $\mathbb{Z}$-basis of the cohomology ring $H^*\left(\widehat{X};\mathbb{Z}\right)$. Moreover, if $\widehat{X}$ happens to be isomorphic to $\mathcal{H}^G(\mathbb{C}^r)$, then these sheaves enjoy very good algebraic-geometric properties (they are generated by their global sections, are vector bundles, their first Chern classes induce nef linear systems etc., cf. [63], 5.5). In particular, in this case $\mathcal{H}^G(\mathbb{C}^r)$ is birationally distinguished among all the other projective crepant resolutions of X.

• As both **A** and **B** are clarified in dimension 2, let us recall what is known for $r = 3$. Next theorem is due to Markushevich [47], [48], Ito [35], [36], [37], and Roan [65], [66], [67], [68].

Theorem 1.6. *The underlying spaces of all 3-dimensional Gorenstein quotient singularities possess crepant resolutions.*

Conjecture 1.7. *Let G be any finite subgroup of $\mathrm{SL}(3,\mathbb{C})$. Then $\mathcal{H}^G(\mathbb{C}^3)$ is a crepant resolution of $\mathbb{C}^3/G$.*

For abelian G's Conjecture 1.7 was proved by Nakamura [53] (for an outline of the proof see also [63], § 7); work on the non-abelian case is in progress. The complete verification of 1.7 would mean that for all quotients $\mathbb{C}^3/G$ there is always a distinguished[1] smooth minimal model available, satisfying all the above mentioned peculiar properties.

• In dimensions $r \geq 4$, however, there are certain additional troubles already from the very beginning and Reid's complementary question ([62], [41], § 4.5, [63], 5.4) still remains an unanswered enigma:

[1]In contrast to dimension 2, in dimension 3 minimal models are unique only up to isomorphisms in codimension 1. Moreover, there exist lots of examples of acting groups G, for which $\mathbb{C}^3/G$ has crepant, full, *non-projective* resolutions (see below 7.5).

Reid's question : Under which conditions on the acting groups

$$G \subset \mathbf{SL}(r, \mathbb{C}), \ r \geq 4,$$

do the quotient spaces $\mathbb{C}^r/G$ have projective crepant desingularizations?

• Note that the existence of *terminal* Gorenstein singularities implies automatically that *not all* Gorenstein quotient spaces $\mathbb{C}^r/G$, $r \geq 4$, can have such desingularizations (cf. Morrison-Stevens [50]).

• Moreover, in contrast to what is valid in the "low" dimensions 2 and 3, the Hilbert scheme $\mathcal{H}^G(\mathbb{C}^r)$ for $r \geq 4$ might be *singular*, even if the quotient $\mathbb{C}^r/G$ being under consideration is known to possess projective, crepant resolutions. (We are indebted to I.Nakamura and M.Reid for this information[2]).

• On the other hand, as it was proved in [11] by making use of Watanabe's classification [82] of all abelian quotient singularities $(\mathbb{C}^r/G, [\mathbf{0}])$, $G \subset \mathrm{SL}(r, \mathbb{C})$, (up to analytic isomorphism) whose underlying spaces are embeddable as complete intersections ("c.i.'s") of hypersurfaces into an affine complex space, and methods of toric and discrete geometry,

Theorem 1.8. *The underlying spaces of all abelian quotient c.i.-singularities admit of torus-equivariant projective, crepant resolutions* (*and therefore* smooth minimal models) *in all dimensions.*

In particular, taking into account the specific structure of these singularities depending on the *free parameters* of the so-called "Watanabe forests", this theorem guarantees the existence of *infinitely many* (isomorphism classes of) Gorenstein quotient singularities *in each dimension* having resolutions with the required properties. Nevertheless, these c.i.-singularities are of special nature, and they form a relatively "sparse" subclass of the class of all Gorenstein abelian quotient singularities. (For instance, all Gorenstein *cyclic* quotient msc-singularities in dimensions ≥ 3 are *not* c.i.'s!). Thus, as "next step" it is natural to ask what happens with respect to the *non-c.i.*'s. Various necessary existence-criteria working quite well in the framework of this most general consideration, partially sufficient conditions, and certain theoretical and algorithmic difficulties which arise from LP-feasibility problems (cf. rem. 6.16 below), as well as further families of non-c.i. abelian quotient singularities for which it is possible to apply a direct, constructive method to obtain the desired resolutions, will be discussed in detail in [12]. In the present paper, we shall study another special (but, again, infinite) series of Gorenstein, non-c.i. (for $r \geq 3$), cyclic quotient singularities admitting a *uniquely determined* torus-equivariant projective, crepant resolution under very simple and absolutely well-controllable (necessary *and* sufficient) number-theoretic conditions. This resolution will be defined as an immediate generalization of the most well-known example in the literature, namely of the "single blow-up" of the affine cone over

[2]Nakamura's typical counterexample is the so-called $(4;2)$-hypersurface-singularity (in the terminology of [11]), with *non-smooth* $\mathcal{H}^G$. This singularity has projective crepant resolutions (cf. [11], cor. 6.3, or [66], § 5).

the l-tuple Veronese embedding of $\mathbb{P}^{r-1}_{\mathbb{C}}$ at the origin. Let us first formulate it explicitly.

Proposition 1.9. *Let G be the finite cyclic group of analytic automorphisms of $\mathbb{C}^r$, $r \geq 2$, of order $l \geq 2$, generated by*

$$g : \mathbb{C}^r \ni (z_1, \ldots, z_r) \mapsto \left(e^{\frac{2\pi\sqrt{-1}}{l}} \, z_1, \ldots, e^{\frac{2\pi\sqrt{-1}}{l}} \, z_r \right) \in \mathbb{C}^r \tag{1.2}$$

(i) $(\mathbb{C}^r/G, [\mathbf{0}])$ *is an isolated singularity, and its underlying space is embedded into* $\mathbb{C}^{\binom{r+l-1}{l}}$ *as the zero-set*

$$\{ t_{\mathbf{u}} \, t_{\mathbf{v}} \, \cdots \, t_{\mathbf{w}} - t_{\mathbf{u}'} \, t_{\mathbf{v}'} \, \cdots \, t_{\mathbf{w}'} = 0 \mid \mathrm{sort}\,(\mathbf{u}\,\mathbf{v}\,\cdots\,\mathbf{w}) = \mathrm{sort}\,(\mathbf{u}'\,\mathbf{v}'\,\cdots\,\mathbf{w}') \} \ ,$$

where $\mathbf{u}, \mathbf{v}, \ldots$ are defined as multiple index sets

$$\mathbf{u} = u_1 \, u_2 \, \ldots \, u_l = \underbrace{1\,1\ldots1}_{i_1\text{-times}} \ \underbrace{2\,2\ldots2}_{i_2\text{-times}} \ \underbrace{3\,3\ldots3}_{i_3\text{-times}} \ \cdots \ \underbrace{r\,r\ldots r}_{i_r\text{-times}}$$

with

$$i_1 + i_2 + \cdots + i_r = l \quad \& \quad 0 \leq i_j \leq l, \ \forall j, \ 1 \leq j \leq r,$$

and $\mathrm{sort}(\,\cdot\,)$ *denotes the sorting of any string of the alphabet* $\{1, 2, 3, \ldots, r\}$ *into weakly increasing order.*

(ii) *The singularity* $(\mathbb{C}^r/G, [\mathbf{0}])$ *is*

$$\begin{cases} \text{terminal} & \Longleftrightarrow \quad r > l \\ \text{canonical} & \Longleftrightarrow \quad r \geq l \\ \text{Gorenstein} & \Longleftrightarrow \quad l \mid r \end{cases}$$

(iii) *If* $\mathbf{Bl_0}(\mathbb{C}^r)$ *denotes the (usual) blow-up of* $\mathbb{C}^r$ *at the origin, then the action of G on $\mathbb{C}^r$ can be extended onto* $\mathbf{Bl_0}(\mathbb{C}^r)$,

$$\mathbf{Bl_0}(\mathbb{C}^r)/G \to \mathbb{C}^r/G \tag{1.3}$$

is a (full) resolution of $[\mathbf{0}] \in \mathbb{C}^r/G$, the exceptional prime divisor D is isomorphic to $\mathbb{P}^{r-1}_{\mathbb{C}}$, and the corresponding relative canonical divisor equals $\left(\frac{r}{l} - 1\right) D$.

(iv) (1.3) *is* (the unique) *crepant resolution of $\mathbb{C}^r/G$ if and only if $r = l$.*

Proof. (i) is an easy exercise (one has just to compute the generators of the ring of invariants and their relations, cf. [42], p. 40, and Sturmfels [73], p. 141, for the explanation); (ii) follows from the general theorems of Reid [59] (cf. (1.5), p. 277, (3.1), p. 292) or directly from (iii). The construction of (1.3) is due to Ueno [75], (see also [76], pp. 199-211), who called it the "canonical resolution" of $\mathbb{C}^r/G$

and used it to obtain generalized Kummer manifolds by resolving special quotient spaces defined by discrete groups acting on complex tori. (iv) was pointed out by Hirzebruch & Höfer in [34], p. 257, and follows from (iii). For the uniqueness (up to isomorphism), see e.g. Roan's comments in [68], Ex. 1, p. 135. □

(c) For *fixed* dimension $r \geq 2$, a "2-parameter" series of cyclic quotient singularities containing (1.2) of prop. 1.9 as its "first member" ($\mu = 1$) is that of *type*

$$\frac{1}{l}\left(\underbrace{1,1,\ldots,1,1}_{(r-1)\text{-times}},\ \mu\right),\ \mu \geq 1 \tag{1.4}$$

(see 5.3 for the definition and notation). A unique "canonical desingularization" for each of its members was given by Ueno [74], § 4, pp. 53-63, and Fujiki [21], pp. 316-318, in the case in which the dimension equals $r = 3$, and turned out to be very useful for the characterization of exceptional fibers belonging to smooth threefolds fibered over a curve and having normally polarized abelian surfaces as generic fibers. (This was, in fact, a direct generalization of Hirzebruch's "continued fraction algorithm" [33], pp. 15-20, to the next coming dimension). Since we are mainly interested in Gorenstein singularities and in the existence (or non-existence) of projective, crepant, full resolutions in dimensions $r \geq 4$, we shall consider for (1.4) only the case where $\mu = l - (r-1)$ (cf. 5.6, 8.1 below) and consequently l as the only available parameter. Our main motivation to work out this series was a recent remark of Reid in [63], 5.4; namely, that the four-dimensional cyclic singularities of type $\frac{1}{l}(1,1,1,l-3)$ with $\gcd(l,3) = 1$ are to be resolved by a crepant morphism if and only if $l \equiv 1 \bmod 3$. We give a generalization in all dimensions, even without assuming the isolatedness of the corresponding singularity, show the uniqueness and projectivity of the crepant morphism, describe the exceptional prime divisors and their intersection numbers, and compute the cohomology dimensions of the desingularized space.

Exactly as in [11], we shall exclusively work with the machinery of toric geometry. More precisely, the paper has the following structuring : In section 2 we give the toric glossary which will be used in the sequel. (The reader who is familiar with this matter may skip it). Sections 3 and 4 are complementary. (In fact, the reason for adding in § 3 some lengthy explanations is that there is a potential for confusion between the usual blow-up of a toric subvariety $V(\tau)$ of an $X(N,\Delta)$ and the starring subdivision w.r.t. τ. These are identical only for smooth $X(N,\Delta)$'s! On the other hand, to lend an algebraic-geometric characterization to even very simple combinatorially motivated cone subdivisions, it is absolutely natural to blow-up also not necessarily reduced subschemes). In § 4 we deal with a high dimensional analogue of the so-called Hirzebruch-surfaces and make certain remarks concerning its embeddings and intersection theory. (It turns out that all but one exceptional prime divisors which will arise later on in our desingularizations are of this sort). Sections 5-6 outline a first systematic approach to the general problem of the existence or non-existence of crepant (preferably projective) resolutions of Gorenstein abelian quotient singularities of dimension ≥ 4 by basic (and coherent) triangulations of the junior simplex. In § 7 we take a closer look at the low dimensions. Sections 8 and 9 contain our main results. Though the singularity-series which we study is rather special, we hope at least that it will become clear how one may apply our techniques to more demanding singularities. In particular, in section 9, the factorization of the desingularizing morphism is reduced to a "game" with the available simplices. Finally, in § 10 we give a foretaste of what may be done for the series generalizing $\frac{1}{7}(1,2,4)$ and state the GPSS-conjecture.

Terminology and general notation. By a *complex variety* is meant an integral, separated algebraic scheme over $\mathbb{C}$. A complex variety is therefore an irreducible, reduced ringed space $(X, \mathcal{O}_X)$ with structure sheaf $\mathcal{O}_X$ which is locally determined by the canonical structure sheaf of the spectrum of an affine complex coordinate ring. $\mathrm{Sing}(X)$ denotes the *singular locus* of X, i.e., the set of all points $x \in X$ with $\mathcal{O}_{X,x}$ a non-regular local ring. Analogously, $x \in X$ is normal, Cohen-Macaulay, Gorenstein etc., if $\mathcal{O}_{X,x}$ is of this type. A *subvariety* Y of X is a closed integral subscheme of X. If $\mathrm{codim}_X(Y) = 1$, then Y is especially called a *prime divisor*. By $\mathrm{CDiv}(X)$, $\mathrm{WDiv}(X)$, $\mathrm{Pic}(X)$ and $A_\bullet(X) = \oplus_{k\geq 0} A_k(X)$ we denote the groups of Cartier and Weil divisors, the Picard group, and the graded Chow ring of X, respectively. (For X smooth, $A^\bullet(X) = \oplus_{k\geq 0} A^k(X)$, with $A^k(X) = A_{\dim X - k}(X)$). Just as in [3], [11], by a *desingularization* (or *resolution of singularities*) $f : \widehat{X} \to X$ of a non-smooth X, we mean a "full" or "overall" desingularization (if not mentioned), i.e., $\mathrm{Sing}\left(\widehat{X}\right) = \varnothing$. When we deal with *partial* desingularizations, we mention it explicitly. A birational morphism $f : X' \to X$ is *projective* if X' admits an f-ample Cartier divisor. The *intersection numbers* of Cartier divisors are defined as in [22] (see below § 2 (i)).

2. Preliminaries from toric geometry

We recall some basic facts from the theory of toric varieties and fix the notation which will be used in the sequel. For details the reader is referred to the standard textbooks of Oda [54], Fulton [23], and Ewald [17], and to the lecture notes [42].

(a) The *linear hull,* the *affine hull*, the *positive hull* and *the convex hull* of a set B of vectors of $\mathbb{R}^r$, $r \geq 1$, will be denoted by $\mathrm{lin}(B)$, $\mathrm{aff}(B)$, $\mathrm{pos}(B)$ (or $\mathbb{R}_{\geq 0}\, B$) and $\mathrm{conv}(B)$ respectively. The *dimension* $\dim(B)$ of a $B \subset \mathbb{R}^r$ is defined to be the dimension of its affine hull.

(b) *Polyhedral cones.* Let $N \cong \mathbb{Z}^r$ be a free $\mathbb{Z}-$module of rank $r \geq 1$. N can be regarded as a *lattice* in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r$. (For fixed identification, we shall represent the elements of $N_{\mathbb{R}}$ by column-vectors in $\mathbb{R}^r$). If $\{n_1, \ldots, n_r\}$ is a $\mathbb{Z}$-basis of N, then

$$\det(N) := |\det(n_1, \ldots, n_r)|$$

is the *lattice determinant*. An $n \in N$ is called *primitive* if $\mathrm{conv}(\{\mathbf{0}, n\}) \cap N$ contains no other points except $\mathbf{0}$ and n.

Let $N \cong \mathbb{Z}^r$ be as above, $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual lattice, $N_{\mathbb{R}}, M_{\mathbb{R}}$ their real scalar extensions, and $\langle .,. \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ the natural $\mathbb{R}$-bilinear pairing. (For fixed identification $M_{\mathbb{R}} \cong \mathbb{R}^r$, we analogously represent the elements of $M_{\mathbb{R}}$ by row-vectors in $\mathbb{R}^r$). A subset σ of $N_{\mathbb{R}}$ is called *strongly convex polyhedral cone* (*s.c.p.c.*, for short), if there exist $n_1, \ldots, n_k \in N_{\mathbb{R}}$, such that $\sigma = \mathrm{pos}(\{n_1, \ldots, n_k\})$ and $\sigma \cap (-\sigma) = \{\mathbf{0}\}$. Its *relative interior* $\mathrm{int}(\sigma)$ (resp. its *relative boundary* $\partial\sigma$) is the usual topological interior (resp. the usual topological boundary) of it, considered as subset of $\mathrm{lin}(\sigma)$. The *dual cone* of σ is defined by

$$\sigma^{\vee} := \{\mathbf{x} \in M_{\mathbb{R}} \mid \langle \mathbf{x}, \mathbf{y} \rangle \geq 0, \ \forall \mathbf{y}, \ \mathbf{y} \in \sigma\}$$

and satisfies: $\sigma^{\vee} + (-\sigma^{\vee}) = M_{\mathbb{R}}$ and $\dim(\sigma^{\vee}) = r$. A subset τ of a s.c.p. cone σ is called a *face* of σ (notation: $\tau \prec \sigma$), if $\tau = \{\mathbf{y} \in \sigma \mid \langle m_0, \mathbf{y} \rangle = 0\}$, for some $m_0 \in \sigma^{\vee}$. A s.c.p.c $\sigma = \mathrm{pos}(\{n_1, \ldots, n_k\})$ is called *simplicial* (resp. *rational*) if $n_1, \ldots, n_k$ are $\mathbb{R}$-linearly independent (resp. if $n_1, \ldots, n_k \in N_{\mathbb{Q}}$, where $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$).

(c) *Monoids.* If $\sigma \subset N_{\mathbb{R}}$ is a rational s.c.p. cone, then σ has $\mathbf{0}$ as its apex and the subsemigroup $\sigma \cap N$ of N is a monoid. The following two propositions describe the fundamental properties of this monoid $\sigma \cap N$ and their proofs go essentially back to Gordan [26], Hilbert [30] and van der Corput [77], [78].

Proposition 2.1 (Gordan's lemma). *$\sigma \cap N$ is finitely generated as additive semigroup, i.e. there exist*

$$n_1, n_2, \ldots, n_\nu \in \sigma \cap N \ , \quad \textit{such that} \quad \sigma \cap N = \mathbb{Z}_{\geq 0}\, n_1 + \mathbb{Z}_{\geq 0}\, n_2 + \cdots + \mathbb{Z}_{\geq 0}\, n_\nu \ .$$

Proposition 2.2 (Minimal generating system). *Among all systems of generators of $\sigma \cap N$, there is a system $\mathbf{Hlb}_N(\sigma)$ of* minimal cardinality, *which is uniquely determined (up to the ordering of its elements) by the following characterization* :

$$\boxed{\mathbf{Hlb}_N(\sigma) = \left\{ n \in \sigma \cap (N \smallsetminus \{\mathbf{0}\}) \;\middle|\; \begin{array}{c} n \text{ cannot be expressed} \\ \text{as the sum of two other} \\ \text{vectors belonging} \\ \text{to } \ \sigma \cap (N \smallsetminus \{\mathbf{0}\}) \end{array} \right\}} \qquad (2.1)$$

Proof. See e.g. Schrijver [70], p. 233. □

Definition 2.3. $\mathbf{Hlb}_N(\sigma)$ is often called *the Hilbert basis of σ w.r.t. N.*

About *algorithms* for the determination of Hibert bases of pointed rational cones, we refer to Pottier [56], [57], Sturmfels [73] (13.2, p. 128), and Henk-Weismantel [28], and to the other references therein.

(d) *Algebraic tori defined via N.* Let $\mathbb{C}^*$ be the multiplicative group of non-zero complex numbers. For $N \cong \mathbb{Z}^r$ we define an r-dimensional *algebraic torus* $T_N \cong (\mathbb{C}^*)^r$ by :

$$T_N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^* .$$

Every $m \in M$ assigns a character $\mathbf{e}(m) : T_N \to \mathbb{C}^*$ with

$$\mathbf{e}(m)(t) := t(m), \quad \forall t, \quad t \in T_N.$$

We have :

$$\mathbf{e}(m + m') = \mathbf{e}(m) \cdot \mathbf{e}(m') \ , \quad \text{for} \quad m, m' \in M \ , \ \text{ and } \ \mathbf{e}(\mathbf{0}) = 1 \ .$$

Moreover, for each $n \in N$, we define an 1-parameter subgroup

$$\gamma_n : \mathbb{C}^* \to T_N \quad \text{with} \quad \gamma_n(\lambda)(m) := \lambda^{\langle m,n \rangle}, \quad \text{for} \quad \lambda \in \mathbb{C}^*, \ m \in M,$$

$(\gamma_{n+n'} = \gamma_n \circ \gamma_{n'}, \quad \text{for} \quad n, n' \in N)$. We can therefore identify M with the character group of T_N and N with the group of 1-parameter subgroups of T_N. If $\{n_1, \ldots, n_r\}$ is a $\mathbb{Z}$-basis of N and $\{m_1, \ldots, m_r\}$ the dual basis (of M) and if we set $u_j := \mathbf{e}(m_j)$, $\forall j$, $1 \leq j \leq r$, then there exists an isomorphism

$$T_N \ni t \overset{\cong}{\longmapsto} (u_1(t), \ldots, u_r(t)) \in (\mathbb{C}^*)^r \ ,$$

and $\{u_1, \ldots, u_r\}$ plays the role of a coordinate system of T_N. Hence, to an

$$m = \sum_{j=1}^{r} a_j m_j \in M \qquad (\text{ resp. to an } n = \sum_{j=1}^{r} b_j n_j \ )$$

we associate the character ("Laurent monomial") $\quad \mathbf{e}(m) = u_1^{a_1} \cdot u_1^{a_2} \cdots \ u_r^{a_r}$

$$(\text{ resp. the 1-parameter subgroup } \gamma_n : \mathbb{C}^* \ni \lambda \mapsto \left(\lambda^{b_1}, \ldots, \lambda^{b_r}\right) \in (\mathbb{C}^*)^r \ ) \ .$$

On the other hand, for a rational full-dimensional s.c.p.c. σ with

$$M \cap \sigma^\vee = \mathbb{Z}_{\geq 0}\, m_1 + \mathbb{Z}_{\geq 0}\, m_2 + \cdots + \mathbb{Z}_{\geq 0}\, m_k,$$

we associate to the finitely generated, normal, monoidal $\mathbb{C}$-subalgebra $\mathbb{C}[M \cap \sigma^\vee]$ of $\mathbb{C}[M]$ an affine complex variety

$$U_\sigma := \text{Max-Spec}\,(\mathbb{C}[M \cap \sigma^\vee]),$$

which can be identified with the set of semigroup homomorphisms :

$$U_\sigma = \left\{ u : M \cap \sigma^\vee \ \to \mathbb{C} \ \middle| \ \begin{array}{c} u(\mathbf{0}) = 1, \ u(m + m') = u(m) \cdot u(m'), \\ \text{for all} \quad m, m' \in M \cap \sigma^\vee \end{array} \right\} ,$$

where $\mathbf{e}(m)(u) := u(m)$, $\forall m, \ m \in M \cap \sigma^\vee$ and $\forall u, \ u \in U_\sigma$.

Proposition 2.4 (Embedding by binomials). *In the analytic category, U_σ, identified with its image under the injective map $(\mathbf{e}(m_1), \ldots, \mathbf{e}(m_k)) : U_\sigma \hookrightarrow \mathbb{C}^k$, can be regarded as an analytic set determined by a system of equations of the form:*

(monomial) = (monomial). *This analytic structure induced on U_σ is independent of the semigroup generators $\{m_1, \ldots, m_k\}$ and each map $\mathbf{e}(m)$ on U_σ is holomorphic w.r.t. it. In particular, for $\tau \prec \sigma$, U_τ is an open subset of U_σ. Moreover, if $\#(\mathbf{Hlb}_M(\sigma^\vee)) = d \ (\leq k)$, then d is nothing but the* embedding dimension *of U_σ, i.e. the* minimal *number of generators of the maximal ideal of the local $\mathbb{C}$-algebra $\mathcal{O}_{U_\sigma, (\mathbf{0} \in \mathbb{C}^d)}$.*

Proof. See Oda [54] prop. 1.2 and 1.3., pp. 4-7. □

(e) *Fans.* A *fan* w.r.t. $N \cong \mathbb{Z}^r$ is a finite collection Δ of rational s.c.p. cones in $N_\mathbb{R}$, such that :

(i) any face τ of $\sigma \in \Delta$ belongs to Δ, and

(ii) for $\sigma_1, \sigma_2 \in \Delta$, the intersection $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

The union $|\Delta| := \cup\{\sigma \mid \sigma \in \Delta\}$ is called the *support* of Δ. Furthermore, we define

$$\Delta(i) := \{\sigma \in \Delta \mid \dim(\sigma) = i\} \ , \text{ for } \quad 0 \leq i \leq r \ .$$

If $\varrho \in \Delta(1)$, then there exists a unique primitive vector $n(\varrho) \in N \cap \varrho$ with $\varrho = \mathbb{R}_{\geq 0}\, n(\varrho)$ and each cone $\sigma \in \Delta$ can be therefore written as

$$\sigma = \sum_{\substack{\varrho \in \Delta(1) \\ \varrho \prec \sigma}} \mathbb{R}_{\geq 0}\, n(\varrho) \quad .$$

The set $\mathrm{Gen}(\sigma) := \{n(\varrho) \mid \varrho \in \Delta(1), \varrho \prec \sigma\}$ is called the *set of minimal generators* (within the pure first skeleton) of σ. For Δ itself one defines analogously

$$\mathrm{Gen}(\Delta) := \bigcup_{\sigma \in \Delta} \mathrm{Gen}(\sigma) \ .$$

(f) *Toric varieties, orbits and stars.* The *toric variety* $X(N, \Delta)$ associated to a fan Δ w.r.t. the lattice N is by definition the identification space

$$X(N, \Delta) := \left(\left(\bigcup_{\sigma \in \Delta} U_\sigma \right) \ / \ \sim \right)$$

with $U_{\sigma_1} \ni u_1 \sim u_2 \in U_{\sigma_2}$ if and only if there is a $\tau \in \Delta$, such that $\tau \prec \sigma_1 \cap \sigma_2$ and $u_1 = u_2$ within U_τ (cf. lemma 3.1 below). As complex variety, $X(N, \Delta)$ turns out to be irreducible, normal, Cohen-Macaulay and to have at most rational singularities (cf. [23], p. 76, and [54], thm. 1.4, p. 7, and cor. 3.9, p. 125). $X(N, \Delta)$ is called *simplicial* if all cones of Δ are simplicial.

• $X(N, \Delta)$ admits a canonical T_N-action which extends the group multiplication of $T_N = U_{\{\mathbf{0}\}}$:

$$T_N \times X(N, \Delta) \ni (t, u) \longmapsto t \cdot u \in X(N, \Delta) \tag{2.2}$$

where, for $u \in U_\sigma$, $(t \cdot u)(m) := t(m) \cdot u(m)$, $\forall m,\ m \in M \cap \sigma^\vee$. The orbits w.r.t. the action (2.2) are parametrized by the set of all the cones belonging to Δ. For a $\tau \in \Delta$, we denote by orb(τ) (resp. by $V(\tau)$) the orbit (resp. the closure of the orbit) which is associated to τ. The spaces orb(τ) and $V(\tau)$ have the following properties (cf. [23], pp. 52-55, [54], § 1.3) :

(i) For $\tau \in \Delta$, it is

$$V(\tau) = \coprod \{\mathrm{orb}(\sigma) \mid \sigma \in \Delta, \tau \prec \sigma\}$$

and

$$\mathrm{orb}(\tau) = V(\tau) \smallsetminus \bigcup \{V(\sigma) \mid \tau \precneqq \sigma\} .$$

(ii) If $\tau \in \Delta$, then $V(\tau) = X(N(\tau), \mathrm{Star}(\tau;\Delta))$ is itself a toric variety w.r.t.

$$N(\tau) := N/N_\tau , \quad N_\tau := N \cap \mathrm{lin}(\tau) , \quad \mathrm{Star}(\tau;\Delta) := \{\overline{\sigma} \mid \sigma \in \Delta,\ \tau \prec \sigma\} ,$$

where $\overline{\sigma} = (\sigma + (N_\tau)_{\mathbb{R}}) / (N_\tau)_{\mathbb{R}}$ denotes the image of σ in $N(\tau)_{\mathbb{R}} = N_{\mathbb{R}}/(N_\tau)_{\mathbb{R}}$.

(iii) For $\tau \in \Delta$, the closure $V(\tau)$ is equipped with an affine open covering

$$\{U_\sigma(\tau) \mid \tau \prec \sigma\}$$

consisting of "intermediate" subvarieties

$$U_\tau(\tau) = \mathrm{orb}(\tau) \hookrightarrow U_\sigma(\tau) \hookrightarrow U_\sigma$$

being defined by : $U_\sigma(\tau) :=$ Max-Spec($\mathbb{C}[\overline{\sigma}^\vee \cap M(\tau)]$), with $M(\tau)$ denoting the dual of $N(\tau)$.

(g) *Smoothness and compactness criterion.* Let $N \cong \mathbb{Z}^r$ be a lattice of rank r and $\sigma \subset N_{\mathbb{R}}$ a simplicial, rational s.c.p.c. of dimension $k \leq r$. σ can be obviously written as $\sigma = \varrho_1 + \cdots + \varrho_k$, for distinct 1-dimensional cones $\varrho_1, \ldots, \varrho_k$. We denote by

$$\mathbf{Par}(\sigma) := \left\{ \mathbf{y} \in (N_\sigma)_{\mathbb{R}} \;\middle|\; \mathbf{y} = \sum_{j=1}^{k} \varepsilon_j\ n(\varrho_j),\ \text{with}\ \ 0 \leq \varepsilon_j < 1,\ \forall j,\ 1 \leq j \leq k \right\}$$

the *fundamental* (*half-open*) *parallelotope* which is associated to σ. The *multiplicity* mult($\sigma; N$) of σ with respect to N is defined as

$$\mathrm{mult}(\sigma; N) := \#(\mathbf{Par}(\sigma) \cap N_\sigma) = \mathrm{Vol}(\mathbf{Par}(\sigma); N_\sigma) ,$$

where Vol($\mathbf{Par}(\sigma)$) denotes the usual volume of $\mathbf{Par}(\sigma)$ and

$$\mathrm{Vol}(\mathbf{Par}(\sigma); N_\sigma) := \frac{\mathrm{Vol}(\mathbf{Par}(\sigma))}{\det(N_\sigma)}$$

its the relative volume w.r.t. N_σ.

Proposition 2.5. *The affine toric variety U_σ is smooth iff* $\mathrm{mult}(\sigma; N) = 1$. (*Correspondingly, an arbitrary toric variety $X(N, \Delta)$ is smooth if and only if it is simplicial and each s.c.p. cone $\sigma \in \Delta$ satisfies this condition.*)

Proof. It follows from [54], thm. 1.10, p. 15. $\square$

• For the systematic study of toric singularities it is useful to introduce the notion of the "splitting codimension" of the closed point $\mathrm{orb}(\sigma)$ of an U_σ. For the germ $(U_\sigma, \mathrm{orb}(\sigma))$ of an affine r-dimensional toric variety w.r.t. a *singular point* $\mathrm{orb}(\sigma)$, the *splitting codimension* $\mathrm{splcod}(\mathrm{orb}(\sigma); U_\sigma)$ of $\mathrm{orb}(\sigma)$ in U_σ is defined as :

$$\mathrm{splcod}(\mathrm{orb}(\sigma); U_\sigma) := \min \left\{ \varkappa \in \{2, \ldots, r\} \;\middle|\; \begin{array}{c} U_\sigma \cong U_{\sigma'} \times \mathbb{C}^{r-\varkappa}, \text{ s.t.} \\ \dim(\sigma') = \varkappa \text{ and} \\ \mathrm{Sing}(U_{\sigma'}) \neq \varnothing \end{array} \right\}$$

If $\mathrm{splcod}(\mathrm{orb}(\sigma); U_\sigma) = r$, then $\mathrm{orb}(\sigma)$ will be called an *msc-singularity*, i.e. a singularity having the maximum splitting codimension.

• Next theorem gives a necessary and sufficient condition for $X(N, \Delta)$ to be compact.

Theorem 2.6. *A toric variety $X(N, \Delta)$ is compact if and only if Δ is a* complete fan, *i.e.*, $|\Delta| = N_{\mathbb{R}}$.

Proof. See Oda [54], thm. 1.11, p. 16. $\square$

(h) *Order functions, support functions and divisors.* If $X(N, \Delta)$ is a (not necessarily compact) toric variety associated to a fan Δ w.r.t. a lattice N, M its dual, and $\iota : T_N \hookrightarrow X(N, \Delta)$ the canonical inclusion, then $\iota_*(\mathcal{O}_{T_N})$ is a T_N-invariant quasi-coherent sheaf of $\mathcal{O}_{X(N,\Delta)}$-modules canonically embedded into the constant sheaf $\mathbb{C}(X(N, \Delta))$ of rational functions of $X(N, \Delta)$. Let $\mathcal{F} \neq \underline{0}$ be a T_N-invariant coherent sheaf of fractional ideals over $X(N, \Delta)$ contained in $\iota_*(\mathcal{O}_{T_N})$. Fix a s.c.p. cone $\sigma \in \Delta$, $n \in N \cap \sigma$, and consider the corresponding 1-parameter group[3]:

$$\gamma_n : \mathrm{Spec}\left(\mathbb{C}\left[w, w^{-1}\right]\right) = \mathbb{C}^* \longrightarrow T_N = \mathrm{Spec}(\mathbb{C}[M])$$

Since $\exists \lim\limits_{\lambda \to 0} \gamma_n(\lambda) \in U_\sigma$ (cf. [54], 1.6.(v), p. 10), γ_n is extendable to a map

$$\overline{\gamma_n} : \mathrm{Spec}(\mathbb{C}[w]) = \mathbb{A}^1_{\mathbb{C}} \longrightarrow U_\sigma = \mathrm{Spec}(\mathbb{C}[M \cap \sigma^\vee]) \ .$$

The coherence of $\mathcal{F}$ implies that $\mathcal{F}|_{U_\sigma}$ is of type $J_\sigma^\sim$ (cf. [27], pp. 110-111), with J_σ being an M-graded complex vector-subspace of

$$H^0(T_N, \mathcal{O}_{T_N}) = \mathbb{C}[M] = \bigoplus_{m \in M} \mathbb{C}\,\mathbf{e}(m)$$

[3] Here we drop the prefix Max- because we do not work only with closed points but also with closed subsets, and the $\mathbb{C}$-scheme structure is essential for the arguments.

on the one hand, and a finitely generated $\mathbb{C}[M \cap \sigma^{\vee}]$-module

$$J_{\sigma} = \sum_{j=1}^{q_{\sigma}} \mathbb{C}[M \cap \sigma^{\vee}] \cdot \mathbf{e}\left(m_j^{(\sigma)}\right), \text{ for some } m_1^{(\sigma)}, \ldots, m_{q_\sigma}^{(\sigma)} \in M, \text{ and } q_{\sigma} \in \mathbb{N},$$

on the other. The pullback $\overline{\gamma_n}^{\,*} \mathcal{F}\,|_{U_\sigma}$ is realized via the finitely generated $\mathbb{C}[w]$-module

$$\sum_{j=1}^{q_{\sigma}} \mathbb{C}[w] \cdot w^{\left\langle m_j^{(\sigma)}, n \right\rangle} \subset \mathbb{C}(w).$$

Define the *order-function* w.r.t. $\mathcal{F}$ by

$$\mathrm{ord}_{\mathcal{F}}(n) := \inf \left\{ \left\langle m_j^{(\sigma)}, n \right\rangle \mid 1 \leq j \leq q_{\sigma} \right\} \in \mathbb{Z}.$$

This $\mathrm{ord}_{\mathcal{F}}(n)$ is exactly the image $\mathrm{ord}_0(\overline{\gamma_n}^{\,*} \mathcal{F}\,|_{U_\sigma})$ under the usual order function

$$\mathrm{ord}_0 : \mathbb{C}(w)^* \to \mathbb{Z}$$

of the discrete valuation ring $\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}},0}$ with w as uniformizing parameter. Since the above definition depends only on σ, one extends $\mathrm{ord}_{\mathcal{F}}$ to the entire $|\Delta|$ by setting

$$\mathrm{ord}_{\mathcal{F}}(\mathbf{y}) := \inf \left\{ \left\langle m_j^{(\sigma)}, \mathbf{y} \right\rangle \mid 1 \leq j \leq q_{\sigma} \right\}, \ \forall \mathbf{y}, \ \ \mathbf{y} \in \sigma, \ \ \sigma \in \Delta.$$

The order function $\mathrm{ord}_{\mathcal{F}}$ is $\mathbb{R}$-valued and has the following characteristic properties:

(i) it is *positively homogeneous*, i.e. $\mathrm{ord}_{\mathcal{F}}(c\,\mathbf{y}) = c\ \mathrm{ord}_{\mathcal{F}}(\mathbf{y})$, for all $c \in \mathbb{R}_{\geq 0}$,

(ii) $\mathrm{ord}_{\mathcal{F}}|_{\sigma}$ is piecewise linear on each $\sigma \in \Delta$,

(iii) $\mathrm{ord}_{\mathcal{F}}(N \cap |\Delta|) \subset \mathbb{Z}$, and

(iv) for all $\sigma \in \Delta$, $\mathrm{ord}_{\mathcal{F}}|_{\sigma}$ is *upper convex*, i.e.,

$$\mathrm{ord}_{\mathcal{F}}|_{\sigma}(\mathbf{y} + \mathbf{y}') \geq \mathrm{ord}_{\mathcal{F}}|_{\sigma}(\mathbf{y}) + \mathrm{ord}_{\mathcal{F}}|_{\sigma}(\mathbf{y}'), \quad \text{for any pair } \ \mathbf{y}, \mathbf{y}' \in \sigma.$$

Definition 2.7. Let $X(N, \Delta)$ be a toric variety. A function $\psi : |\Delta| \to \mathbb{R}$ is called *integral PL-support function* if it satisfies the above properties (i)-(iv). We define

$$\text{PL-SF}(N, \Delta) := \left\{ \begin{array}{c} \text{all integral PL-support} \\ \text{functions defined on } |\Delta| \end{array} \right\}$$

and the sets of *integral* (Δ-*linear*) *support functions*[4]:

[4] An integral Δ-linear support function ψ is called *strictly upper convex* if it is upper convex on $|\Delta|$ and if for any two distinct maximal-dimensional cones σ and σ', the linear functions m_σ, $m_{\sigma'} \in M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \subset \mathrm{Hom}_{\mathbb{R}}(N_{\mathbb{R}}, \mathbb{R}) = M_{\mathbb{R}}$ defining $\psi\,|_{\sigma} = \langle m_\sigma, \bullet \rangle$ and $\psi\,|_{\sigma'} = \langle m_{\sigma'}, \bullet \rangle$ are different.

$$\begin{array}{rl} \mathrm{SF}\,(N,\Delta) := & \{\psi \in \text{PL-SF}\,(N,\Delta) \ \mid \psi\,|_{\sigma} \ \text{ linear on each } \ \sigma \in \Delta \ \} \\ & \bigcup \\ \mathrm{UCSF}\,(N,\Delta) := & \left\{ \psi \in \mathrm{SF}\,(N,\Delta) \ \middle| \ \begin{array}{c} \psi \ \text{ upper convex} \\ \text{on the whole } \ |\Delta| \end{array} \right\} \\ & \bigcup \\ \mathrm{SUCSF}\,(N,\Delta) := & \left\{ \psi \in \mathrm{UCSF}\,(N,\Delta) \ \middle| \ \begin{array}{c} \psi \ \text{ strictly upper} \\ \text{convex on } \ |\Delta| \end{array} \right\}. \end{array}$$

Analogously, one defines the sets $\mathrm{SF}_{\mathbb{Q}}\,(N,\Delta)$, $\mathrm{UCSF}_{\mathbb{Q}}\,(N,\Delta)$, $\mathrm{SUCSF}_{\mathbb{Q}}\,(N,\Delta)$ of *rational* support functions by modifying property (iii) into : $\psi\,(N_{\mathbb{Q}} \cap |\Delta|) \subset \mathbb{Q}$. All the above sets are equipped with the usual additive group structure.

Theorem 2.8. *For a function* $\psi \in$ PL-SF(N,Δ), *and an arbitrary cone* $\sigma \in \Delta$ *define*

$$(J_{\psi})_{\sigma} := \bigoplus_{m \in M} \{\mathbb{C}\,\mathbf{e}\,(m) \ \mid \langle m, \mathbf{y} \rangle \geq \psi\,(\mathbf{y})\,, \ \forall \mathbf{y}, \ \mathbf{y} \in \sigma\}\,.$$

The family of T_N*-invariant sheaves* $\left\{(J_{\psi})_{\sigma}^{\sim} \ \mid \sigma \in \Delta\right\}$ *being associated to the family of ideals* $\left\{(J_{\psi})_{\sigma} \ \mid \sigma \in \Delta\right\}$ *can be glued together* (cf. [27], Ex.II.1.22, p. 69) *to construct a coherent sheaf* $\mathcal{F}_{\psi}$ *of* T_N*-invariant fractional ideals over* $X\,(N,\Delta)$ *contained in* $\boldsymbol{\iota}_*\,(\mathcal{O}_{T_N})$. *Moreover,*

(i) $\mathrm{ord}_{\mathcal{F}_{\psi}} = \psi$,

(ii) $\mathcal{F}_{\mathrm{ord}_{\mathcal{F}}}$ *is the* completion *of* $\mathcal{F}$ (in Zariski's sence);

(iii) *mapping* $\mathcal{F} \longmapsto \mathrm{ord}_{\mathcal{F}}$ *and* $\psi \longmapsto \mathcal{F}_{\psi}$ *one obtains a bijection*

$$\text{PL-SF}\,(N,\Delta) \underset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{coherent sheaves of } T_N\text{-invariant} \\ \text{complete fractional ideals over } \ X\,(N,\Delta) \end{array} \right\} ,$$

(iv) $\mathcal{F} \subset \mathcal{F}_{\psi} \Leftrightarrow \mathrm{ord}_{\mathcal{F}} \geq \psi$, $\ \mathrm{ord}_{\mathcal{F}_1 \cdot \mathcal{F}_2} = \mathrm{ord}_{\mathcal{F}_1} + \mathrm{ord}_{\mathcal{F}_2}$, *and*

(v) $\mathcal{F}_{\psi_1} \cong \mathcal{F}_{\psi_2}$ (*as* $\mathcal{O}_{X(N,\Delta)}$*-module sheaves*) $\Leftrightarrow \psi_1 - \psi_2$ *is linear.*

Proof. See Saint-Donat [42], ch. I, § 2, thm. 9, pp. 28-31. □

Definition 2.9. Let $X\,(N,\Delta)$ be a toric variety and $\sigma \in \Delta$. The *convex interpolation* ψ_{ϑ} of a function $\vartheta : \mathrm{Gen}(\sigma) \to \mathbb{Z}$ is defined by

$$\sigma \ni \mathbf{y} \longmapsto \psi_{\vartheta}\,(\mathbf{y}) := \inf \left\{ \langle m, \mathbf{y} \rangle \ \middle| \ \begin{array}{c} m \in M \ \text{ and } \ \langle m, \mathbf{y} \rangle \geq \vartheta\,(n\,(\varrho))\,, \\ \forall \varrho, \ \ \varrho \in \Delta\,(1)\,, \ \text{ with } \varrho \prec \sigma \end{array} \right\} \in \mathbb{R}.$$

Correspondingly, by the *convex interpolation* of a real function $\vartheta : \mathrm{Gen}(\Delta) \to \mathbb{Z}$ is meant a function $\psi_{\vartheta} : |\Delta| \to \mathbb{R}$, such that $\psi_{\vartheta}\,|_{\sigma}$ is the convex interpolation of $\vartheta\,|_{\sigma}$ (in the above sence) for all $\sigma \in \Delta$. Obviously, such a ψ_{ϑ} belongs to PL-SF(N,Δ), and conversely each integral PL-support function has this form.

• Let X be any r-dimensional *normal* complex variety and $D \in \mathrm{WDiv}(X)$. The correspondence[5]:

$$A_{r-1}(X) \ni \{D\} \overset{\boldsymbol{\delta}}{\longmapsto} \{\mathcal{O}_X(D)\} \in \left\{ \begin{array}{c} \textit{reflexive} \text{ coherent} \\ \text{(i.p. torsion-free) sheaves} \\ \text{of fractional ideals} \\ \text{over } X \text{ having rank one} \end{array} \right\} / \, H^0(X, \mathcal{O}_X^*)$$

with $\mathcal{O}_X(D)$ defined by sending every non-empty open subset U of X onto

$$U \longmapsto \mathcal{O}_X(D)(U) := \left\{ \varphi \in \mathbb{C}(X)^* \mid (\mathrm{div}(\varphi) + D)\,|_U \geq 0 \right\},$$

induces a $\mathbb{Z}$-module isomorphism (cf. Reid [59], App. to § 1); in fact, to avoid torsion, one defines this $\mathbb{Z}$-module structure by setting:

$$\boldsymbol{\delta}(D_1 + D_2) := (\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2))^{\vee\vee}, \ \boldsymbol{\delta}(\kappa D) := \mathcal{O}_X(D)^{[\kappa]} = \mathcal{O}_X(\kappa D)^{\vee\vee},$$

for any $D, D_1, D_2 \in \mathrm{WDiv}(X)$ and $\kappa \in \mathbb{Z}$.

• For $X = X(N, \Delta)$ any r-dimensional toric variety let now

$$T_N\text{-WDiv}(X) \quad \text{and} \quad T_N\text{-CDiv}(X) = T_N\text{-WDiv}(X) \cap \mathrm{CDiv}(X)$$

denote the groups of T_N-invariant Weil and Cartier divisors, respectively. If Δ is not contained in any proper subspace of $N_{\mathbb{R}}$, then[6]

$$\begin{array}{ccc} 0 & & 0 \\ \uparrow & & \uparrow \\ \mathrm{Pic}(X) & \hookrightarrow & A_{r-1}(X) \\ \uparrow & & \uparrow \\ T_N\text{-CDiv}(X) & \hookrightarrow & T_N\text{-WDiv}(X) \\ \uparrow & & \uparrow \\ M & = & M \\ \uparrow & & \uparrow \\ 0 & & 0 \end{array}$$

is a commutative diagram with exact columns. T_N-WDiv(X) has as $\mathbb{Z}$-basis:

$$T_N\text{-WDiv}(X) = \bigoplus \mathbb{Z}\,\{V(\varrho) \mid \varrho \in \Delta(1)\}.$$

---

[5] *Reflexive* coherent sheaves $\mathcal{F}$ are those which are isomorphic to their biduals $\mathcal{F}^{\vee\vee}$ (where $\mathcal{F}^{\vee} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$). For $\mathcal{F} \subset \mathbb{C}(X)$ of rank one they are also called *divisorial*.

[6] In particular, for X smooth and compact, Pic(X) is torsion free and the Picard number equals $\#\Delta(1) - r$.

Theorem 2.10 (Divisors and support functions). *There exist one-to-one correspondences:*

$$\begin{array}{ccc}
\text{PL-SF}\,(N,\Delta) & \longleftrightarrow & \left\{ \begin{array}{c} \text{evaluation functions} \\ \vartheta : \text{Gen}\,(\Delta) \to \mathbb{Z} \end{array} \right\} \\
\updownarrow & & \updownarrow \\
T_N\text{-WDiv}\,(X) & \longleftrightarrow & \left\{ \begin{array}{c} \text{reflexive coherent (i.p. torsion-free)} \\ \text{sheaves of rank 1 of} \\ T_N\text{-invariant, complete} \\ \text{fractional ideals} \\ \text{over } \ X = X\,(N,\Delta) \end{array} \right\}
\end{array}$$

(*in fact*, $\mathbb{Z}$*-module isomorphisms*), *induced by mapping*

$$\vartheta \longmapsto \psi_\vartheta, \quad \psi = \psi_\vartheta \longmapsto D = D_\psi \ ,$$

and

$$D = D_\psi \overset{\boldsymbol{\delta}}{\longmapsto} \mathcal{F}_\psi = \mathcal{O}_X\,(D) \longmapsto \text{ord}_{\mathcal{F}_\psi} \ ,$$

with $\boldsymbol{\delta}$ *as above and*

$$D_{\psi_\vartheta} := - \sum_{\varrho \in \Delta(1)} \vartheta\,(n\,(\varrho))\; V\,(\varrho) \ \ .$$

Moreover,

$$\left\{ \begin{array}{c} D = D_\psi \in T_N\text{-CDiv}\,(X) \\ (\text{i.e., } \mathcal{F}_\psi = \mathcal{O}_X\,(D) \ \text{ is invertible}) \end{array} \right\} \Longleftrightarrow \ \psi = \text{ord}_{\mathcal{F}_\psi} \in \text{SF}\,(N,\Delta) \ .$$

Proof. See Saint-Donat [42], ch. I, § 2, thm. 9, pp. 28-31. □

Theorem 2.11 (H^0-generated). *If* $X = X\,(N,\Delta)$ *is a compact toric variety and* $\psi \in \text{SF}(N,\Delta)$*, then*

$$\mathcal{O}_X\,(D_\psi) \,\text{is generated by its global sections} \ \Longleftrightarrow \psi \in \text{UCSF}\,(N,\Delta)\,.$$

Proof. See Oda [54], thm. 2.7, p. 76. □

Theorem 2.12 (Ampelness). *Let* $X = X\,(N,\Delta)$ *be a (not necessarily compact) toric variety. Then a divisor*

$$D = D_\psi \in T_N\text{-CDiv}\,(X) \qquad (\text{resp. } \ D = D_\psi \in T_N\text{-CDiv}\,(X) \otimes_{\mathbb{Z}} \mathbb{Q})$$

is ample *if and and only if*

$$\psi \in \text{SUCSF}\,(N,\Delta) \qquad (\text{resp. } \ \psi \in \text{SUCSF}_{\mathbb{Q}}\,(N,\Delta)\,)$$

Proof. See Kempf [42], ch. I, § 3, thm. 13, p. 48. □

Corollary 2.13 (Quasiprojectivity). *Let $X(N,\Delta)$ be a toric variety (resp. a compact toric variety). Then $X(N,\Delta)$ is quasiprojective (resp. projective) if and only if*

$$\mathrm{SUCSF}(N,\Delta) \neq \varnothing \qquad (\text{or equivalently } \mathrm{SUCSF}_{\mathbb{Q}}(N,\Delta) \neq \varnothing)\ .$$

(i) *Intersection numbers of Cartier divisors on toric varieties.* If X is a normal complex variety of dimension r, and $D_1, \ldots, D_r$ Cartier divisors on X, such that $W := \bigcap_{i=1}^{r} (\mathrm{supp}(D_i))$ is compact, then their *intersection number* is defined to be the degree

$$(D_1 \cdots D_r) := \deg_W (\{D_1 \cdots D_r\}) \in \mathbb{Z}$$

of the zero-cycle

$$\{D_1 \cdot D_2 \cdots D_r\} := \{D_1 \cdot (D_2 \cdots D_r)\} \in A_0(W)$$

determined inductively as usual (i.e. probably after passing to the corresponding pseudodivisors). See Fulton [22], Ch. 1-2; in particular, I.1.4, p. 13, and pp. 38-39. For $X = X(N,\Delta)$ a *smooth* toric variety, $D_i = V(\varrho_i)$, for all i, $1 \leq i \leq r$, and ϱ_i's pairwise distinct rays, we have

$$(D_1 \cdots D_r) = \begin{cases} 1 & , \ \text{if} \quad \varrho_1 + \cdots + \varrho_r \in \Delta \\ 0 & , \ \text{otherwise} \end{cases} \tag{2.3}$$

Moreover, one obtains by general techniques:

Lemma 2.14. *Let $X(N,\Delta)$ be an r-dimensional smooth toric variety and*

$$D_1 = V(\varrho_1), \ldots, D_r = V(\varrho_r)$$

divisors on X. Suppose that either X itself or at least W is compact, and that $\varrho_1 = \varrho_2$, while all the other rays are distinct. Then :

(i) *If $\tau := \varrho_2 + \cdots + \varrho_r \notin \Delta$, the intersection number $(D_1 \cdots D_r)$ vanishes.*

(ii) *If $\tau \in \Delta$, there exist rays $\varrho', \varrho'' \in \Delta(1)$ and integers $\kappa_2, \ldots, \kappa_r$, such that*

$$n(\varrho') + n(\varrho'') + \sum_{j=2}^{r} \kappa_j\, n(\varrho_j) = 0, \qquad \varrho' + \tau \in \Delta(r), \qquad \varrho'' + \tau \in \Delta(r).$$

In this case

$$\left(D_2^2 \cdot D_3 \cdots D_r\right) = \kappa_2 \tag{2.4}$$

Proof. See Oda [54], p. 81. □

There are also easy generalizations for Δ simplicial but we shall not use them because we shall work exclusively with intersection numbers of divisors on smooth $X(N,\Delta)$'s. Another method for the evaluation of intersection numbers is based on mixed polytope volumes.

Proposition 2.15. *Let $X(N,\Delta)$ be an r-dimensional compact toric variety. If $\psi \in \mathrm{UCSF}(N,\Delta)$, then the self-intersection number of D_ψ equals*

$$D_\psi^r = (r!) \cdot \mathrm{Vol}(P_\psi) \tag{2.5}$$

where

$$P_\psi := P_{D_\psi} = \{\mathbf{x} \in M_\mathbb{R} \mid \langle \mathbf{x}, n(\varrho)\rangle \geq \psi(n(\varrho))\}$$

is the lattice polytope associated to the divisor $D = D_\psi$ defined in 2.10. *More generally, for r upper convex functions $\psi_1,\ldots,\psi_r$, one has*

$$\left(D_{\psi_1} \cdots D_{\psi_r}\right) = (r!) \cdot \mathrm{Vol}\left(P_{\psi_1},\ldots, P_{\psi_r}\right) \tag{2.6}$$

where $\mathrm{Vol}\left(P_{\psi_1},\ldots, P_{\psi_r}\right)$ *denotes the mixed volume of the polytopes* $P_{\psi_1},\ldots, P_{\psi_r}$.

Proof. See Oda [54], prop. 2.10, p. 79. □

(j) *Euler-Poincaré characteristic.* The topological Euler-Poincaré characteristic of a (not necessarily compact) toric variety can be easily read off from the maximal cones of the defining fan.

Proposition 2.16. *Let $X(N,\Delta)$ be an r-dimensional toric variety associated to Δ. Then the topological Euler-Poincaré characteristic*

$$\chi(X(N,\Delta)) = \sum_{i=0}^{2r} (-1)^i \dim_\mathbb{Q} H^i(X(N,\Delta);\mathbb{Q})$$

of $X(N,\Delta)$ is equal to the number of r-dimensional cones, i.e.

$$\chi(X(N,\Delta)) = \#(\Delta(r)) \tag{2.7}$$

Proof. See Fulton [23], p. 59. □

(k) *Maps of fans.* A map of fans $\varpi : (N',\Delta') \to (N,\Delta)$ is a $\mathbb{Z}$-linear homomorphism $\varpi : N' \to N$ whose scalar extension $\varpi : N'_\mathbb{R} \to N_\mathbb{R}$ satisfies the property:

$$\forall \sigma',\ \sigma' \in \Delta' \quad \exists\, \sigma,\ \sigma \in \Delta \quad \text{with} \quad \varpi(\sigma') \subset \sigma\ .$$

$\varpi \otimes_\mathbb{Z} \mathrm{id}_{\mathbb{C}^*} : T_{N'} = N' \otimes_\mathbb{Z} \mathbb{C}^* \to T_N = N \otimes_\mathbb{Z} \mathbb{C}^*$ is a homomorphism from $T_{N'}$ to T_N and the scalar extension $\varpi^\vee : M_\mathbb{R} \to M'_\mathbb{R}$ of the dual $\mathbb{Z}$-linear map $\varpi^\vee : M \to M'$ induces an *equivariant holomorphic map* $\varpi_* : X(N',\Delta') \to X(N,\Delta)$ as follows: If $\varpi(\sigma') \subset \sigma$ for $\sigma \in \Delta, \sigma' \in \Delta'$, then obviously $\varpi^\vee(M \cap \sigma^\vee) \subset M' \cap (\sigma')^\vee$, and the holomorphic map

$$\varpi_* : U'_{\sigma'} \to U_\sigma \quad \text{with} \quad \varpi_*(u')(m) := u'(\varpi^\vee(m))\,,\ \forall m,\ m \in M \cap \sigma^\vee,$$

is equivariant because

$$\begin{aligned}&\varpi_*(t' \cdot u')(m) = (t' \cdot u')(\varpi^\vee(m)) = \\ &= t'(\varpi^\vee(m)) \cdot u'(\varpi^\vee(m)) = \varpi_*(t')(m) \cdot \varpi_*(u')(m)\,,\end{aligned}$$

for all $t' \in T_{N'}$, $m \in M \cap \sigma^\vee$. After gluing together the affine charts of Δ and of Δ' we determine a well-defined map $\varpi_* : X(N',\Delta') \to X(N,\Delta)$.

Theorem 2.17 (Properness). *If $\varpi : (N', \Delta') \to (N, \Delta)$ is a map of fans, ϖ_* is* proper *if and only if $\varpi^{-1}(|\Delta|) = |\Delta'|$. In particular, if $N = N'$ and Δ' is a refinement of Δ, i.e. if each cone of Δ is a union of cones of Δ', then the holomorphic map $\mathrm{id}_* : X(N, \Delta') \to X(N, \Delta)$ is* proper *and* birational.

Proof. See Oda [54], thm. 1.15, p. 20, and cor. 1.18, p. 23. □

3. Blow-ups and resolutions of toric varieties

One of the most fundamental cornerstones of various significant constructions of birational morphisms between complex varieties is the blowing up along subvarieties or -more general- along closed subschemes.

(a) *Local construction.* Let $U = \text{Max-Spec}(R)$ be (the closed point set of) an affine noetherian scheme, I an ideal of R, $Z = \text{Max-Spec}(R/I)$ and $S(R, I) := \bigoplus_{d \geq 0} I^d$. The homogeneous spectrum $\text{Proj}(S(R, I))$ of $S(R, I)$, together with the structure morphism

$$\mathrm{Bl}_I(R) := \mathrm{Bl}_Z^I(U) := \mathrm{Proj}(S(R, I)) \xrightarrow{\beta} U$$

is called the blow-up of U w.r.t. I or *the blow-up* of U *along* Z (or of U with *center* Z). If $\{h_0, \ldots, h_\mu\}$ is a set of generators of I, then

$$\mathrm{Bl}_Z^I(U) = \bigcup_{i=0}^{\mu} \text{Max-Spec}\left(R\left[\frac{h_0}{h_i}, \ldots, \frac{h_\mu}{h_i}\right]\right)$$

with $R\left[\frac{h_0}{h_i}, \ldots, \frac{h_\mu}{h_i}\right]$ viewed as an R-subalgebra of R_{h_i}. The exceptional locus of β is $\mathrm{Exc}(\beta) := \beta^{-1}(Z)$ and its contraction locus $= Z$. Moreover, $\beta^{-1}(I^\sim)$ is invertible (with $I^\sim$ denoting here the sheaf being associated to the ideal I, and with I regarded as an R-module, cf. [27], p. 110), and $\beta^{-1}(I^\sim) \cong S(R, I)(1)^\sim$.

(b) *Globalization by gluing lemma.* Let $\{\mathfrak{X}_j \mid j \in J\}$ be a family of schemes.

Lemma 3.1 (Gluing schemes). *If there exists a collection $\{\mathfrak{X}_{j,k} \mid j, k \in J\}$ of open sets $\mathfrak{X}_{j,k}$ of $\mathfrak{X}_j$ and isomorphisms of schemes*

$$\eta_{k,j} : (\mathfrak{X}_{j,k}, \mathcal{O}_{\mathfrak{X}_j}|_{\mathfrak{X}_{j,k}}) \xrightarrow{\cong} (\mathfrak{X}_{k,j}, \mathcal{O}_{\mathfrak{X}_j}|_{\mathfrak{X}_{k,j}})$$

satisfying the conditions :

- $\mathfrak{X}_j = \mathfrak{X}_{j,j}$ *and* $\eta_{j,j} = \mathrm{id}_{\mathfrak{X}_j}$,
- $\eta_{j,k} \circ \eta_{k,j} = \mathrm{id}_{\mathfrak{X}_{j,k}}$, *and*
- $(\eta_{j,k} \circ \eta_{j,\xi})|_{\mathfrak{X}_{j,k,\xi}} = \eta_{j,\xi}|_{\mathfrak{X}_{j,k,\xi}}$, *where* $\mathfrak{X}_{j,k,\xi} := \mathfrak{X}_{j,k} \cap \eta_{j,\xi}^{-1}(\mathfrak{X}_{k,j})$,

then there exists a scheme W, an open cover $\{W_j \mid j \in J\}$ of W, and a collection of isomorphisms $\left\{f_j : W_j \xrightarrow{\cong} \mathfrak{X}_j \mid j \in J\right\}$, such that $\eta_{k,j} = f_k \circ f_j^{-1}|_{\mathfrak{X}_{j,k}}$ for all $i, j \in J$.

The scheme W obtained by gluing the members of the family $\{\, \mathfrak{X}_j \mid j \in J\}$ via the above isomorphisms will be denoted by

$$W = \coprod_{\eta_{k,j}} \mathfrak{X}_j \quad \text{or simply by} \quad W = \overset{\blacktriangledown}{\coprod} \mathfrak{X}_j$$

if the gluing isomorphisms are self-evident from the context.

Proof. See e.g. [27], Ex. II.2.12, p. 80. □

Let now $\mathcal{I}$ be a coherent (non-zero) sheaf of $\mathcal{O}_X$-ideals over a complex variety X and

$$\{U_j = \text{ Max-Spec}\,(R_j) \mid j \in J\}$$

an affine cover of X. For every $j \in J$, we have $\mathcal{I}\,|_{U_j} \cong I_j^{\sim}$ for some ideal I_j of R_j (cf. [27], II.5.4). Considering

$$\left\{\beta_j : \operatorname{Bl}_{Z_j}^{I_j}(U_j) \longrightarrow U_j \subset X \mid j \in J\right\}$$

as in (a), where $Z_j = \text{Max-Spec}(R_j/I_j)$, and Z the closed subscheme of X defined by

$$Z = \operatorname{supp}(\mathcal{O}_X/\mathcal{I}) := \{\mathbf{x} \in X \mid (\mathcal{O}_X/\mathcal{I})_{\mathbf{x}} \neq 0\} \ ,$$

we determine natural isomorphisms

$$\beta_j^{-1}(U_j \cap U_k) \xrightarrow[\eta_{k,j}]{\cong} \beta_k^{-1}(U_j \cap U_k) \ .$$

Applying lemma 3.1 to the family $\left\{\beta_j^{-1}(U_j) \mid j \in J\right\}$ we construct a birational, proper, surjective morphism $\pi = \pi_{\mathcal{I}}$

$$\boxed{\mathbf{Bl}_Z^{\mathcal{I}}(X) := \coprod_{\eta_{k,j}} \beta_j^{-1}(U_j) = \mathbf{Proj}\left(\bigoplus_{d \geq 0} \mathcal{I}^{\,d}\right) \xrightarrow{\pi} X}$$

as the natural projection induced by $\mathcal{O}_X \overset{\cong}{\to} \mathcal{I}^0 \hookrightarrow \bigoplus_{d \geq 0} \mathcal{I}^{\,d}$, with $\mathbf{Proj}$ denoting the *global* homogeneous spectrum (as in [27], p. 160). $\left(\mathbf{Bl}_Z^{\mathcal{I}}(X), \pi\right)$ is *the blow-up* of X *along* $\mathcal{I}$ or *the monoidal transformation* w.r.t. $\mathcal{I}$ (with *center* Z). Let us recall its main properties :

- π induces an isomorphism $\mathbf{Bl}_Z^{\mathcal{I}}(X) \smallsetminus \pi^{-1}(Z) \overset{\cong}{\to} X \smallsetminus Z$, i.e. $\text{Exc}(\pi) = \pi^{-1}(Z)$.
- The algebraic scheme $\mathbf{Bl}_Z^{\mathcal{I}}(X)$ is a complex variety (cf. [27], II. 7.16 (a), p. 166).
- If both X and Z are smooth, then $\mathbf{Bl}_Z^{\mathcal{I}}(X)$ is smooth too.
- The preimage sheaf of $\mathcal{I}$, $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_X := \operatorname{Im}\left(\pi^*\mathcal{I} \to \mathcal{O}_{\mathbf{Bl}_Z^{\mathcal{I}}(X)}\right)$, is invertible and

determines $\mathrm{Exc}(\pi)$. Hence, $\mathrm{Exc}(\pi)$ represents a Cartier (not necessarily prime) divisor of $\mathbf{Bl}_Z^{\mathcal{I}}(X)$ which is isomorphic to the projectivization $\mathbb{P}\left(\mathrm{NC}_Z^{\mathcal{I}}(X)\right)$ of *the normal cone*

$$\mathrm{NC}_Z^{\mathcal{I}}(X) := \mathrm{Spec}\left(\bigoplus_{d\geq 0} \mathcal{I}^d / \mathcal{I}^{d+1}\right)$$

of X *along* Z. The relation between $\mathbf{Bl}_Z^{\mathcal{I}}(X)$ and $\mathrm{Exc}(\pi)$ is described by the isomorphisms

$$\mathcal{N}_{\mathrm{Exc}(\pi)\,|\,\mathbf{Bl}_Z^{\mathcal{I}}(X)} \cong \mathcal{O}_{\mathbf{Bl}_Z^{\mathcal{I}}(X)}\left(\mathrm{Exc}\,(\pi)\right)\big|_{\mathrm{Exc}(\pi)} \cong \mathcal{O}_{\mathbb{P}\left(\mathrm{NC}_Z^{\mathcal{I}}(X)\right)}(-1)\ .$$

In particular, if Z is a local complete intersection in X, the canonical epimorphism from the d-th symmetrizer sheaf of $\mathcal{I}/\mathcal{I}^2$ onto the d-th part of the normal cone graded algebra

$$\mathrm{Sym}^d\left(\mathcal{I}/\mathcal{I}^2\right) \twoheadrightarrow \mathcal{I}^d/\mathcal{I}^{d+1}$$

becomes an isomorphism and therefore

$$\mathrm{Exc}\,(\pi) \cong \mathbf{Proj}\left(\bigoplus_{d\geq 0} \mathrm{Sym}^d\left(\mathcal{I}/\mathcal{I}^2\right)\right) \cong \mathbb{P}\left(\mathcal{I}/\mathcal{I}^2\right) = \mathbb{P}\left(\mathcal{N}_{Z\,|\,X}^{\vee}\right)\ .$$

(Here $\mathcal{N}_{\ldots}$, $\mathcal{N}_{\ldots}^{\vee}$ denote the corresponding normal and conormal sheaves.)

(c) *Universal property of blowing up.* If $g : X' \to X$ is any morphism, and Z (resp. Z') is defined by the ideal sheaf $\mathcal{I}$ (resp. by $\mathcal{I}' := \pi^{-1}\mathcal{I}\cdot\mathcal{O}_{X'}$), where π denotes the blow-up-morphism of X along Z, then composing the morphism $\mathbf{Bl}_{Z'}^{\mathcal{I}'}(X') \to \mathbf{Bl}_Z^{\mathcal{I}}(X)\times_X X'$ with the projection to X', we get the commutative diagram:

$$\begin{array}{ccccc} & & \mathbf{Bl}_Z^{\mathcal{I}}(X) & \xrightarrow{\pi} & X \\ & \nearrow & \uparrow & \nwarrow^{h} & \uparrow\, g \\ \mathbf{Bl}_{Z'}^{\mathcal{I}'}(X') & \to & \mathbf{Bl}_Z^{\mathcal{I}}(X)\times_X X' & \longrightarrow & X' \end{array}$$

If $g^{-1}\mathcal{I}\cdot\mathcal{O}_{X'}$ is invertible, then it is easy to deduce an isomorhism $\mathbf{Bl}_{Z'}^{\mathcal{I}'}(X') \cong X'$. Hence, there is a unique morphism h factorizing g. This means that $\left(\mathbf{Bl}_Z^{\mathcal{I}}(X), \pi\right)$ is universal among all pairs (X', g) having invertible ideal sheaves $g^{-1}\mathcal{I}\cdot\mathcal{O}_{X'}$.

(d) *Blowing up intermediate subschemes.* Let $Z \subsetneqq X$, $W \subsetneqq X$ be two closed subschemes of X which are defined by the ideal sheaves $\mathcal{I}$ and $\mathcal{J}$ respectively, such that $Z\cap W = \mathrm{supp}(\mathcal{O}_X/\mathcal{I}+\mathcal{J}) \neq \varnothing$ and $Z\cap W$ is nowhere dense in W, and let

$$\pi = \pi_{\mathcal{I}} : \mathbf{Bl}_Z^{\mathcal{I}}(X) \longrightarrow X, \quad \pi\,|_{\mathrm{restr.}} : \mathbf{Bl}_{Z\cap W}^{\mathcal{I}+\mathcal{J}}(W) \longrightarrow W$$

denote the corresponding blow-ups. Then, by the above mentioned universality of the morphism π, one verifies easily the following isomorphism:

$$\begin{array}{ccccc} & & \mathbf{Bl}_Z^{\mathcal{I}}(X) & \stackrel{\pi}{\longrightarrow} & X \\ & & \cup & & \cup \\ \left\{ \begin{array}{c} \text{scheme-theoretic closure of the preimage} \\ \pi^{-1}(W \smallsetminus (Z \cap W)) \quad \text{within} \quad \mathbf{Bl}_Z^{\mathcal{I}}(X) \end{array} \right\} & \cong & \mathbf{Bl}_{Z\cap W}^{\mathcal{I}+\mathcal{J}}(W) & \stackrel{\pi|_{\text{restr.}}}{\longrightarrow} & W \end{array}$$

This closure is called *the strict transform* $\mathrm{STR}(W, \mathcal{J}, \pi)$ of (W, π) *under* π. If we assume that $W \nsubseteq Z$ and $Z \nsubseteq W$, then $\mathrm{STR}(Z, \mathcal{I}, \pi_{\mathcal{I}}) \cap \mathrm{STR}(W, \mathcal{J}, \pi_{\mathcal{J}}) = \varnothing$ (cf. [27], Ex. 7.12, p. 171).

(e) *Normalization process.* Even if the complex variety X itself is normal and $Z = \mathrm{supp}(\mathcal{O}_X \,/\, \mathcal{I})$ smooth, with $Z \cap \mathrm{Sing}(X) \neq \varnothing$, $\mathbf{Bl}_Z^{\mathcal{I}}(X)$ is *not* necessarily normal. Using an affine cover of X

$$\{U_j = \text{ Max-Spec}\,(R_j) \mid j \in J\} \quad \text{and} \quad Z_j = \text{Max-Spec}\,(R_j/I_j)$$

(as in (b)) and the (finite) normalization morphisms onto $\mathrm{Bl}_{Z_j}^{I_j}(U_j)$:

$$\boldsymbol{\nu}_j : \mathrm{Norm}\left[\mathrm{Bl}_{Z_j}^{I_j}(U_j)\right] \longrightarrow \mathrm{Bl}_{Z_j}^{I_j}(U_j) \ ,$$

we define *the normalized blow-up* $\left(\mathrm{Norm}\left[\mathbf{Bl}_Z^{\mathcal{I}}(X)\right], \pi_{\mathcal{I}} \circ \boldsymbol{\nu}_{\mathcal{I}}\right)$ of X w.r.t. Z (or with *center* Z) by patching the affine pieces together :

$$\boxed{\mathrm{Norm}\left[\mathbf{Bl}_Z^{\mathcal{I}}(X)\right] := \overset{\blacktriangledown}{\coprod} \mathrm{Norm}\left[\mathrm{Bl}_{Z_j}^{I_j}(U_j)\right] \stackrel{\boldsymbol{\nu}_{\mathcal{I}}}{\longrightarrow} \mathbf{Bl}_Z^{\mathcal{I}}(X) \stackrel{\pi_{\mathcal{I}}}{\longrightarrow} X}$$

The combination of the universal property of the normalization morphism $\boldsymbol{\nu}_{\mathcal{I}}$ (see [27], Ex. 3.8, p. 91) with that of $\pi_{\mathcal{I}}$ (cf. (c)) leads to the universal property of normalized blowing up: If $g : X' \to X$ is any proper morphism, X' normal and $g^{-1}\mathcal{I} \cdot \mathcal{O}_{X'}$ is invertible, then there exists a unique morphism h factorizing g :

$$\begin{array}{ccc} \mathrm{Norm}\left[\mathbf{Bl}_Z^{\mathcal{I}}(X)\right] & \stackrel{\pi_{\mathcal{I}} \circ \boldsymbol{\nu}_{\mathcal{I}}}{\longrightarrow} & X \\ & \nwarrow^{h} & \uparrow g \\ & & X' \end{array}$$

(f) *Some warnings.* To blow up arbitrary subschemes Z of X requires great care.

- Sometimes, for different coherent ideal $\mathcal{O}_X$-sheaves $\mathcal{I}$, $\mathcal{I}'$ over X and

$$Z := \mathrm{supp}\,(\mathcal{O}_X \,/\, \mathcal{I}) \,, \quad Z' := \mathrm{supp}\,(\mathcal{O}_X \,/\, \mathcal{I}') \,,$$

it is possible to have an isomorphism $\mathbf{Bl}_Z^{\mathcal{I}}(X) \cong \mathbf{Bl}_{Z'}^{\mathcal{I}'}(X)$. For instance, for $\mathcal{I}$ an arbitrary coherent sheaf of ideals and $\mathcal{I}'$ an invertible sheaf of ideals, $\mathcal{I}$ and $\mathcal{I}\cdot\mathcal{I}'$ give isomorphic blow-ups. The same remains true for $\mathcal{I}$ and $\mathcal{I}^d$ with $d \geq 2$ (see [27], Ex. 7.11, p. 171).

• On the other hand, if Z is endowed with two different scheme structures

$$Z = \operatorname{supp}(\mathcal{O}_X / \mathcal{I}) = \operatorname{supp}(\mathcal{O}_X / \mathcal{J}), \quad \mathcal{I} \neq \mathcal{J},$$

then $\mathbf{Bl}_Z^{\mathcal{I}}(X)$ and $\mathbf{Bl}_Z^{\mathcal{J}}(X)$ are in general too far from being isomorphic to each other.

Example 3.2. Perhaps the simplest example is to consider $X = \mathbb{A}^2_{\mathbb{C}}$ to be the affine complex plane with coordinate system $\{x, y\}$, $Z = \{(0,0)\}$ the zero point, $\mathcal{I} = (x, y)\cdot\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}$ the maximal ideal of $\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}},(0,0)}$ and $\mathcal{J} = \left(x, y^2\right)\cdot\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}$. Then

$$\mathbf{Bl}^{\mathcal{I}}_{\{(0,0)\}}\left(\mathbb{A}^2_{\mathbb{C}}\right) \xrightarrow{\pi_{\mathcal{I}}} \mathbb{A}^2_{\mathbb{C}}$$

is the usual blow-up with exceptional set

$$\pi_{\mathcal{I}}^{-1}(\{(0,0)\}) \cong \mathbb{P}^1_{\mathbb{C}} \quad \text{and} \quad \operatorname{Sing}\left(\mathbf{Bl}^{\mathcal{I}}_{\{(0,0)\}}\left(\mathbb{A}^2_{\mathbb{C}}\right)\right) = \varnothing.$$

In contrast to the usual blow-up case, and although the blow-up

$$\mathbf{Bl}^{\mathcal{J}}_{\{(0,0)\}}\left(\mathbb{A}^2_{\mathbb{C}}\right) \xrightarrow{\pi_{\mathcal{J}}} \mathbb{A}^2_{\mathbb{C}}$$

w.r.t. the prime ideal $\mathcal{J}$ has also a smooth rational curve as exceptional set, the singular locus of $\mathbf{Bl}^{\mathcal{J}}_{\{(0,0)\}}\left(\mathbb{A}^2_{\mathbb{C}}\right)$ is non-empty. As one may easily verify, it consists of a single ordinary double point which lies on this rational curve. The real parts $\pi_{\mathcal{I}}^{-1}(\mathfrak{D})$ and $\pi_{\mathcal{J}}^{-1}(\mathfrak{D})$ over a small disk $\mathfrak{D} \subset \mathbb{A}^2_{\mathbb{R}} \subset \mathbb{A}^2_{\mathbb{C}}$ centered at $(0,0)$ are illustrated in figures **1** (a) and **2**. The real part of $\pi_{\mathcal{I}}^{-1}(\mathfrak{D})$ is used to be viewed as a *spiral staircase* (whose stairs extend in both directions). Away from the origin we get an isomorphism, and the points of $\operatorname{Exc}(\pi_{\mathcal{I}})$ are in 1-1 correspondence with the set of straight lines passing through $(0,0)$ (see fig.**1** (a)). One may, of course, think of it just as an *enlargement* (*Aufblasung*) of the origin spreading out the "tangent directions" (see fig.**1** (b)). In fact, it can be shown that the topological space of the real part of $\pi_{\mathcal{I}}^{-1}(\mathfrak{D})$ is homeomorphic to a Möbius strip, while, in the analogous setting, $\pi_{\mathcal{J}}^{-1}(\mathfrak{D})$ has a marked "twisted" point which corresponds to the occuring singularity (see fig.**2**). In both cases the dotted meridian line indicates the exceptional set.

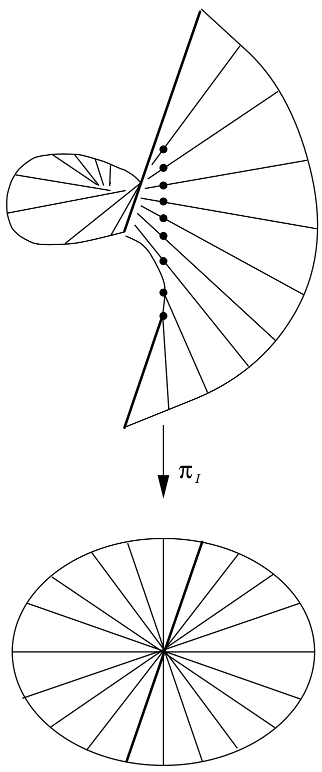


Fig. 1 (a)

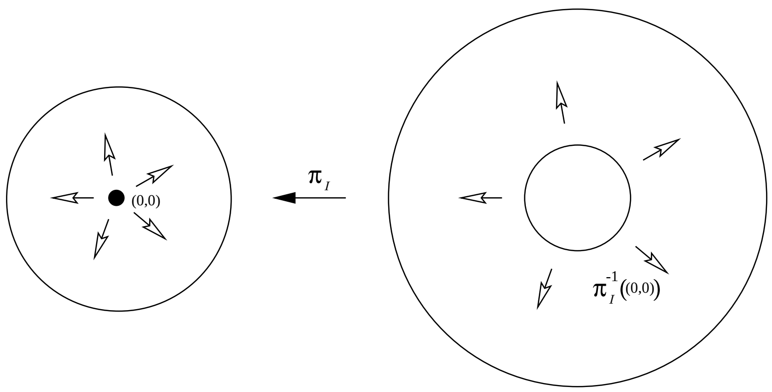


Fig. 1 (b)

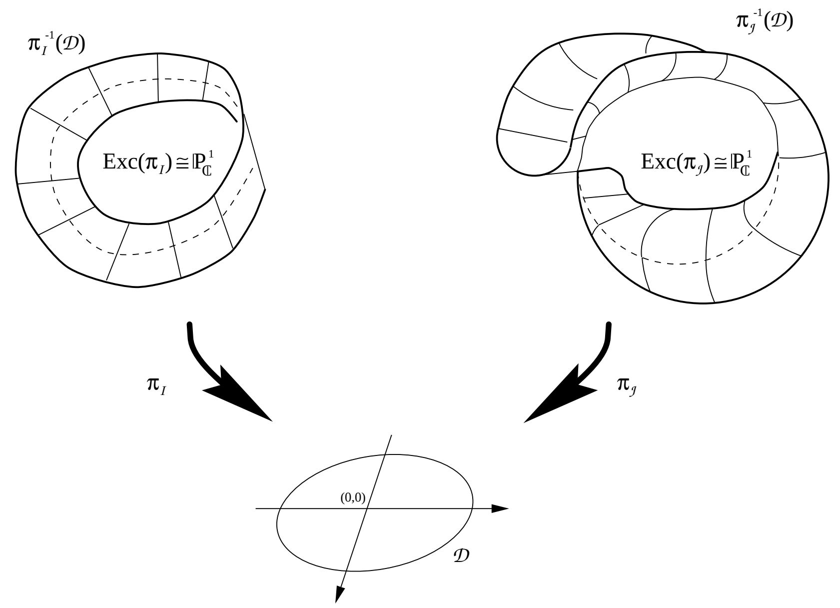


Fig. 2

• The closed subscheme Z of X can be in general equipped with lots of scheme structures $\mathcal{O}_Z$, with $\mathcal{O}_Z = \mathcal{O}_X \,/\, \mathcal{I}$ for different $\mathcal{I}$'s. There is, however, a unique scheme structure above them, which is the "smallest one"; namely, the reduced induced structure $\mathcal{O}_Z \,/\, \mathfrak{R}_Z$ (supporting the same underlying topological space) where $\mathfrak{R}_Z$ is the $\mathcal{O}_Z$-ideal sheaf defined by

$$\{\text{open sets of } Z\} \ni U \longmapsto \mathfrak{R}_Z\,(U) := \{\text{the nilradical of } \mathcal{O}_Z\,(U)\}$$

(see [27], II.3.2.6, p. 86). Let $Z_{\mathbf{red}} = \mathrm{supp}(\mathcal{O}_Z \,/\, \mathfrak{R}_Z)$ denote the reduced subscheme associated to Z. The singular locus[7] of Z can be written as

$$\mathrm{Sing}\,(Z) = \mathrm{Sing}\,(Z_{\mathbf{red}}) \cup \{\text{non-reduced points of } Z\}\ .$$

From now on, the uniquely determined blow-up $\mathbf{Bl}_{Z_{\mathbf{red}}}^{\mathcal{I}|_Z \,/\mathfrak{R}_Z}\,(X)$ of X along this reduced subscheme $Z_{\mathbf{red}}$ will be denoted simply by $\mathbf{Bl}_Z^{\mathrm{red}}\,(X)$ and will be called the *usual* blow-up of X along Z.

• Another notable property of a blow-up is the projectivity of its defining morphism $\pi_{\mathcal{I}}$ (and consequently the projectivity of $\pi_{\mathcal{I}} \circ \boldsymbol{\nu}_{\mathcal{I}}$, because $\boldsymbol{\nu}_{\mathcal{I}}$ is finite).

[7] Since we allow Z to carry a non-reduced structure (in contrast to X which is assumed to be a complex variety, and therefore always reduced), $\mathrm{Sing}(Z)$ might become very "big". In particular, in the above example 3.2, for $Z = \{(0,0)\} = \mathrm{supp}\Big(\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}} \,/\, \mathcal{J}\Big)$, we obtain $Z = \mathrm{Sing}(Z)$!

Proposition 3.3. *If X is quasiprojective (resp. projective), then both* $\mathbf{Bl}_Z^{\mathcal{I}}(X)$ *and* Norm $\left[\mathbf{Bl}_Z^{\mathcal{I}}(X)\right]$ *will be quasiprojective (resp. projective) as well.*

Proof. See Hartshorne [27], prop. II.7.16, p. 166. □

Corollary 3.4. *Let X be a quasiprojective (resp. projective) complex variety and $Z = \mathrm{supp}(\mathcal{O}_X \,/\, \mathcal{I})$ a closed subscheme of X. If $X_0 := X$, $Z_0 := Z$, $\mathcal{I}_0 := \mathcal{I}$ and*

$$\left\{\pi_{\mathcal{I}_{j-1}} : X_j = \mathbf{Bl}_{Z_{j-1}}^{\mathcal{I}_{j-1}}(X_{j-1}) \longrightarrow X_{j-1} \mid 1 \le j \le l\right\}$$

is a finite sequence of blow-ups with $Z_j = \mathrm{supp}(\mathcal{O}_{X_j} \,/\, \mathcal{I}_j)$, then X_l is also quasiprojective (resp. projective).

• The troubles with the projectivity begin whenever one wishes to blow up (once or more times) "overlapping one another pieces" $U_1, \ldots, U_k$ of a fixed closed (possibly reducible, non-reduced or singular) subscheme Z of a quasiprojective complex variety with $Z = \bigcup_{i=1}^{k} U_i$, and try to glue the blown up (non-disjoint) overlying parts, say $W_1, \ldots, W_k$, together. Even if this gluing procedure is absolutely natural, the resulting morphism

$$f : X' = \overset{\blacktriangledown}{\coprod} W_i \longrightarrow X$$

(with contraction locus Z) is *not* always projective. The two classical examples of this kind, having as starting-point two smooth curves meeting trasversally at two points and a curve with an ordinary double point on a smooth complete threefold, respectively, are due to Hironaka. See Hartshorne [27], pp. 443-445, for the details of the construction. The non-projectivity of such an f implies automatically not only the non-quasiprojectivity of X' but also the fact that f cannot be expressed as a composite of finitely many blow-ups (by cor. 3.4). For another simple example, see rem. 7.5 below.

(g) *Desingularization by successive blow-ups with smooth centers.* We just formulate here the famous theorem of Hironaka which guarantees the *existence* of (full) desingularizations by performing a finite number of monoidal transformations with smooth centers. (Of course, we should again stress, that *not every* desingularization can be composed of finitely many blow-ups.)

Theorem 3.5. (**Hironaka's Theorem**[8], [31]). *Let X be any complex variety. Then there exists always a finite sequence of blow-ups*

$$\left\{\pi_{\mathcal{I}_{j-1}} : X_j = \mathbf{Bl}_{Z_{j-1}}^{\mathcal{I}_{j-1}}(X_{j-1}) \longrightarrow X_{j-1} \mid 1 \le j \le l\right\} ,$$

such that $X_0 = X$ and $\mathrm{Sing}(X_l) = \varnothing$, where Z_j is a smooth *subvariety of* $\mathrm{Sing}(X_j)$, $\forall j,\ 0 \le j \le l-1$,

[8] The original result of Hironaka is more general. It is valid for any equicharacteristic zero excellent scheme X. The centers Z_j are normally flat in the ambient space.

(There are meanwhile considerable simplifications of the original proof of thm. 3.5, like those due to Villamayor [79], [80] and Bierstone & Milman [4], [5], who introduced an appropriate discrete local invariant for points of X whose maximum locus determines a center of blow-up, leading to constructive desingularization.)

(h) *Toric blow-ups.* Working within the category of toric varieties, both blow-up and resolution are much more easier as they can be translated into purely combinatorial operations involving specific cones of our fans.

Theorem 3.6 (Toric normalized blow-up). *Let $X(N,\Delta)$ be a toric variety, $\mathcal{I} \neq \underline{0}$ a T_N-invariant coherent ideal sheaf defining a subscheme Z of $X(N,\Delta)$ with $\mathcal{F} = \left(\mathcal{O}_{X(N,\Delta)} / \mathcal{I}\right)|_Z$ contained in $\boldsymbol{\iota}_*(\mathcal{O}_{T_N})$ (where $\boldsymbol{\iota} : T_N \hookrightarrow X(N,\Delta)$ denotes the canonical injection), and*

$$\mathrm{Norm}\left[\mathbf{Bl}_Z^{\mathcal{I}}(X(N,\Delta))\right] \overset{\pi_{\mathcal{I}} \circ \boldsymbol{\nu}_{\mathcal{I}}}{\longrightarrow} X(N,\Delta)$$

the normalized blow-up of $X(N,\Delta)$ along Z. Then $\pi_{\mathcal{I}} \circ \boldsymbol{\nu}_{\mathcal{I}}$ is the T_N-equivariant holomorphic map and the overlying space is identified with the toric variety

$$\boxed{\mathrm{Norm}\left[\mathbf{Bl}_Z^{\mathcal{I}}(X(N,\Delta))\right] = X(N,\Delta_{\mathbf{bl}}(\mathcal{I};Z))}$$

where $\Delta_{\mathbf{bl}}(\mathcal{I};Z)$ denotes the following refinement of Δ :

$$\Delta_{\mathbf{bl}}(\mathcal{I};Z) := \left\{ \begin{array}{c} \text{the fan in } N_{\mathbb{R}} \text{ defined by rational s.c.p. cones} \\ \text{which constitute the maximal subdivision} \\ \text{(w.r.t. usual inclusion) of the cones of } \Delta, \\ \text{so that the restriction of } \mathrm{ord}(\mathcal{F}) \\ \text{on each of them becomes an integral} \\ \textit{linear} \text{ support function} \end{array} \right\}$$

Proof. Since $\Delta_{\mathbf{bl}}(\mathcal{I};Z)$ is a refinement of the initial fan Δ, the toric map

$$\mathrm{id}_* : X(N,\Delta_{\mathbf{bl}}(\mathcal{I};Z)) \longrightarrow X(N,\Delta)$$

is a proper birational morphism by thm. 2.17. Since $\mathcal{F} = \left(\mathcal{O}_{X(N,\Delta)} / \mathcal{I}\right)|_Z$ is T_N-invariant, T_N acts on $\mathbf{Bl}_Z^{\mathcal{I}}(X(N,\Delta))$ too. $\mathcal{F}$ is invertible over the open set T_N of $X(N,\Delta)$. Hence, the normalized blow-up is an isomorphism over T_N,

$$T_N \hookrightarrow \mathbf{Bl}_Z^{\mathcal{I}}(X(N,\Delta))$$

is an equivariant immersion and $\pi_{\mathcal{I}} \circ \boldsymbol{\nu}_{\mathcal{I}}$ a torus-equivariant map. Combining the universal property of the normalized blow-up (see (e)) with the fact, that for any $\sigma \in \Delta$, the inverse image of $\mathcal{I}$ restricted onto U_σ is an invertible sheaf if and only if $\mathrm{ord}_{\mathcal{F}}|_\sigma$ is linear (by thm. 2.10), we deduce that $\mathrm{id}_* = \pi_{\mathcal{I}} \circ \boldsymbol{\nu}_{\mathcal{I}}$ (up to isomorphism over $X(N,\Delta)$). $\square$

Remark 3.7. (i) If $X(N,\Delta)$ is *smooth,*

$$\Delta(1) = \{\varrho_1, \ldots, \varrho_k\}, \quad \mathrm{Gen}(\Delta) = \{n_1, \ldots, n_k\},$$

with $n_i = n(\varrho_i)$, for $1 \le i \le k$,

$$n_0 := n_1 + \cdots + n_k$$

and

$$\tau_j := \begin{cases} \mathrm{pos}(n_0, n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_k) & , \quad \text{for} \quad 1 \le j \le k-1 \\ \mathrm{pos}(n_0, n_1, \ldots, n_{k-1}) & , \quad \text{for} \quad j = k \end{cases}$$

then, fixing a cone $\tau \in \Delta$, every cone $\sigma \in \Delta$ with $\tau \prec \sigma$ can be written as

$$\sigma = \tau + \sigma', \quad \text{for some cone } \sigma' \in \Delta, \quad \text{with } \sigma' \cap \tau = \{\mathbf{0}\} .$$

Setting $\sigma_j := \tau_j + \sigma'$, for all j, $1 \le j \le k$, and

$$\Delta^*(\tau) := (\Delta \smallsetminus \{\sigma \in \Delta \mid \tau \prec \sigma\}) \cup \left\{ \text{faces of } \sigma_j \;\middle|\; \begin{array}{c} \sigma \in \Delta, \ \tau \prec \sigma \\ \text{for all } j, 1 \le j \le k \end{array} \right\}$$

we get

$$X(N, \Delta^*(\tau)) = \mathbf{Bl}^{\mathrm{red}}_{V(\tau)}(X(N,\Delta))$$

with $\Delta^*(\tau) = \Delta_{\mathbf{bl}}(\mathcal{I}; V(\tau))$ the fan "starring" τ, where $\mathcal{I}$ is now the usual ideal (with no nilpotent elements) defining the closure $V(\tau)$ as subvariety of $X(N,\Delta)$ (cf. Oda [54], prop. 1.26, pp. 38-39; see also Ewald [17], § VI.7, for an equivalent combinatorial characterization in terms of "stellar subdivisions".)

(ii) If $X(N,\Delta)$ is *not* smooth, then performing the above starring subdivision w.r.t. τ, we get a normalized blow-up of $X(N,\Delta)$ along $V(\tau)$, with $V(\tau)$ being equipped with a *not* necessarily reduced scheme structure! For a simple example, see 7.2 below.

(iii) Even if $X(N,\Delta)$ *is* smooth, applying theorem 3.6 for a *non-reduced* subscheme Z as center, the resulting normal complex variety Norm $\left[\mathbf{Bl}^{\mathcal{I}}_Z(X(N,\Delta))\right]$ is *not* necessarily smooth. The simplest example is to take $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, i.e., the standard lattice with e_1, e_2 the unit vectors, $M = \mathbb{Z}e_1^\vee \oplus \mathbb{Z}e_2^\vee$ its dual,

$$\sigma = \mathrm{pos}(e_1, e_2), \; U_\sigma \cong \mathbb{C}^2, \; Z = (0,0) = \mathrm{orb}(\sigma), \; \text{ and } \; \mathcal{I} = (\mathbf{e}(e_1^\vee), \mathbf{e}(2e_2^\vee)).$$

Then the blown up space corresponds to the fan consisting of the cones

$$\sigma_1 = \mathrm{pos}((2,1)^\intercal, e_2), \quad \sigma_2 = \mathrm{pos}(e_1, (2,1)^\intercal),$$

together with their faces. Obviously, $\mathrm{mult}(\sigma_1; N) = 2$, and we rediscover the second example of 3.2. (In fact, this is nothing but the so-called *weighted blow-up* of the origin w.r.t. $(2,1)^\intercal$ in Reid's terminology; see [59], p. 297).

(iv) On the other hand, the normalization of the *usual* blow-up of a *not* necessarily smooth, affine toric variety $U_\sigma = \text{Max-Spec}(\mathbb{C}[M \cap \sigma^\vee])$ along a subvariety can be described intrinsically by making use of thm. 3.6 and arguments coming from an embedding. Next proposition treats of the case in which one blows up U_σ at $\text{orb}(\sigma)$.

Proposition 3.8 (Usual normalized blow-up at the closed point).
Let N be a lattice of rank r, σ an r-dimensional rational s.c.p. cone in $N_{\mathbb{R}}$ and

$$U_\sigma = \text{Max-Spec}\left(\mathbb{C}[M \cap \sigma^\vee]\right)$$

the associated affine toric variety with $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Moreover, let

$$\mathbf{Hlb}_M(\sigma^\vee) = \{m_1, \ldots, m_d\}, \quad r \leq d,$$

denote a fixed enumeration of the members of the Hilbert basis of $\sigma^\vee$ w.r.t. M (cf. prop. 2.2). *Then*

$$\text{Norm}\left[\mathbf{Bl}^{\text{red}}_{\text{orb}(\sigma)}(U_\sigma)\right] = X\left(N, \Delta_{\mathbf{bl}}[\text{orb}(\sigma)]\right),$$

where

$$\Delta_{\mathbf{bl}}[\text{orb}(\sigma)] := \Delta_{\mathbf{bl}}\left(\left(\begin{array}{c}\text{maximal}\\ \text{ideal}\end{array}\right); \text{orb}(\sigma)\right) = \left\{\begin{array}{c}\{\sigma_j \mid 1 \leq j \leq d\} \text{ together}\\ \text{with all their faces}\end{array}\right\},$$

with

$$\sigma_j := \{\mathbf{y} \in \sigma \mid \langle m_i - m_j, \mathbf{y}\rangle \geq 0, \ \forall i, \ i \in \{1, \ldots, j-1, j+1, \ldots, d\}\},$$

for all j, $1 \leq j \leq d$. (Warning. Though the union of the above σ_j's forms always a fan, it might happen that $\sigma_j = \sigma_{j'}$ or that σ_j is a proper face of $\sigma_{j'}$, for certain distinct indices j, j' belonging to $\{1, \ldots, d\}$. In this case, we just ignore the superfluous cones and introduce a new index-enumeration for the rest, preferably by considering only the maximal-dimensional ones.)

Proof. Under the above assumptions, we may use prop. 2.4 and embed U_σ by $(\mathbf{e}(m_1), \ldots, \mathbf{e}(m_d))$ "minimally" into the affine complex space

$$\mathbb{C}^d = \text{Max-Spec}\left(\mathbb{C}\left[\widetilde{M} \cap \widetilde{\sigma}^\vee\right]\right),$$

where

$$\widetilde{\sigma}^\vee := \text{pos}\left(\{m_1, \ldots, m_d\}\right) \subset \left(\widetilde{M}\right)_{\mathbb{R}}, \quad \widetilde{M} := \mathbb{Z}\, m_1 \oplus \cdots \oplus \mathbb{Z}\, m_d\,.$$

$\text{orb}(\sigma)$ is mapped onto $\mathbf{0} = \text{orb}(\widetilde{\sigma}) \in \mathbb{C}^d$, with

$$\widetilde{\sigma} = \widetilde{\sigma}^{\vee\vee} \subset \left(\widetilde{N}\right)_{\mathbb{R}}, \quad \widetilde{N} = \text{Hom}_{\mathbb{Z}}\left(\widetilde{M}, \mathbb{Z}\right) \cong \mathbb{Z}^d.$$

Using the embedding $N \hookrightarrow \widetilde{N}$, it is possible to describe $\mathbf{Bl}^{\mathrm{red}}_{\mathrm{orb}(\sigma)}(U_\sigma)$ as the strict transform of U_σ under the usual blow-up morphism of $\mathbb{C}^d$ at the origin (i.e. just by applying what we mentioned in (d) for $Z = \mathrm{orb}(\sigma)$, $W = U_\sigma$ and $X = \mathbb{C}^d$). Hence, we obtain the following commutative diagrams of torus-equivariant holomorphic maps:

$$\begin{array}{lclcll}
 & & \mathrm{Norm}\left[\mathbf{Bl}^{\mathrm{red}}_{\mathrm{orb}(\sigma)}(U_\sigma)\right] & & & \\
 & & \downarrow & \searrow & & \\
\mathrm{Exc}(\pi\,|_{\mathrm{restr.}}) & \in & \mathbf{Bl}^{\mathrm{red}}_{\mathrm{orb}(\sigma)}(U_\sigma) & \hookrightarrow & \mathbf{Bl}^{\mathrm{red}}_{\mathbf{0}}(\mathbb{C}^d) & \ni \ \mathrm{Exc}(\pi) \\
 & & \downarrow\ \pi\,|_{\mathrm{restr.}} & & \downarrow\ \pi & \\
\mathrm{orb}(\sigma) & \in & U_\sigma & \hookrightarrow & \mathbb{C}^d & \ni \ \mathbf{0}
\end{array}$$

Since $\mathbf{Bl}^{\mathrm{red}}_{\mathbf{0}}(\mathbb{C}^d)$ is realized as toric variety by the d cones of the barycentric subdivision of $\widetilde{\sigma}$,

$$\tau_j := \{\mathbf{y} \in \widetilde{\sigma} \mid \langle m_i - m_j, \mathbf{y}\rangle \geq 0,\ \forall i,\ i \in \{1, \ldots, j-1, j+1, \ldots, d\}\},\ 1 \leq j \leq d,$$

(cf. 3.7 (i)), the above defined σ_j's are exactly the restrictions of τ_j's on σ and determine obviously $\Delta_{\mathbf{bl}}[\mathrm{orb}(\sigma)]$ as it was given initially in thm. 3.6. □

(i) *Resolutions of toric singularities.* To resolve toric singularities is actually equivalent to subdividing simplicial cones into others of smaller multiplicity.

Theorem 3.9 (Resolution of toric singularities. Weak version).
If $X(N,\Delta)$ is an arbitrary toric variety, then there exists a refinement $\widetilde{\Delta}$ of Δ, such that

$$f = \mathrm{id}_* : X\left(N, \widetilde{\Delta}\right) \longrightarrow X(N,\Delta)$$

is a (full) desingularization of $X(N,\Delta)$.

Sketch of proof. Considering the multiplicity of a simplicial cone σ as a volume and using the well-behaved volume properties under subdivisions w.r.t. lattice points of $\mathbf{Par}(\sigma) \cap N_\sigma$, one can easily desingularize equivariantly any toric variety $X(N,\Delta)$:

• At first we refine the cones of Δ in order to make it simplicial. That this is always possible follows basically from Carathéodory's theorem concerning convex polyhedral cones (cf. [70], p. 94; for a simple proof see also G.Ewald [17], III 2.6, p. 75, and V 4.2, p. 158).

• In the second step this new simplicial Δ will be subdivided further into subcones of strictly smaller multiplicities than those of the cones of the starting-point. After finitely many subdivisions of this kind one can construct a refinement Δ' of Δ, so that $f = \mathrm{id}_*$ gives rise to a resolution of singularities of $X(N,\Delta)$ (by thm. 2.17 and prop. 2.5). □

If fact, for toric varieties, a *single* normalized blow-up of a suitable ideal sheaf is able to provide a full resolution.

Theorem 3.10 (Resolution of toric singularities. Strong version). *If $X(N,\Delta)$ is an arbitrary toric variety, then there exists a T_N-invariant coherent sheaf $\mathcal{I}$ of $\mathcal{O}_{X(N,\Delta)}$-ideals with*

$$\operatorname{Sing}(X(N,\Delta)) = \operatorname{supp}\left(\mathcal{O}_{X(N,\Delta)} \,/\, \mathcal{I}\right),$$

such that

$$\operatorname{Norm}\left[\mathbf{Bl}^{\mathcal{I}}_{\operatorname{Sing}(X(N,\Delta))}(X(N,\Delta))\right] \overset{\pi_{\mathcal{I}} \circ \boldsymbol{\nu}_{\mathcal{I}} = \mathrm{id}_*}{\longrightarrow} X(N,\Delta)$$

forms a (full, projective) desingularization of $X(N,\Delta)$. ($\mathcal{I}$ is in general not uniquely determined by this property and might contain nilpotent elements).

Proof. See Saint-Donat [42], thm. 11, pp. 32-35, and Brylinski [10], pp. 273-279.□

4. Toric $\mathbb{P}^k$-bundles over projective spaces

This section is a brief excursus to a part of the theory of toric bundles over projective spaces which will be used later on (in § 8) for the precise description of the exceptional prime divisors occuring in our desingularizations.

(a) An equivariant holomorphic map $\varpi_* : X(N,\Delta) \to X(N',\Delta')$ of toric varieties, induced by a map of fans $\varpi : (N,\Delta) \to (N',\Delta')$ (cf. §2 (k)), can be viewed as the projection map of an equivariant fiber bundle (*toric bundle*) over $X(N',\Delta')$ with typical fiber $X(N'',\Delta'')$, $N'' = \operatorname{Ker}(\varpi : N \to N')$, if and only if $\varpi : N \to N'$ is surjective and the cones of Δ are representable as "joins" of the cones of a fan Δ'' ($|\Delta''| \subset N''_{\mathbb{R}}$) with those of another fan $\widetilde{\Delta}' \subset \Delta$, so that the supports $\left|\widetilde{\Delta}'\right|$ and $|\Delta'|$ are homeomorphic to each other. (See Oda [54], prop. 1.33, p. 58, and Ewald [17], thm. VI.6.7, p. 246). In the case in which its total space is assumed to be smooth and compact, this criterion can be considerably simplified by means of the notion of "primitive collections" introduced by Batyrev [1].

Definition 4.1. If $X(N,\Delta)$ is an r-dimensional smooth, compact toric variety, then a non-empty subset $\mathfrak{N} = \{n_1, n_2, \ldots, n_k\}$ of $\operatorname{Gen}(\Delta)$, $k \geq 2$, is defined to be a *primitive collection* if it satisfies anyone of the following equivalent conditions :

(i) For each $n_i \in \mathfrak{N}$, $1 \leq i \leq k$, one has $\mathfrak{N} \smallsetminus \{n_i\} = \operatorname{Gen}(\sigma_i)$, for some σ_i belonging to $\Delta(k-1)$, while $\mathfrak{N}$ itself cannot be the set of minimal generators of any k-dimensional cone of Δ.

(ii) For each subset of indices $\{j_1, \ldots, j_q\} \subset \{1, \ldots, k\}, 1 \leq q < k$, the set $\left\{n_{j_1}, \ldots, n_{j_q}\right\}$ coincides with the set of minimal generators of a q-dimensional cone of Δ, while $\mathfrak{N}$ itself cannot be the set of minimal generators of any k-dimensional cone of Δ.

Proposition 4.2 (Characterization of toric $\mathbb{P}^k_{\mathbb{C}}$-bundles). *Let $X(N,\Delta)$ be a smooth, compact toric variety of dimension r and k a positive integer $\leq r$. $X(N,\Delta)$*

is the total space of a toric $\mathbb{P}^k_{\mathbb{C}}$-bundle over a smooth $(r-k)$-dimensional toric variety if and only if there exists a primitive collection

$$\mathfrak{N} = \{n_1, n_2, \ldots, n_{k+1}\} \subset \mathrm{Gen}\,(\Delta) \ ,$$

such that

(i) $n_1 + n_2 + \cdots + n_{k+1} = \mathbf{0}_N$, *and*

(ii) $\mathfrak{N} \cap \mathfrak{N}' = \varnothing$, *for all primitive collections* $\mathfrak{N}' \subset \mathrm{Gen}(\Delta)$ *with* $\mathfrak{N} \neq \mathfrak{N}'$.

Definition 4.3. Let $\mathrm{X}(N, \Delta)$ be a smooth, compact toric variety. The fan Δ is called a *splitting fan* if any two different primitive collections within $\mathrm{Gen}(\Delta)$ have no common elements.

Theorem 4.4. *If Δ is a splitting fan, then $X\,(N, \Delta)$ is a projectivization of a decomposable bundle over a toric variety being associated to a splitting fan of smaller dimension.*

Proof. See Batyrev [1], thm. 4.3, p. 577. □

(b) The projectivized decomposable bundles over projective spaces, having only twisted hyperplane bundles as summands, can be easily described as toric bundles in terms of splitting fans with exactly two disjoint primitive collections. In particular, applying 4.2 and 4.4, we obtain:

Lemma 4.5. *Let k and s be two positive integers, $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ a k-tuple of non-negative integers and $N_{(r;\lambda_1,\lambda_2,\ldots,\lambda_k)}$ (resp. N'_s) a lattice of rank $r = k+s$ (resp. of rank s) generated by $\mathfrak{N} \cup \mathfrak{N}'$ (resp. by $\mathfrak{N}'$) where*

$$\mathfrak{N} := \{\, n_1, n_2, \ldots, n_{k+1}\}\,, \quad \mathfrak{N}\,' := \left\{n'_1, n'_2, \ldots, n'_s, n'_{s+1}\right\}$$

with the two relations

$$n_1 + n_2 + \cdots + n_k + n_{k+1} = \mathbf{0}, \quad n'_1 + n'_2 + \cdots + n'_{s+1} = \lambda_1 n_1 + \cdots + \lambda_k n_k \ .$$

If we define the r-dimensional, smooth, compact toric variety

$$\boxed{\quad Y\,(r; \lambda_1, \lambda_2, \ldots, \lambda_k) := X\left(N_{(r;\lambda_1,\lambda_2,\ldots,\lambda_k)},\ \Delta_{(r;\lambda_1,\lambda_2,\ldots,\lambda_k)}\right) \quad}$$

by means of the fan

$$\Delta_{(r;\lambda_1,\lambda_2,\ldots,\lambda_k)} := \left\{ \begin{array}{l} \mathrm{pos}\left((\mathfrak{N} \cup \mathfrak{N}') \smallsetminus \{n_i, n'_j\}\right), \\ \text{for all } \ (i,j) \in \{1, \ldots, k+1\} \times \{1, \ldots, s+1\}, \\ \text{together with all their faces} \end{array} \right\} ,$$

then it is isomorphic to the total space of the $\mathbb{P}^k_{\mathbb{C}}$*-bundle*

$$Y(r;\lambda_1,\lambda_2,\ldots,\lambda_k)\cong\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^s_{\mathbb{C}}}\oplus\mathcal{O}_{\mathbb{P}^s_{\mathbb{C}}}(\lambda_1)\oplus\mathcal{O}_{\mathbb{P}^s_{\mathbb{C}}}(\lambda_2)\oplus\cdots\oplus\mathcal{O}_{\mathbb{P}^s_{\mathbb{C}}}(\lambda_k)\right)\to\mathbb{P}^s_{\mathbb{C}}$$

over $\mathbb{P}^s_{\mathbb{C}}=X(N'_s,\Delta'_s)$, *with*

$$\Delta'_s:=\left\{\begin{array}{l}\text{pos}\left(\mathfrak{N}'\smallsetminus\{n'_j\}\right)\text{ in }(N'_s)_{\mathbb{R}},\\ \text{for all } j\in\{1,\ldots,s+1\},\\ \text{together with all their faces}\end{array}\right\}.$$

Proof. By construction, $\mathfrak{N}$ and $\mathfrak{N}'$ are the only primitive collections within $\text{Gen}\left(\Delta_{(r;\lambda_1,\ldots,\lambda_k)}\right)$, and $\mathfrak{N}\cap\mathfrak{N}'=\varnothing$. Hence, $\Delta_{(r;\lambda_1,\ldots,\lambda_k)}$ is a splitting fan. Since $\#(\text{Gen}(\Delta'_s))=\#(\Delta'_s(1))=s+1$, the base-space of the smooth toric $\mathbb{P}^k_{\mathbb{C}}$-bundle $Y(r;\lambda_1,\ldots,\lambda_k)$ has to be isomorphic to $\mathbb{P}^s_{\mathbb{C}}$. Finally, the decomposable bundle over $X(N'_s,\Delta'_s)$ (whose projectivization gives the total space $Y(r;\lambda_1,\ldots,\lambda_k)$) is isomorphic to $\mathcal{O}_{\mathbb{P}^s_{\mathbb{C}}}\oplus\mathcal{O}_{\mathbb{P}^s_{\mathbb{C}}}(\lambda_1)\oplus\cdots\oplus\mathcal{O}_{\mathbb{P}^s_{\mathbb{C}}}(\lambda_k)$ because the fan corresponding to the typical fiber consists of cones which are the images of the cones of Δ'_s under the linear map

$$(N'_s)_{\mathbb{R}}\hookrightarrow\left(N_{(r;\lambda_1,\ldots,\lambda_k)}\right)_{\mathbb{R}}$$

sending a $\mathbf{y}'$ to $(\mathbf{y}',-(\lambda_1n_1+\cdots+\lambda_kn_k+n_{k+1}))$. □

Theorem 4.6 (Classification theorem of Kleinschmidt).
Every smooth, compact r*-dimensional toric variety* $(r\geq 2)$ *with Picard number 2, is isomorphic to one of the varieties* $Y(r;\lambda_1,\lambda_2,\ldots,\lambda_k)$.

Proof. See [44], § 2, pp. 256-261. □

Example 4.7 (Hirzebruch surfaces). Setting $k=1,\lambda=\lambda_1$, and $r=2$, one gets the rational scrolls

$$\mathbb{F}_\lambda:=Y(2;\lambda)\cong\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}\oplus\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\lambda)\right)$$

over $\mathbb{P}^1_{\mathbb{C}}$, i.e. the so-called *Hirzebruch surfaces* whose topological, analytic and birational properties were studied in the early fifties in [32]. (Certain birational properties of them were already investigated by Segre and Del Pezzo around the end of the last century in connection with other types of scrolls and ruled surfaces. See e.g. Segre [71]). It is well-known that all $\mathbb{F}_\lambda$'s, $\lambda\neq 1$, together with $\mathbb{P}^2_{\mathbb{C}}$, exhaust the class of all minimal, smooth, rational, projective complex surfaces, and that $\mathbb{F}_\lambda$ can be considered as the zero-set :

$$\left\{([z_0:z_1:z_2],[t_1,t_2])\in\mathbb{P}^2_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}\;\middle|\;z_1\cdot t_1^\lambda-z_2\cdot t_2^\lambda=0\right\}\tag{4.1}$$

$\mathbb{F}_0$ is therefore $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$, $\mathbb{F}_1$ is isomorphic to the usual blow-up of $\mathbb{P}^2_{\mathbb{C}}$ at a ($T_{N_{(2;\lambda)}}$-fixed) point, and for $\lambda\geq 2$ there exists a natural λ-sheeted ramified covering $\mathbb{F}_\lambda\to\mathbb{F}_1$ over $\mathbb{F}_1$ (see [32], p. 82). The differential-topological and diffeomorphism classification theory, and the deformation theory of the $\mathbb{P}^k_{\mathbb{C}}$-bundles $Y(k+1;\lambda_1,\ldots,\lambda_k)$ over $\mathbb{P}^1_{\mathbb{C}}$ were developed in Brieskorn's work [7] in the sixties. (For another, purely geometric approach to the theory of rational scrolls over $\mathbb{P}^1_{\mathbb{C}}$, see Reid [64]).

Generalizing the embedding (4.1) for arbitrary k and s, one obtains :

Proposition 4.8 (Bihomogeneous binomial representation). *The toric variety $Y(r;\lambda_1,\ldots,\lambda_k)$ is embeddable into the projective space $\mathbb{P}_{\mathbb{C}}^{(s+1)k}\times\mathbb{P}_{\mathbb{C}}^{s}$; in fact, if*

$$\{z_0,z_{1,1},z_{2,1},\ldots,z_{s+1,1},z_{1,2},\ldots,z_{s+1,2},\ldots,z_{1,k},\ldots,z_{s+1,k};t_1,t_2,\ldots,t_{s+1}\}$$

denote bihomogeneous coordinates, it is representable as the zero-set

$$\left\{ z_{\nu,i}\cdot t_{\mu}^{\lambda_i}-z_{\mu,i}\cdot t_{\nu}^{\lambda_i}=0 \;\middle|\; \begin{array}{l}\text{for all triples } (\mu,\nu,i) \text{ with}\\ 1\le i\le k,\ 1\le\mu,\nu\le s+1,\ \mu\neq\nu\end{array} \right\}$$

It is possible to embed these, sometimes called *Hirzebruch-Kleinschmidt varieties*, $Y(r;\lambda_1,\ldots,\lambda_k)$, into a single projective space by using the Segre-embedding

$$\mathbb{P}_{\mathbb{C}}^{(s+1)k}\times\mathbb{P}_{\mathbb{C}}^{s}\hookrightarrow\mathbb{P}_{\mathbb{C}}^{((s+1)k+1)(s+1)-1}$$

but as it was proved by Ewald and Schmeinck in [18], this can be done in a more economical way (w.r.t. the degrees of the defining homogeneous binomials), namely by only considering quadrics within a suitably higher dimensional projective space.

Theorem 4.9 (Representation by quadrics).
The toric varieties $Y(r;\lambda_1,\ldots,\lambda_k)$ are embeddable into the projective space $\mathbb{P}_{\mathbb{C}}^{d}$ of dimension

$$d=r-k+\sum_{i=1}^{k}\binom{\lambda_i+r-k+1}{r-k}$$

(depending on λ_i's) and their defining ideals (w.r.t. homogeneous coordinates $\{z_0,z_1,\ldots,z_d\}$ of $\mathbb{P}_{\mathbb{C}}^{d}$) are generated by the quadratic binomials

$$z_{\nu_1}\,z_{\nu_2}-z_{\nu_3}\,z_{\nu_4},\quad\text{for all}\quad(\nu_1,\nu_2,\nu_3,\nu_4)\in(\{0,1,\ldots,d\})^4\ .$$

(c) The intersection theory on the varieties $Y(r;\lambda_1,\ldots,\lambda_k)$ is more complicated compared with that on $\mathbb{F}_\lambda$'s. For example, although for $s=1$, the Chern numbers of the total spaces of these $\mathbb{P}_{\mathbb{C}}^{k}$-bundles over $\mathbb{P}_{\mathbb{C}}^{1}$ for fixed dimension $r=k+1$ are constant (cf. Brieskorn [7], Satz 2.4.(ii), p. 348), this is no more true for decomposable $\mathbb{P}_{\mathbb{C}}^{k}$-bundles over $\mathbb{P}_{\mathbb{C}}^{s}$ with $s\ge2$, because there is an obvious dependence on λ_i's, i.e. on the given "twisting numbers". On the other hand, the isomorphism for the Picard group

$$\mathrm{Pic}\,(Y(r;\lambda_1,\ldots,\lambda_k))\cong\ \mathrm{Pic}\,(\mathbb{P}_{\mathbb{C}}^{s})\times\mathbb{Z}\cong\mathbb{Z}\times\mathbb{Z}$$

is in general valid (cf. [27], Ex. 7.9, p. 170), and is useful as long as one makes a specific choice of a $\mathbb{Z}$-basis and expresses each member of any examined r-tuple of divisors as concrete $\mathbb{Z}$-linear combinations of its two elements. (The most natural choice is to consider the classes in Pic corresponding to a typical fiber of the bundle map and to a hyperplane section under the embedding of $Y(r;\lambda_1,\ldots,\lambda_k)$ in $\mathbb{P}^d_{\mathbb{C}}$ of thm. 4.9, respectively). Let us now give, in broad outline, three complementary practical methods for the computation of intersection numbers.

• *First method.* For arbitrary k and $s \geq 1$, there exist two towers of birational morphisms

$$\begin{array}{lcl} Y_0 \longleftarrow Y_1 \longleftarrow Y_2 \longleftarrow \cdots \longleftarrow Y_{\nu-1} \longleftarrow & & Y_\nu = W_\mu \\ \| & & \downarrow \\ Y(r;\lambda_1,\ldots,\lambda_k) & & W_{\mu-1} \\ & & \downarrow \\ & & \vdots \\ & & \downarrow \\ & & W_1 \\ & & \downarrow \\ & & W_0 = \mathbb{P}^r_{\mathbb{C}} \end{array}$$

which are nothing but usual toric blow-ups with toric subvarieties of codimension ≥ 2 as centers. (For the precise description of this procedure in terms of convex geometry, i.e. via barycentric stellar subdivisions of cones of $\Delta_{(r;\lambda_1,\lambda_2,\ldots,\lambda_k)}$, and the algorithmic determination of ν, $\mu \geq 0$, the reader is referred to Kleinschmidt [44], pp. 264-265). So the evaluation of intersection numbers of divisors on $Y(r;\lambda_1,\ldots,\lambda_k)$ can be reduced to another one w.r.t. divisors sitting on $\mathbb{P}^r_{\mathbb{C}}$. The problem here is that one blows up and down subvarieties of varying dimensions and must therefore control carefully the intersection behaviour of the proper transforms of divisors in each step. The simplest example is $Y(r;1)$ (with $k=1$, $r=s+1$) which is $\mathbb{P}^r_{\mathbb{C}}$ blown up at a ($T_{N(r;1)}$-fixed) point (with $\nu=0$, $\mu=1$); but, for instance, already $Y(3;2)$ is the blow-up of $\mathbb{P}^3_{\mathbb{C}}$ along an entire ($T_{N(3;2)}$-fixed) curve followed by the contraction of another ($T_{N(3;2)}$-fixed) curve (i.e., $\nu=\mu=1$, in this case).

• *Second method.* This method can be applied to any projective toric variety (or even to any compact toric variety), but demands familiarity with mixed volumes of "virtual" lattice polytopes. (Virtual polytopes are defined to be finite families of suitably translated dual cones of cones of a given fan, which are equipped with a $\mathbb{Z}$-module structure w.r.t. formal addition and scalar multiplication, though their intersection might be not a polytope in the usual sence). If all the line bundles, being associated to the Cartier divisors whose intersection number is to be computed, are generated by their global sections (in other words, if the corresponding integral support functions are upper convex, cf. thm. 2.11), then we may apply formula (2.6) and evaluate the normalized mixed volumes of the arising lattice polytopes w.r.t. the dual lattice. For arbitrary Cartier divisors, however, we need the combinatorial version of "moving lemma", i.e., to write down each of their associated support functions as the difference of two upper convex (or even strictly upper convex) support functions, and to calculate afterwards the desired intersection numbers as the mixed volume of the difference of two virtual polytopes. (See in Ewald's book [17], thm. V. 5.15, pp. 175-177, and thm. VII. 6.3, pp. 292-295. For an intrinsic, algorithmic characterization of moving lemma of this kind, we refer to Wessels' thesis [83], Satz 3.2.4, p. 83, and 3.2.18, p. 89. In contrast to [17], he works directly with "generalized" mixed volumes of virtual polytopes the normalizations of which take integer, but not necessarily only non-negative values). A simple example: the self-intersection number of the canonical divisor of $Y(3;2)$. Since $Y(3;2)$ is a Fano 3-fold, $-K_{Y(3;2)}$ is ample, and it is easy to verify that the lattice 3-polytope $P_{-K_{Y(3;2)}}$ (3-polytope, in the usual sence), induced by the anticanonical integral strictly upper convex support function, can be realized (up to an affine integral transformation) by

$$P_{-K_{Y(3;2)}} = \mathrm{conv}\left(\left\{\begin{array}{c} \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \\ \\ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \end{array}\right\}\right) \subset \mathbb{R}^3 ,$$

i.e., by the polar of a lattice bypiramid over a triangle or, equivalently, by a *triangular lattice prism.* Fig. **3** shows $-P_{-K_{Y(3;2)}}$.
An immediate calculation gives

$$\mathrm{Vol}\left(P_{-K_{Y(3;2)}}\right) = \frac{31}{3} \Longrightarrow K^3_{Y(3;2)} = -\left(-K_{Y(3;2)}\right)^3 = -62$$

(by formula (2.5)).

• *Third method.* If $\mathcal{E}$ is a locally free sheaf with $q = \mathrm{rk}(\mathcal{E})$ defined over a smooth projective complex variety Z and

$$\pi : \mathbb{P}(\mathcal{E}) = \mathbf{Proj}\,(\mathrm{Sym}^\bullet\,(\mathcal{E})) \longrightarrow Z$$

the associated projective bundle, then by Grothendieck's direct construction of the Chern classes of $\mathcal{E}$,

$$c_i(\mathcal{E}) \in A^i(Z), \quad i \in \{0, 1, \ldots, q\},$$

i.e., by setting $c_0(\mathcal{E}) = 1$ and

$$\sum_{i=0}^{q} (-1)^i \ \pi^* c_i(\mathcal{E}) \cdot \left[E^{q-i}\right] = 0 \tag{4.2}$$

(cf. Hartshorne [27], p. 429, or Fulton [22], 3.2.4, p. 55), the Chow ring $A^\bullet(\mathbb{P}(\mathcal{E}))$ can be viewed as a free $A^\bullet(Z)$-module generated by the classes $[1]$, $[E]$, ..., $\left[E^{q-1}\right]$,

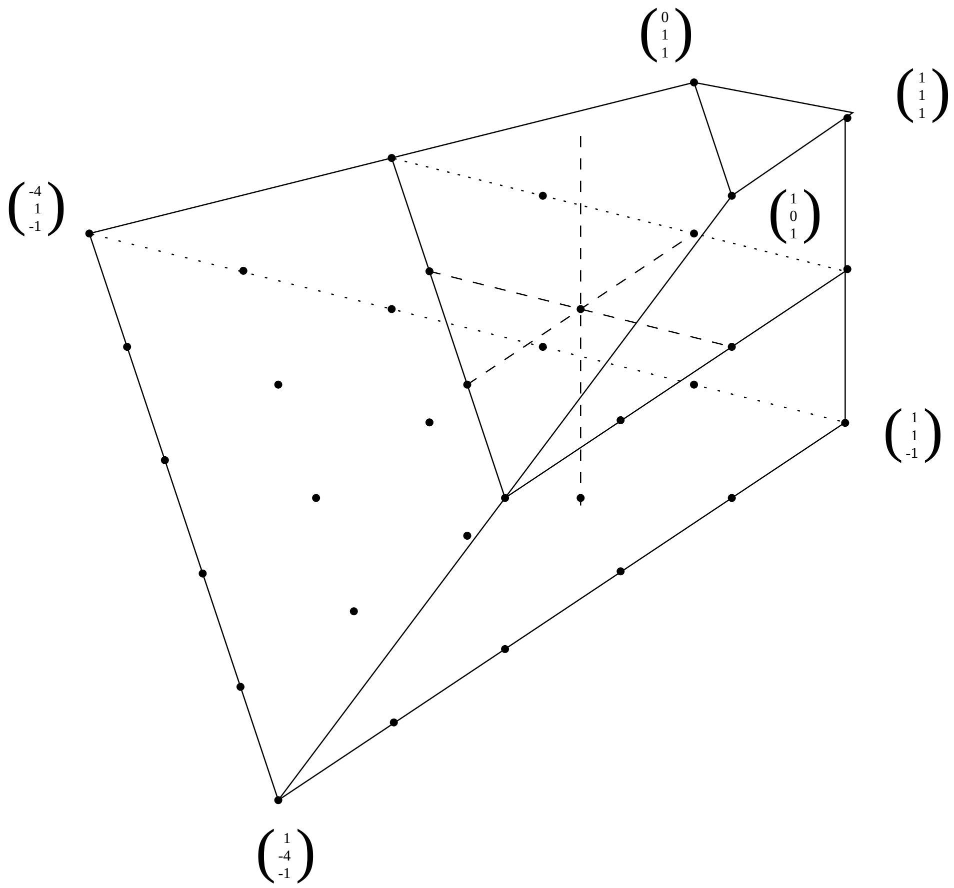

$\begin{pmatrix}0\\1\\1\end{pmatrix}$
$\begin{pmatrix}1\\1\\1\end{pmatrix}$
$\begin{pmatrix}-4\\1\\-1\end{pmatrix}$
$\begin{pmatrix}1\\0\\1\end{pmatrix}$
$\begin{pmatrix}1\\1\\-1\end{pmatrix}$
$\begin{pmatrix}1\\-4\\-1\end{pmatrix}$

Fig. 3

where here E denotes the divisor on $\mathbb{P}(\mathcal{E})$ corresponding to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. The relationship between the Chern polynomial of the tangent bundle of $\mathbb{P}(\mathcal{E})$ and that of the pullback of the tangent bundle of Z follows from the relative tangent bundle exact sequence (see Fulton [22], 3.2.11, p. 59). In particular, computing the first Chern class, we deduce *the canonical bundle formula* :

$$K_{\mathbb{P}(\mathcal{E})} \underset{\text{lin}}{\sim} \pi^* \left(K_Z + \det(\mathcal{E})\right) - qE \tag{4.3}$$

The equation (4.2) for $Z = \mathbb{P}^s_{\mathbb{C}}$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^s_{\mathbb{C}}} \oplus \mathcal{O}_{\mathbb{P}^s_{\mathbb{C}}}(\lambda_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^s_{\mathbb{C}}}(\lambda_k)$ turns out to be a quite powerful tool for manipulating intersection numbers. In the next proposition we compute two basic self-intersection numbers for $Y(s+1;\lambda)$ (as $\mathbb{P}^1_{\mathbb{C}}$-bundle over $\mathbb{P}^s_{\mathbb{C}}$) which will be used in § 8, and leave to the reader as exercise to examine further (and more general) examples of various r-tuples of divisors by applying the above mentioned methods.

Proposition 4.10. *For $k=1$, $r=s+1$, $\lambda_1=\lambda\neq 0$, the self-intersection number of the canonical divisor of the total space of the $\mathbb{P}^1_{\mathbb{C}}$-bundle $Y(s+1;\lambda)\to\mathbb{P}^s_{\mathbb{C}}$ is given by the formula*

$$K^r_{Y(r;\lambda)} = \sum_{i=0}^{r-1} \binom{r}{i} \ (-2)^{r-i} \ (\lambda - r)^i \ \lambda^{r-i-1} \tag{4.4}$$

Moreover, the self-intersection number of the divisor $E = V(\mathrm{pos}(\{n_2\}))$ (in the notation of 4.5) equals

$$E^r = \lambda^{r-1} \tag{4.5}$$

Proof. For $Z=\mathbb{P}^s_{\mathbb{C}}$, $\mathcal{E} = \mathcal{O}_{\mathbb{P}^s_{\mathbb{C}}} \oplus \mathcal{O}_{\mathbb{P}^s_{\mathbb{C}}}(\lambda)$, and H a divisor of $\mathbb{P}(\mathcal{E})$ associated to the pullback $\pi^* c_1(\mathcal{E})$, we have

$$E \cdot (E - \lambda H) = E^2 - \lambda (H \cdot E) = 0 \tag{4.6}$$

(by (4.2) & $c_2(\mathcal{E}) = 0$) and

$$H^r = 0, \quad \left(H^{r-1}\cdot E\right) = 1 \tag{4.7}$$

(by definition). Furthermore, by the canonical bundle formula (4.3),

$$K_{Y(r;\lambda)} \underset{\text{lin}}{\sim} \pi^*\left(K_{\mathbb{P}^s_{\mathbb{C}}} + \det(\mathcal{E})\right) - 2E \underset{\text{lin}}{\sim} -(s+1)H + \lambda H - 2E = (\lambda - r) H - 2E \ .$$

For the self-intersection number we get

$$K^r_{Y(r;\lambda)} = \left((\lambda - r)H - 2E\right)^r = \sum_{i=0}^{r-1} \binom{r}{i} \ (-2)^{r-i} \ (\lambda - r)^i \ \left(H^i \cdot E^{r-i}\right) \tag{4.8}$$

(by the first of equations (4.7)), and applying successively (4.6) and the second of the equations (4.7),

$$\begin{aligned} \left(H^i \cdot E^{r-i}\right) &= \left(H^i \cdot (\lambda H)\cdot E^{r-i-1}\right) = \left(H^i \cdot (\lambda H)^2 \cdot E^{r-i-2}\right) = \cdots \\ \cdots\ \cdots &= \left(H^i \cdot (\lambda H)^{r-i-1}\cdot E\right) = \lambda^{r-i-1}\left(H^{r-1}\cdot E\right) = \lambda^{r-i-1} \ . \end{aligned}$$

Thus, formula (4.4) follows directly from (4.8). Finally, (4.5) is proved similarly via (4.6) and (4.7). □

5. Toric description of abelian quotient singularities

Abelian quotient singularities can be investigated by means of the theory of toric varieties in a direct manner. If G is a finite subgroup of $\mathrm{GL}(r, \mathbb{C})$, then $(\mathbb{C}^*)^r / G$ is automatically an algebraic torus embedded in $\mathbb{C}^r / G$.

Notation. We shall henceforth use the following extra notation. For $\nu \in \mathbb{N}$, $\mu \in \mathbb{Z}$, we denote by $[\mu]_\nu$ the (uniquely determined) integer for which

$$0 \leq [\mu]_\nu < \nu, \quad \mu \equiv [\mu]_\nu \,(\mathrm{mod}\ \nu)\,.$$

If $q \in \mathbb{Q}$, we define $\lfloor q \rfloor$ to be the greatest integer number $\leq q$. "gcd" will be abbreviation for greatest common divisor, and $\mathrm{diag}(\mathfrak{z}_1, \ldots, \mathfrak{z}_r)$ for complex diagonal $r \times r$ matrices having $\mathfrak{z}_1, \ldots, \mathfrak{z}_r$ as diagonal elements. Furthermore, for integers $\nu \geq 2$, we denote by $\zeta_\nu := e^{\frac{2\pi\sqrt{-1}}{\nu}}$ the "first" ν-th primitive root of unity.

(a) Let G be a finite subgroup of $\mathrm{GL}(r, \mathbb{C})$ which is *small*, i.e. with no pseudoreflections, acting linearly on $\mathbb{C}^r$, and $p : \mathbb{C}^r \to \mathbb{C}^r / G$ the quotient map. Denote by $(\mathbb{C}^r / G, [\mathbf{0}])$ the (germ of the) corresponding quotient singularity with $[\mathbf{0}] := p(\mathbf{0})$.

Proposition 5.1 (Singular locus). *If G is a small finite subgroup of $\mathrm{GL}(r, \mathbb{C})$, then*

$$\mathrm{Sing}\,(\mathbb{C}^r / G) = p\,(\{\mathbf{z} \in \mathbb{C}^r \mid G_{\mathbf{z}} \neq \{\mathrm{Id}\}\})$$

where $G_{\mathbf{z}} := \{g \in G \mid g \cdot \mathbf{z} = \mathbf{z}\}$ is the isotropy group of $\mathbf{z} = (z_1, \ldots, z_r) \in \mathbb{C}^r$.

Theorem 5.2 (Prill's group-theoretic isomorphism criterion). *Let G_1, G_2 be two small finite subgroups of $\mathrm{GL}(r, \mathbb{C})$. Then there exists an analytic isomorphism*

$$(\mathbb{C}^r / G_1, [\mathbf{0}]) \cong (\mathbb{C}^r / G_2, [\mathbf{0}])$$

if and only if G_1 and G_2 are conjugate to each other within $\mathrm{GL}(r, \mathbb{C})$.

Proof. See Prill [58], thm. 2, p. 382, and Brieskorn [8], Satz 2.3, p. 341. □

(b) Let G be a finite, small, *abelian* subgroup of $\mathrm{GL}(r, \mathbb{C})$, $r \geq 2$, having order $l = |G| \geq 2$. Define

$$\{e_1 = (1, 0, \ldots, 0, 0)^\intercal, \ldots, e_r = (0, 0, \ldots, 0, 1)^\intercal\}$$

to denote the standard basis of $\mathbb{Z}^r$, $N_0 := \sum_{i=1}^r \mathbb{Z} e_i$ the standard lattice, M_0 its dual, and

$$T_{N_0} := \text{Max-Spec}\left(\mathbb{C}\left[\mathfrak{x}_1^{\pm 1}, \ldots, \mathfrak{x}_r^{\pm 1}\right]\right) = (\mathbb{C}^*)^r \ .$$

Clearly,

$$T_{N_G} := \text{Max-Spec}\left(\mathbb{C}\left[\mathfrak{x}_1^{\pm 1}, \ldots, \mathfrak{x}_r^{\pm 1}\right]^G\right) = (\mathbb{C}^*)^r / G$$

is an r-dimensional algebraic torus with 1-parameter group N_G and with group of characters M_G. Using the exponential map

$$(N_0)_{\mathbb{R}} \ni (y_1, \ldots, y_r)^{\intercal} = \mathbf{y} \longmapsto \exp(\mathbf{y}) := \left(e^{\left(2\pi\sqrt{-1}\right)y_1}, \ldots, e^{\left(2\pi\sqrt{-1}\right)y_r}\right)^{\intercal} \in T_{N_0}$$

and the injection $\boldsymbol{\iota} : T_{N_0} \hookrightarrow \mathrm{GL}(r, \mathbb{C})$ defined by

$$T_{N_0} \ni (t_1, \ldots, t_r)^{\intercal} = \mathbf{t} \hookrightarrow \boldsymbol{\iota}(\mathbf{t}) := \mathrm{diag}(t_1, \ldots, t_r) \in \mathrm{GL}(r, \mathbb{C}) \ ,$$

we have obviously

$$N_G = (\boldsymbol{\iota} \circ \exp)^{-1}(G) \qquad (\text{ and determinant } \det(N_G) = \frac{1}{l} \ )$$

(as long as we choose eigencoordinates to diagonalize the action of the elements of G on $\mathbb{C}^r$) with

$$M_G = \left\{ m \in M_0 \;\middle|\; \begin{array}{c} \mathfrak{x}^m = \mathfrak{x}_1^{\mu_1} \cdots \mathfrak{x}_r^{\mu_r} \quad \text{is a } G\text{-invariant} \\ \text{Laurent monomial } (m = (\mu_1, \ldots, \mu_r)) \end{array} \right\} \text{ (and } \det(M_G) = l).$$

• If we define

$$\boxed{\sigma_0 := \mathrm{pos}(\{e_1, .., e_r\})}$$

to be the r-dimensional positive orthant, and Δ_G to be the fan

$$\boxed{\Delta_G := \{\sigma_0 \text{ together with its faces}\}}$$

then by the exact sequence

$$0 \to G \cong N_G/N_0 \to T_{N_0} \to T_{N_G} \to 0$$

induced by the canonical duality pairing

$$M_0/M_G \times N_G/N_0 \to \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}^*$$

(cf. [23], p. 34, and [54], pp. 22-23), we get as projection map :

$$\mathbb{C}^r = X(N_0, \Delta_G) \to X(N_G, \Delta_G) \ ,$$

where

$$\boxed{X(N_G, \Delta_G) = U_{\sigma_0} = \mathbb{C}^r/G = \text{Max-Spec}\left(\mathbb{C}[\mathfrak{x}_1, \ldots, \mathfrak{x}_r]^G\right) \hookleftarrow T_{N_G}}$$

• Formally, we identify $[\mathbf{0}]$ with $\mathrm{orb}(\sigma_0)$. Moreover, in these terms, the singular locus of $X(N_G, \Delta_G)$ can be written (by 5.1 and 2.5) as the union

$$\mathrm{Sing}\,(X(N_G,\Delta_G)) = \ \mathrm{orb}\,(\sigma_0) \cup \left(\bigcup \left\{ U_{\sigma_0}(\tau) \ \middle| \ \begin{array}{c} \tau \precneqq \sigma_0, \dim(\tau) \geq 2 \\ \text{and } \ \mathrm{mult}\,(\tau; N_G) \geq 2 \end{array} \right\} \right) .$$

• In particular, if the acting group G is *cyclic*, then, fixing diagonalization of the action on $\mathbb{C}^r$, we may assume that G is generated by the element

$$\mathrm{diag}\,(\zeta_l^{\alpha_1}, \ldots, \zeta_l^{\alpha_r})$$

for r integers $\alpha_1, \ldots, \alpha_r \in \{0, 1, \ldots, l-1\}$, at least 2 of which are $\neq 0$. This r-tuple $(\alpha_1, \ldots, \alpha_r)$ of *weights* is unique only up to the usual conjugacy relations (see 5.5 below), and N_G is to be identified with the so-called *lattice of weights*

$$N_G = N_0 + \mathbb{Z} \left(\frac{1}{l} (\alpha_1, \ldots, \alpha_r)^{\intercal} \right)$$

containing all lattice points representing the elements of

$$G = \left\{ \mathrm{diag} \left(\zeta_l^{[\lambda \alpha_1]_l}, \ldots, \zeta_l^{[\lambda \alpha_r]_l} \right) \ \mid \ \lambda \in \mathbb{Z}, \ \ 0 \leq \lambda \leq l-1 \right\} .$$

Definition 5.3. Under these conditions, we say that the quotient singularity $(X(N_G, \Delta_G), \mathrm{orb}\,(\sigma_0))$ is *of type*

$$\boxed{\frac{1}{l}(\alpha_1, \ldots, \alpha_r)} \tag{5.1}$$

(This is identical with the definition given in [61], § 4.2, p. 370, up to the predeterminated fixing of the primitive root ζ_l of unity to be the "first" one. In fact, this extra assumption is not a significant restriction, and by fixing in advance the isomorphism $G \cong N_G/N_0$ one just simplifies certain technical arguments. Even if we would let ζ_l denote an arbitrary primitive root of unity, all results would remain the same up to an obvious multiplication of the exponents of the diagonalized elements of G by a suitable integer which would be relatively prime to l; see also the comments in [41], at the top of p. 225.)

• The existence of torus-equivariant resolutions of cyclic quotient singularities was proved by Ehlers [16], § I.3 & III.1-3, along the same lines as the more general theorem 3.9, i.e., by appropriate subdivisions of σ_0 into smaller cones of multiplicity 1. (Essentially the same result, expressed in the past language of gluings of affine pieces, is due to Fujiki [21], § 1.3).

• Note that, since G is small, $\gcd(l, \alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \alpha_r) = 1$, for all i, $1 \leq i \leq r$. (The symbol $\widehat{\alpha_i}$ means here that α_i is omitted.)

Lemma 5.4. (i) *A cyclic quotient singularity of type* (5.1) *has splitting codimension* $\varkappa \in \{2, \ldots, r-1\}$ *if and only if there exists an index-subset*

$$\{\nu_1, \nu_2, \ldots, \nu_{r-\varkappa}\} \subset \{1, \ldots, r\},$$

such that

$$\alpha_{\nu_1} = \alpha_{\nu_2} = \cdots = \alpha_{\nu_{r-\varkappa}} = 0 \ ,$$

which is, in addition, maximal *w.r.t. this property.*
(ii) *A cyclic quotient msc-singularity of type* (5.1) *is isolated if and only if*

$$\gcd(\alpha_i, l) = 1, \ \forall i, \ \ 1 \le i \le r \ .$$

Proof. It is immediate by the way we let G act on $\mathbb{C}^r$. □

(c) For two integers l, $r \geq 2$, we define

$$\Lambda(l;r) := \left\{ (\alpha_1, .., \alpha_r) \in \{0,1,2,..,l-1\}^r \ \middle| \ \begin{array}{c} \gcd(l, \alpha_1, .., \widehat{\alpha_i}, .., \alpha_r) = 1, \\ \text{for all } i, \ \ 1 \le i \le r \end{array} \right\}$$

and for $((\alpha_1, \ldots, \alpha_r), (\alpha'_1, \ldots, \alpha'_r)) \in \Lambda(l;r) \times \Lambda(l;r)$ the relation

$$(\alpha_1, \ldots, \alpha_r) \backsim (\alpha'_1, \ldots, \alpha'_r) :\Longleftrightarrow \left\{ \begin{array}{l} \text{there exists a permutation} \\ \theta : \{1, \ldots, r\} \to \{1, \ldots, r\} \\ \text{and an integer } \lambda, \ 1 \le \lambda \le l-1, \\ \text{with } \gcd(\lambda, l) = 1, \text{ such that} \\ \alpha'_{\theta(i)} = [\lambda \cdot \alpha_i]_l \ , \ \forall i, \ 1 \le i \le r \end{array} \right\}$$

It is easy to see that $\backsim$ is an equivalence relation on $\Lambda(l;r) \times \Lambda(l;r)$.

Corollary 5.5 (Isomorphism criterion for cyclic acting groups).
Let G, G' *be two small, cyclic finite subgroups of* $\mathrm{GL}(r, \mathbb{C})$ *acting on* $\mathbb{C}^r$, *and let the corresponding quotient singularities be of type* $\frac{1}{l}\,(\alpha_1, \ldots, \alpha_r)$ *and* $\frac{1}{l'}\,(\alpha'_1, \ldots, \alpha'_r)$ *respectively. Then there exists an analytic (torus-equivariant) isomorphism*

$$(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0)) \cong (X(N_{G'}, \Delta_{G'}), \mathrm{orb}(\sigma_0))$$

if and only if $l = l'$ *and* $(\alpha_1, \ldots, \alpha_r) \backsim (\alpha'_1, \ldots, \alpha'_r)$ *within* $\Lambda(l;r)$.

Proof. It follows easily from 5.2 (cf. Fujiki [21], lemma 2, p. 296). □

Proposition 5.6 (Gorenstein-condition).
Let $(\mathbb{C}^r/G, [\mathbf{0}]) = (X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ *be an abelian quotient singularity. Then the following conditions are equivalent :*
(i) $X(N_G, \Delta_G) = U_{\sigma_0} = \mathbb{C}^r/G$ *is Gorenstein,*
(ii) $G \subset \mathrm{SL}(r, \mathbb{C})$,
(iii) $\langle (1, 1, \ldots .1, 1), n \rangle \geq 1$, *for all* n, $n \in \sigma_0 \cap (N_G \smallsetminus \{\mathbf{0}\})$,

(iv) $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ *is a canonical singularity of index* 1.
In particular, if $(\mathbb{C}^r/G, [\mathbf{0}])$ *is cyclic of type* $\frac{1}{l}(\alpha_1, \ldots, \alpha_r)$, *then* (i)-(iv) *are equivalent to*

$$\sum_{j=1}^{r} \alpha_j \equiv 0 \pmod{l}$$

Proof. See Watanabe [81] and Reid [59] . □

• If $X(N_G, \Delta_G)$ is Gorenstein, then the cone $\sigma_0 = \mathrm{pos}(\mathfrak{s}_G)$ is supported by the so-called *junior lattice simplex*

$$\boxed{\mathfrak{s}_G = \mathrm{conv}(\{e_1, .., e_r\})}$$

(w.r.t. N_G, cf. [41], [3]). Note that up to $\mathbf{0}$ there is no other lattice point of $\sigma_0 \cap N_G$ lying "under" the affine hyperplane of $\mathbb{R}^r$ containing $\mathfrak{s}_G$. Moreover, the lattice points representing the $l-1$ non-trivial group elements are exactly those belonging to the intersection of a dilation $\lambda\,\mathfrak{s}_G$ of $\mathfrak{s}_G$ with $\mathbf{Par}(\sigma_0)$, for some integer λ, $1 \leq \lambda \leq r-1$.

6. Lattice triangulations and crepant projective resolutions of Gorenstein abelian quotient singularities

In this section we briefly formulate (and only partially prove) some general theorems concerning the study of projective, crepant resolutions of Gorenstein abelian quotient singularities in terms of appropriate lattice triangulations of the junior simplex. (For detailed expositions we refer to [11], [12]). Diagrams and formulae in boxes outline actually the only essential prerequisites for the reading of the rest of the paper.

(a) By $\mathrm{vert}(\mathcal{S})$ we denote the set of vertices of a polyhedral complex $\mathcal{S}$. By a *triangulation* $\mathcal{T}$ of a polyhedral complex $\mathcal{S}$ we mean a geometric simplicial subdivision of $\mathcal{S}$ with $\mathrm{vert}(\mathcal{S}) \subset \mathrm{vert}(\mathcal{T})$. A polytope P will be, as usual, identified with the polyhedral complex consisting of P itself together with all its faces.

(b) A triangulation $\mathcal{T}$ of an r-dimensional polyhedral complex $\mathcal{S}$ is called *coherent* (or *regular*) if there exists a strictly upper convex $\mathcal{T}$-support function $\psi : |\mathcal{T}| \to \mathbb{R}$, i.e. a piecewise-linear real function defined on the underlying space $|\mathcal{T}|$ of $\mathcal{T}$, for which

$$\psi(t\,\mathbf{x} + (1-t)\ \mathbf{y}) \geq t\,\psi(\mathbf{x}) + (1-t)\ \psi(\mathbf{y}),\ \text{for all}\ \ \mathbf{x}, \mathbf{y} \in |\mathcal{T}|,\ \text{and}\ \ t \in [0,1]\ ,$$

so that for every maximal simplex $\mathbf{s}$ of $\mathcal{T}$, there is a linear function $\eta_{\mathbf{s}} : |\mathbf{s}| \to \mathbb{R}$ satisfying $\psi(\mathbf{x}) \leq \eta_{\mathbf{s}}(\mathbf{x})$, for all $\mathbf{x} \in |\mathcal{T}|$, with equality being valid only for those $\mathbf{x}$ belonging to $\mathbf{s}$. The set of all strictly upper convex $\mathcal{T}$-support functions will be denoted by $\mathrm{SUCSF}_{\mathbb{R}}(\mathcal{T})$. A useful lemma to create a new "global" strictly upper convex support function by gluing together given "local" ones, is the following :

Lemma 6.1 (Patching Lemma). *Let $P \subset \mathbb{R}^r$ be a r-polytope, $\mathcal{T} = \{\mathbf{s}_i \mid i \in I\}$ (with I a finite set) a coherent triangulation of P, and $\mathcal{T}_i = \{\mathbf{s}_{i,j} \mid j \in J_i\}$ (J_i finite, for all $i \in I$) a coherent triangulation of $\mathbf{s}_i$, for all $i \in I$. If $\psi_i : |\mathcal{T}_i| \to \mathbb{R}$ denote strictly upper convex $\mathcal{T}_i$-support functions, such that*

$$\boxed{\psi_i \left|_{\mathbf{s}_i \cap \mathbf{s}_{i'}} \right. = \psi_{i'} \left|_{\mathbf{s}_i \cap \mathbf{s}_{i'}} \right.}$$

for all $(i, i') \in I \times I$, then

$$\widetilde{\mathcal{T}} := \{\text{all the simplices } \mathbf{s}_{i,j} \mid \forall j, \ \ j \in J_i, \text{ and } \ \forall i, \ i \in I\}$$

forms a coherent triangulation *of the initial polytope P (because the above ψ_i's can be canonically "patched together" to construct an element ψ of* $\mathrm{SUCSF}_{\mathbb{R}}\left(\widetilde{\mathcal{T}}\right)$).

Proof. See Bruns-Gubeladge-Trung [9], lemma 2.2.2, pp. 143-145. □

(c) Let N denote an r-dimensional lattice. By a *lattice polytope* (w.r.t. N) is meant a polytope in $N_{\mathbb{R}} \cong \mathbb{R}^r$ with vertices belonging to N. If $\{n_0, n_1, \ldots, n_k\}$ is a set of $k \leq r$ affinely independent lattice points, $\mathbf{s}$ the lattice k-dimensional simplex $\mathbf{s} = \mathrm{conv}(\{n_0, n_1, n_2, \ldots, n_k\})$, and $N_{\mathbf{s}} := \mathrm{lin}(\{n_1 - n_0, \ldots, n_k - n_0\}) \cap N$, then
• we say that $\mathbf{s}$ is an *elementary simplex* if

$$\{\mathbf{y} - n_0 \mid \mathbf{y} \in \mathbf{s}\} \cap N_{\mathbf{s}} = \{\mathbf{0}, \, n_1 - n_0, \ldots, \, n_k - n_0\}.$$

• $\mathbf{s}$ is *basic* if it has anyone of the following equivalent properties:
(i) $\{n_1 - n_0, n_2 - n_0, \ldots, n_k - n_0\}$ is a $\mathbb{Z}$-basis of $N_{\mathbf{s}}$,
(ii) $\mathbf{s}$ has relative volume $\mathrm{Vol}(\mathbf{s}; N_{\mathbf{s}}) = \dfrac{\mathrm{Vol}\,(\mathbf{s})}{\det\,(N_{\mathbf{s}})} = \dfrac{1}{k!}$ (w.r.t. $N_{\mathbf{s}}$) .

Lemma 6.2. (i) *Every basic lattice simplex is elementary.*
(ii) *Elementary lattice simplices of dimension ≤ 2 are basic.*

Proof. (i) Let $\mathbf{s} = \mathrm{conv}(\{n_0, n_1, \ldots, n_k\})$ be a basic lattice simplex. Since

$$\{n_1 - n_0, n_2 - n_0, \ldots, n_k - n_0\}$$

is a $\mathbb{Z}$-basis of $N_{\mathbf{s}}$, if $n' \in \mathbf{s} \cap (N_{\mathbf{s}} \smallsetminus \{\mathbf{0}\})$, then obviously $n' = n_i$ for some index i, $1 \leq i \leq k$. Thus,

$$\{\mathbf{y} - n_0 \mid \mathbf{y} \in \mathbf{s}\} \cap N_{\mathbf{s}} = \{\mathbf{0}, \, n_1 - n_0, \ldots, \, n_k - n_0\}.$$

(ii) The relative volume of an elementary lattice 1-simplex (resp. of an elementary lattice 2-simplex) is always equal to 1 (resp. equal to 1/2). □

Example 6.3. The lattice r-simplex

$$\mathbf{s} = \operatorname{conv}(\{\mathbf{0}, e_1, e_2, \ldots, e_{r-2}, e_{r-1}, (1,1,\ldots,1,1,2)^{\intercal}\}) \subset \mathbb{R}^r, \; r \geq 3,$$

(w.r.t. $\mathbb{Z}^r$) serves as example of an elementary but non-basic simplex because $\mathbf{s} \cap \mathbb{Z}^r = \operatorname{vert}(\mathbf{s})$ and

$$r!\operatorname{Vol}(\mathbf{s}; \mathbb{Z}^r) = |\det(e_1, \ldots, e_{r-1}, (1,1,\ldots,1,1,2)^{\intercal})| = 2 \neq 1 \; .$$

Definition 6.4. A triangulation $\mathcal{T}$ of a lattice polytope $P \subset N_{\mathbb{R}} \cong \mathbb{R}^r$ (w.r.t. N) is called *lattice triangulation* if $\operatorname{vert}(P) \subset \operatorname{vert}(\mathcal{T}) \subset N$. The set of all lattice triangulations of a lattice polytope P (w.r.t. N) will be denoted by $\mathbf{LTR}_N(P)$.

Definition 6.5. A lattice triangulation $\mathcal{T}$ of $P \subset N_{\mathbb{R}} \cong \mathbb{R}^r$ (w.r.t. N) is called *maximal triangulation* if $\operatorname{vert}(\mathcal{T}) = N \cap P$. A lattice triangulation $\mathcal{T}$ of P is obviously maximal if and only if each simplex $\mathbf{s}$ of $\mathcal{T}$ is elementary. A lattice triangulation $\mathcal{T}$ of P is said to be *basic* if $\mathcal{T}$ consists of exclusively basic simplices. We define :

$$\mathbf{LTR}_N^{\max}(P) := \{\mathcal{T} \in \mathbf{LTR}_N(P) \mid \mathcal{T} \text{ is a maximal triangulation of } P\} \; ,$$

$$\mathbf{LTR}_N^{\text{basic}}(P) := \{\mathcal{T} \in \mathbf{LTR}_N^{\max}(P) \mid \mathcal{T} \text{ is a basic triangulation of } P\} \; .$$

(Moreover, adding the prefix **Coh-** to anyone of the above sets, we shall mean the subsets of their elements which are coherent). The *hierarchy of lattice triangulations* of a P (as above) is given by the following inclusion-diagram:

$$\boxed{\begin{array}{ccccc} \mathbf{LTR}_N^{\text{basic}}(P) & \subset & \mathbf{LTR}_N^{\max}(P) & \subset & \mathbf{LTR}_N(P) \\ \bigcup & & \bigcup & & \bigcup \\ \textbf{Coh-}\mathbf{LTR}_N^{\text{basic}}(P) & \subset & \textbf{Coh-}\mathbf{LTR}_N^{\max}(P) & \subset & \textbf{Coh-}\mathbf{LTR}_N(P) \end{array}}$$

Proposition 6.6. *For any lattice polytope $P \subset N_{\mathbb{R}} \cong \mathbb{R}^r$ (w.r.t. N) the set of maximal coherent triangulations* $\textbf{Coh-}\mathbf{LTR}_N^{\max}(P) \subset \textbf{Coh-}\mathbf{LTR}_N(P)$ *of P is non-empty.*

Proof. Consider a s.c.p. cone supported by P in $\mathbb{R}^{r+1}$, and then make use of [55], cor. 3.8, p. 394. □

Remark 6.7. (i) Already in dimension $r = 2$, there are lots of examples of P's with

$$\mathbf{LTR}_N^{\text{basic}}(P) \smallsetminus \textbf{Coh-}\mathbf{LTR}_N^{\text{basic}}(P) \neq \varnothing \; .$$

See, for instance, 7.5 below.
(ii) In addition, for lattice polytopes P, $\mathbf{LTR}_N^{\text{basic}}(P) \neq \varnothing$ does not imply necessarily

$$\textbf{Coh-}\mathbf{LTR}_N^{\text{basic}}(P) \neq \varnothing \; .$$

As it was proved recently by Hibi and Ohsugi [29], there exists a 9-dimensional 0/1-polytope (with 15 vertices) possessing basic triangulations but no coherent *and* basic ones.

(d) To pass from triangulations to desingularizations we need to introduce some extra notation.

Definition 6.8. Let $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ be an r-dimensional abelian Gorenstein quotient singularity ($r \geq 2$), and $\mathfrak{s}_G$ the $(r-1)$-dimensional junior simplex. For any simplex $\mathbf{s}$ of a lattice triangulation $\mathcal{T}$ of $\mathfrak{s}_G$ let $\sigma_{\mathbf{s}}$ denote the s.c.p. cone

$$\sigma_{\mathbf{s}} := \{\lambda \mathbf{y} \in (N_G)_{\mathbb{R}} \mid \lambda \in \mathbb{R}_{\geq 0}\ ,\ \mathbf{y} \in \mathbf{s}\} \quad (= \mathrm{pos}(\mathbf{s}) \text{ within } (N_G)_{\mathbb{R}})$$

supporting $\mathbf{s}$. We define the fan

$$\widehat{\Delta_G}(\mathcal{T}) := \{\sigma_{\mathbf{s}} \mid \mathbf{s} \in \mathcal{T}\}$$

of s.c.p. cones in $(N_G)_{\mathbb{R}} \cong \mathbb{R}^r$, and

$$\mathbf{PCDES}(X(N_G, \Delta_G)) := \left\{ \begin{array}{c} \textit{partial crepant } T_{N_G}\text{-equivariant} \\ \text{desingularizations of } X(N_G, \Delta_G) \\ \text{with overlying spaces having} \\ \text{at most (}\mathbb{Q}\text{-factorial) } \textit{canonical} \\ \text{singularities (of index 1)} \end{array} \right\},$$

$$\mathbf{PCDES}^{\max}(X(N_G, \Delta_G)) := \left\{ \begin{array}{c} \textit{partial crepant } T_{N_G}\text{-equivariant} \\ \text{desingularizations of } X(N_G, \Delta_G) \\ \text{with overlying spaces having} \\ \text{at most (}\mathbb{Q}\text{-factorial) } \textit{terminal} \\ \text{singularities (of index 1)} \end{array} \right\},$$

$$\mathbf{CDES}(X(N_G, \Delta_G)) := \left\{ \begin{array}{c} \textit{crepant } T_{N_G}\text{-equivariant (full)} \\ \text{desingularizations of } X(N_G, \Delta_G) \end{array} \right\}.$$

(Whenever we put the prefix **QP**- in the front of anyone of them, we shall mean the corresponding subsets of them consisting of those desingularizations whose overlying spaces are *quasiprojective*.)

Theorem 6.9 (Desingularizing by triangulations).
Let $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ be an r-dimensional abelian Gorenstein quotient singularity ($r \geq 2$). Then there exist one-to-one correspondences :

$$\boxed{\begin{array}{ccc} \mathbf{LTR}^{\mathrm{basic}}_{N_G}(\mathfrak{s}_G) & \overset{1:1}{\longleftrightarrow} & \mathbf{CDES}(X(N_G, \Delta_G)) \\ \cap & & \cap \\ \mathbf{LTR}^{\max}_{N_G}(\mathfrak{s}_G) & \overset{1:1}{\longleftrightarrow} & \mathbf{PCDES}^{\max}(X(N_G, \Delta_G)) \\ \cap & & \cap \\ \mathbf{LTR}_{N_G}(\mathfrak{s}_G) & \overset{1:1}{\longleftrightarrow} & \mathbf{PCDES}(X(N_G, \Delta_G)) \end{array}}$$

as well as

$$\boxed{\begin{array}{ccc} \mathbf{Coh\text{-}LTR}^{\text{basic}}_{N_G}(\mathfrak{s}_G) & \overset{1:1}{\longleftrightarrow} & \mathbf{QP\text{-}CDES}\,(X\,(N_G,\Delta_G)) \\ \bigcap & & \bigcap \\ \mathbf{Coh\text{-}LTR}^{\max}_{N_G}(\mathfrak{s}_G) & \overset{1:1}{\longleftrightarrow} & \mathbf{QP\text{-}PCDES}^{\max}\,(X\,(N_G,\Delta_G)) \\ \bigcap & & \bigcap \\ \mathbf{Coh\text{-}LTR}_{N_G}(\mathfrak{s}_G) & \overset{1:1}{\longleftrightarrow} & \mathbf{QP\text{-}PCDES}\,(X\,(N_G,\Delta_G)) \end{array}}$$

which are realized by crepant T_{N_G}-equivariant birational morphisms of the form

$$\boxed{f_{\mathcal{T}} = \mathrm{id}_* : X\left(N_G,\ \widehat{\Delta_G}\,(\mathcal{T})\right) \longrightarrow X\,(N_G,\Delta_G)} \tag{6.1}$$

induced by mapping

$$\mathcal{T} \longmapsto \widehat{\Delta_G}\,(\mathcal{T})\,, \qquad \widehat{\Delta_G}\,(\mathcal{T}) \longmapsto X\left(N_G,\ \widehat{\Delta_G}\,(\mathcal{T})\right)\ .$$

Sketch of proof. $X\,(N_G,\Delta_G)$ is Gorenstein and has at most rational singularities, i.e. canonical singularities of index 1. Moreover, its dualizing sheaf is trivial. Let

$$f = \mathrm{id}_* : X\left(N_G,\ \widetilde{\Delta_G}\right) \longrightarrow X\,(N_G,\Delta_G)$$

denote an arbitrary partial desingularization. Studying either the behaviour of the highest rational differentials on $X\left(N_G,\ \widetilde{\Delta_G}\right)$ (see [59], § 3, [61], § 4.8, [47], prop. 3, or [11], prop. 4.1), or the support function associated to $K_{X\left(N_G,\ \widetilde{\Delta_G}\right)}$ (cf. [65], § 2), one proves

$$K_{X\left(N_G,\ \widetilde{\Delta_G}\right)} - f^*\left(K_{X(N_G,\Delta_G)}\right) = \sum_{\varrho\in\widetilde{\Delta_G}(1)} \left(\langle(1,\ldots,1)\,,n\,(\varrho)\rangle - 1\right)\ D_{n(\varrho)}\ ,$$

where $D_{n(\varrho)} := V\,(\varrho) = V\,(\mathrm{pos}\,(\{n\,(\varrho)\}))$. Obviously, f is crepant if and only if

$$\mathrm{Gen}\left(\widetilde{\Delta_G}\right) \subset \left\{ \mathbf{y} = (y_1,\ldots,y_r)^{\intercal} \in (N_G)_{\mathbb{R}} \;\middle|\; \sum_{i=1}^{r} y_i = 1 \right\}\ ,$$

and since the number of crepant exceptional prime divisors is independent of the specific choice of f, the first and second 1-1 correspondences (from below) of the first box are obvious by the adjunction-theoretic definition of terminal (resp.

canonical) singularities. In particular, all T_{N_G}-equivariant partial crepant desingularizations of $X(N_G, \Delta_G)$ of the form (6.1) have overlying spaces with at most $\mathbb{Q}$-factorial singularities, and conversely, each partial T_{N_G}-equivariant crepant desingularization with overlying space admitting at most $\mathbb{Q}$-factorial singularities, has to be of this form. ($\mathbb{Q}$-factoriality is here equivalent to the consideration only of triangulations instead of more general polyhedral subdivisions. Furthermore, by maximal triangulations you exhaust all the crepant exceptional prime divisors). The top 1-1 correspondence of the first box follows from the equivalence

$$\mathcal{T} \ni \mathbf{s} \text{ is a basic simplex} \Longleftrightarrow \left\{ \begin{array}{l} \text{mult}(\sigma_{\mathbf{s}}; N_G) = 1 \;\text{ for the cone} \\ \\ \sigma_{\mathbf{s}} \in \widehat{\Delta_G}(\mathcal{T}) \;\text{ supporting it} \end{array} \right\}.$$

It remains to prove the 1-1 correspondences of the second box. As it was explained in [11], § 4, for every $\psi \in \text{SUCSF}_{\mathbb{Q}}\left(N_G,\ \widehat{\Delta_G}(\mathcal{T})\right)$, the restriction $\psi\,|_{\mathcal{T}}$ belongs to $\text{SUCSF}_{\mathbb{R}}(\mathcal{T})$, and conversely, to any support function $\psi \in \text{SUCSF}_{\mathbb{R}}(\mathcal{T})$, one may canonically assign a rational (or even an integral) strictly upper convex support function defined on the entire $\left|\widehat{\Delta_G}(\mathcal{T})\right|$. To finish the proof of the theorem we apply corollary 2.13. $\square$

Remark 6.10. (i) *Flops.* Every pair of triangulations $\mathcal{T}_1, \mathcal{T}_2 \in \textbf{Coh-LTR}^{\max}_{N_G}(\mathfrak{s}_G)$ gives rise to the determination of a birational morphism

$$X\left(N_G,\ \widehat{\Delta_G}(\mathcal{T}_1)\right) \longrightarrow X\left(N_G,\ \widehat{\Delta_G}(\mathcal{T}_2)\right)$$

which is composed of a finite number of flops (cf. [55], § 3). Since all triangulations of $\textbf{Coh-LTR}_{N_G}(\mathfrak{s}_G)$ are parametrized by the vertices of the so-called *secondary polytope* of $\mathfrak{s}_G$, this transition-map is induced by performing successively *bistellar operations*, i.e., by passing from the vertex of the secondary polytope of $\mathfrak{s}_G$ representing $\mathcal{T}_1$ to that one representing $\mathcal{T}_2$ following a (not necessarily uniquely determined) path which connects these two vertices. (In dimension 3 this is nothing but Danilov's theorem [13], cf. 7.4 (vi) below). For detailed presentations of the theory of secondary polytopes we refer to the articles of Billera-Filliman-Sturmfels [6] and Oda-Park [55], and to the treatment of Gelfand-Kapranov-Zelevinsky [24], ch. 7.

(ii) *Factorization.* The birational morphisms corresponding to members of **QP-PCDES** $(X(N_G, \Delta_G))$ can be decomposed into more elementary toric contractions by Reid's toric version of "MMP" ([60], (0.2)-(0.3)). In several cases these contractions can be directly interpreted as inversed (normalized) blow-ups. For concrete examples see rem. 7.2 and § 9 below.

(iii) *Exceptional divisors.* Each irreducible component of an exceptional divisor w.r.t. an $f = f_{\mathcal{T}}$ (as above) is a $\mathbb{Q}$-Cartier prime divisor which carries itself the structure of an $(r-1)$-dimensional toric variety determined by the corresponding star within $\widehat{\Delta_G}(\mathcal{T})$ (cf. above § 2 **(f)**, (ii)). An exceptional prime divisor is compact if and only if the lattice point representing it belongs to $\mathrm{int}(\sigma_0)$ (see thm. 2.6).

Theorem 6.11 (Number-theoretic version of McKay correspondence). *Let $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ be an abelian Gorenstein quotient singularity of dimension $r \geq 2$,*

$$f = \mathrm{id}_* : X\left(N_G, \widehat{\Delta_G}\right) \to X(N_G, \Delta_G)$$

a T_{N_G}-equivariant crepant, full resolution and $\mathbf{F} := f^{-1}([\mathbf{0}])$ the central fiber over the origin $[\mathbf{0}] = \mathrm{orb}(\sigma_0)$:

$$\mathbf{F} = \bigcup_{\varrho \in \widehat{\Delta_G}(1)} \left\{ D_{n(\varrho)} \mid n(\varrho) \in \mathrm{int}(\sigma_0) \cap N_G \right\} \cup$$

$$\cup \left(\bigcup \left\{ V(\sigma) \;\middle|\; \sigma \in \bigcup_{i=2}^{r-1} \widehat{\Delta_G}(i) \;,\; \mathrm{int}(\sigma) \subset \mathrm{int}(\sigma_0) \right\} \right) .$$

Then $\mathbf{F}$ is a strong deformation retract of $X\left(N_G, \widehat{\Delta_G}\right)$, and only the even cohomology groups of $\mathbf{F}$ are non-trivial. Their dimensions (over $\mathbb{Q}$) are given by the formulae :

$$\boxed{\dim_{\mathbb{Q}} H^{2i}(\mathbf{F};\mathbb{Q}) = \begin{cases} 1 & , \text{ if } i = 0 \\ \#\left((i\,\mathfrak{s}_G) \cap \mathbf{Par}(\sigma_0) \cap N_G\right) & , \text{ if } 1 \leq i \leq r-1 \\ 0 & , \text{ otherwise} \end{cases}} \tag{6.2}$$

In particular, the topological Euler-Poincaré characteristic of $X\left(N_G, \widehat{\Delta_G}\right)$ equals:

$$\boxed{\chi\left(X\left(N_G, \widehat{\Delta_G}\right)\right) = \chi(\mathbf{F}) = l = |G|} \tag{6.3}$$

Obviously, the numbers (6.2), (6.3) are independent *of the choice of triangulations $\mathcal{T}$ of $\mathfrak{s}_G$ by means of which we construct $\widehat{\Delta_G}$ $\left(= \widehat{\Delta_G}(\mathcal{T})\right)$* (cf. (6.1)).

Proof. See Batyrev-Dais [3], thm. 5.4, p. 910, and [12] for further comments. □

Corollary 6.12. *If $(\mathbb{C}^r/G, [\mathbf{0}]) = (X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ is a Gorenstein cyclic quotient singularity of type $\frac{1}{l}(\alpha_1, \ldots, \alpha_r)$, then, maintaining the above notation and assumption, we obtain*

$$\boxed{\dim_{\mathbb{Q}} H^{2i}(\mathbf{F};\mathbb{Q}) = \#\left\{ \lambda \in [0, l) \cap \mathbb{Z} \mid \sum_{j=1}^{r} [\lambda\,\alpha_j]_l = i \cdot l \right\}} \tag{6.4}$$

Remark 6.13. The numbers of the right hand side of (6.2) and (6.4) make sence, even without assuming the existence of a T_{N_G}-equivariant crepant, full desingularization of $X(N_G, \Delta_G)$, and were used in [3] as the "correctional terms" for introducing the formal definition of the so-called *string-theoretic Hodge numbers* of Calabi-Yau varieties (or, more general, of Gorenstein compact complex varieties) which have at most abelian quotient singularities.

(e) By theorem 6.9 it is now clear that Reid's question (formulated in §1), restricted to the category of torus-equivariant desingularizations of $X(N_G, \Delta_G)$, can be restated as follows :

Question : For a Gorenstein abelian quotient singularity

$$(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$$

with junior simplex $\mathfrak{s}_G$, what kind of conditions on the acting group G would guarantee the existence of a basic, coherent triangulation of $\mathfrak{s}_G$?

Though (as already mentioned in §1) this question will be treated explicitly in [12], we at least shall explain here how a very simple necessary existence-criterion works, and apply it efficiently in our special singularity-series in § 8.

Lemma 6.14. *Let $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ be a Gorenstein abelian quotient singularity and $\mathfrak{s}_G$ the junior simplex. If $\mathfrak{s}_G$ admits a basic triangulation $\mathcal{T}$, then for all $n \in (N_G \smallsetminus \{\mathbf{0}\}) \cap \sigma_0$ there exist r lattice points $n_1, \ldots, n_r$ belonging to $\mathfrak{s}_G \cap N_G$ so that*

$$n \in \mathbb{Z}_{\geq 0}\, n_1 + \cdots + \mathbb{Z}_{\geq 0}\, n_r$$

Proof. Since $\mathcal{T}$ is a basic triangulation inducing a subdivision of σ_0 into cones of multiplicity 1, n belongs to a subcone σ' of σ_0 of the form

$$\sigma' = \mathbb{R}_{\geq 0}\ n_1 + \cdots + \mathbb{R}_{\geq 0}\ n_r, \qquad \mathrm{mult}(\sigma'; N_G) = 1,$$

and can be therefore written as a linear combination

$$n = \mu_1 n_1 + \cdots + \mu_r n_r, \qquad (\mu_i \in \mathbb{R}_{\geq 0}\,,\ \forall i,\ 1 \leq i \leq r)$$

Now $\{n_1, \ldots, n_r\}$ is a $\mathbb{Z}$-basis of N_G, which means that n can be also written as

$$n = \mu'_1 n_1 + \cdots + \mu'_r n_r, \qquad (\mu'_i \in \mathbb{Z},\ \forall i,\ 1 \leq i \leq r)$$

By linear independence we get $\mu_i = \mu'_i \in \mathbb{Z} \cap \mathbb{R}_{\geq 0} = \mathbb{Z}_{\geq 0}$, $\forall i,\ 1 \leq i \leq r$. □

Theorem 6.15 (Necessary Existence-Criterion).
Let $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ be a Gorenstein abelian quotient singularity. If $\mathfrak{s}_G$ admits a basic triangulation $\mathcal{T}$, then

$$\boxed{\mathbf{Hlb}_{N_G}(\sigma_0) = \mathfrak{s}_G \cap N_G} \qquad (6.5)$$

i.e. all members of the Hilbert basis of σ_0 have to be either "junior" elements or to belong to $\{e_1, \ldots, e_r\}$.

Proof. The inclusion "$\supset$" is always true (without any further assumption about the existence or non-existence of such a triangulation) and is obvious by the definition of Hilbert basis. Now if there were an element $n \in \mathbf{Hlb}_{N_G}(\sigma_0) \smallsetminus (\mathfrak{s}_G \cap N_G)$, then by lemma 6.14 this could be written as a non-negative integer linear combination

$$n = \mu_1 n_1 + \cdots + \mu_r n_r$$

of r elements of $\mathfrak{s}_G \cap N_G$. Since $\mathbf{0} \notin \mathbf{Hlb}_{N_G}(\sigma_0)$, there were at least one index $j = j_\bullet \in \{1, \ldots, r\}$, for which $\mu_{j_\bullet} \neq 0$. If $\mu_{j_\bullet} = 1$ and $\mu_j = 0$ for all j, $j \in \{1, \ldots, r\} \smallsetminus \{j_\bullet\}$, then $n = n_{j_\bullet} \in \mathfrak{s}_G \cap N_G$ which would contradict our assumption. But even the cases in which either $\mu_{j_\bullet} = 1$ and some other μ_j's were $\neq 0$, or $\mu_{j_\bullet} \geq 2$, would be exluded as impossible because of the characterization (2.1) of the Hilbert basis $\mathbf{Hlb}_{N_G}(\sigma_0)$ as the set of additively irreducible vectors of $\sigma_0 \cap (N_G \smallsetminus \{\mathbf{0}\})$. Hence, $\mathbf{Hlb}_{N_G}(\sigma_0) \subset \mathfrak{s}_G \cap N_G$. $\square$

Remark 6.16. (i) For a long time it was expected that condition (6.5) might be sufficient too for the existence of a basic triangulation of $\mathfrak{s}_G$, and, as we shall see in [12], this *is* the case for some special choices of cyclic group actions on $\mathbb{C}^r$. Nevertheless, Firla and Ziegler ([19], §4.2 & [20]) discovered recently the first *counterexamples.* Among them, the counterexample of the 4-dimensional Gorenstein cyclic quotient singularity with the smallest possible acting group-order, fulfilling property (6.5) and admitting no crepant, torus-equivariant resolutions, is that of type $\frac{1}{39}(1, 5, 8, 25)$.
(ii) To apply necessary criterion 6.15 in practice, in order to exclude "candidates" for having crepant, torus-equivariant resolutions, one has first to determine all the elements of the Hilbert basis $\mathbf{Hlb}_{N_G}(\sigma_0)$ and then to test if at least one of them breaks away from the junior simplex or not. From the point of view of complexity theory of algorithms, however, this procedure is in general "NP-hard" (cf. Henk-Weismantel [28], § 3).

Exercise 6.17. For the singularity of type $\frac{1}{9}(1, 2, 3, 3)$ determine explicitly the Hilbert basis $\mathbf{Hlb}_{N_G}(\sigma_0)$ and show that it does not possess any crepant torus-equivariant resolution because condition (6.5) is violated. [Hint. Verify that $\frac{1}{9}(5, 1, 6, 6)^{\intercal} \in \mathbf{Hlb}_{N_G}(\sigma_0)$].

7. Peculiarities of dimensions 2 and 3

In the low dimensions $r \in \{2, 3\}$, we have always

$$\mathbf{CDES}(X(N_G, \Delta_G)) = \mathbf{PCDES}^{\max}(X(N_G, \Delta_G))$$

by lemma 6.2 (ii) and thm. 6.9. This is exactly the lemma which fails in general if $r \geq 4$, and makes the high dimensions so exciting (cf. 6.3). However, low dimensional Gorenstein abelian quotient singularities are still valuable as testing ground for lots of interesting, related problems and pave the way to systematic generalizations.

(a) In dimension 2 we meet only the "classical" A_{l-1}-singularities (i.e. cyclic quotient singularities of type $1/l\,(1, l-1)$).

Lemma 7.1. *All* 2*-dimensional Gorenstein abelian quotient singularities*

$$(X\,(N_G, \Delta_G)\,, \mathrm{orb}\,(\sigma_0))$$

are cyclic of type $\frac{1}{l}\,(1, l-1)$ (with $l \geq 2$). *They admit a unique projective crepant* (= *minimal*) *resolution* (6.1) *induced by the triangulation*

$$\mathcal{T} = \left\{ \mathrm{conv}\left(\left\{ \frac{1}{l}\,(j-1, l-(j-1))^{\intercal}\,, \frac{1}{l}\,(j, l-j)^{\intercal} \right\} \right) \mid 1 \leq j \leq l \right\}$$

and the $l-1$ *exceptional prime divisors*

$$D_j := V\left(\mathrm{pos}\left(\left\{ \frac{1}{l}\,(j, l-j)^{\intercal} \right\} \right) \right), \quad 1 \leq j \leq l-1,$$

are smooth rational curves having intersection numbers

$$(D_i \cdot D_j) = \begin{cases} \;\;1 & , \text{ if } \;\; |i-j| = 1 \\ -2 & , \text{ if } \qquad\;\; i = j \\ \;\;0 & , \text{ if } \;\; |i-j| \geq 2 \end{cases}$$

for all i, j, $1 \leq i \leq j \leq l-1$.

Remark 7.2. Obviously, the subdivision of σ_0 into l cones of multiplicity 1 induced by the above $\mathcal{T}$ can be done successively in k "steps", with $1 \leq k \leq l-1$, i.e., by drawing in k steps the $l-1$ required rays in any order you would wish. (The possibility of drawing more than one rays in a step is not excluded). Such a procedure gives rise to decomposing the full desingularization into a series of partial ones. It should nevertheless be stressed that starting-points of different series of choices correspond to blow-ups of *different* ideal sheaves $\mathcal{I}$ with $\mathrm{orb}(\sigma_0) = \mathrm{supp}\big(\mathcal{O}_{U_{\sigma_0}} / \mathcal{I}\big)$. (Note that in dimension 2 we do not need extra normalizations). Consequently, there are lots of factorizations of the crepant resolution morphism $f = f_{\mathcal{T}}$. Let us illustrate it by considering the example of the A_4-singularity $\frac{1}{5}\,(1, 4)$. The morphism f admits of two different natural factorizations

$$f = g_1 \circ g_2 = h_1 \circ h_2 \circ h_3 \circ h_4$$

which are depicted in figures **4** and **5**, respectively.
(i) Since

$$\mathbf{Hlb}_{M_G}\,(\sigma_0^{\vee}) = \{m_1 = (5, 0)\,,\; m_2 = (1, 1)\,,\; m_3 = (0, 5)\}\;,$$

using the induced embedding $\iota : U_{\sigma_0} \hookrightarrow \mathbb{C}^3$ of $U_{\sigma_0} = X\left(N_G, \Delta_G\right)$,

$$U_{\sigma_0} = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 \;\middle|\; z_2^5 - z_1\, z_3 = 0 \right\}, \quad z_i = \mathbf{e}\left(m_i\right), \; \forall i, \; 1 \leq i \leq 3,$$

and proposition 3.8, we obtain g_1 and g_2 coming from blowing up only maximal 0-dimensional ideals:

$$X\left(N_G, \; \widehat{\Delta_G}\left(\mathcal{T}\right)\right) = \mathbf{Bl}^{\mathrm{red}}_{\mathrm{orb}(\sigma_2)}\left(\mathbf{Bl}^{\mathrm{red}}_{\mathrm{orb}(\sigma_0)}\left(U_{\sigma_0}\right)\right) \xrightarrow{g_2} \mathbf{Bl}^{\mathrm{red}}_{\mathrm{orb}(\sigma_0)}\left(U_{\sigma_0}\right) \xrightarrow{g_1} U_{\sigma_0}\,.$$

More precisely, we first subdivide σ_0 into the three subcones σ_1, σ_2, σ_3, with

$$\sigma_1 = \left\{ \mathbf{y} \in \sigma_0 \;\middle|\; \langle m_2 - m_1, \mathbf{y}\rangle \geq 0 \;\&\; \langle m_3 - m_1, \mathbf{y}\rangle \geq 0 \right\} = \mathrm{pos}\left(\left\{\left(\tfrac{1}{5}, \tfrac{4}{5}\right)^{\mathsf{T}}, e_2\right\}\right),$$

$$\sigma_2 = \left\{ \mathbf{y} \in \sigma_0 \;\middle|\; \langle m_1 - m_2, \mathbf{y}\rangle \geq 0 \;\&\; \langle m_3 - m_2, \mathbf{y}\rangle \geq 0 \right\} = \mathrm{pos}\left(\left(\tfrac{4}{5}, \tfrac{1}{5}\right)^{\mathsf{T}}, \left(\tfrac{1}{5}, \tfrac{4}{5}\right)^{\mathsf{T}}\right),$$

$$\sigma_3 = \left\{ \mathbf{y} \in \sigma_0 \;\middle|\; \langle m_1 - m_3, \mathbf{y}\rangle \geq 0, \langle m_2 - m_3, \mathbf{y}\rangle \geq 0 \right\} = \mathrm{pos}\left(\left\{e_1, \left(\tfrac{4}{5}, \tfrac{1}{5}\right)^{\mathsf{T}}\right\}\right).$$

After that we perform again a usual blow-up, but now for the cone σ_2 (instead of σ_0) which contains the remaining two lattice points.

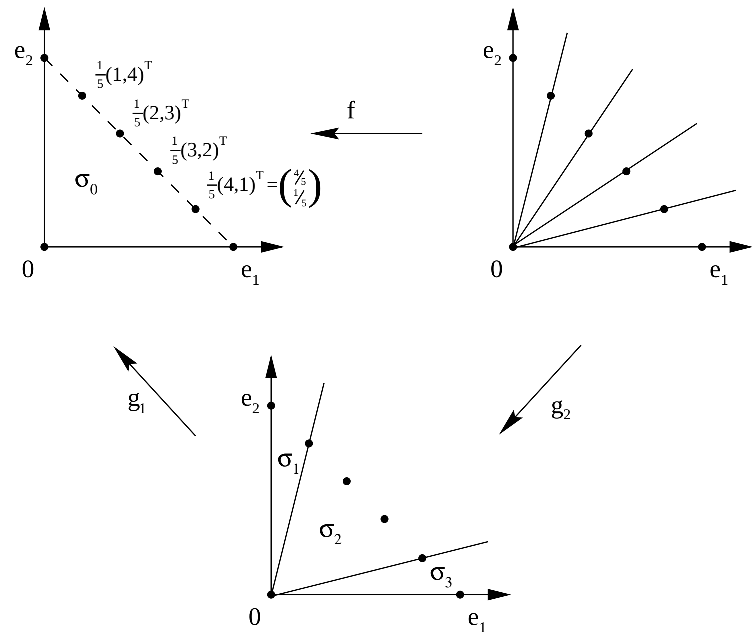


Fig. 4

(ii) In the second factorization, h_1 gives rise to a "starring" subdivision of σ_0 into only two cones

$$\sigma_1 = \mathrm{pos}\left(\left\{\left(\frac{4}{5},\frac{1}{5}\right)^{\intercal}, e_2\right\}\right) \quad \text{and} \quad \sigma_2 = \mathrm{pos}\left(\left\{e_1, \left(\frac{4}{5},\frac{1}{5}\right)^{\intercal}\right\}\right).$$

In fact,

$$h_1 : \mathbf{Bl}^{\mathcal{I}}_{\mathrm{orb}(\sigma_0)}(U_{\sigma_0}) \longrightarrow U_{\sigma_0}$$

is a *directed* blow-up w.r.t. $(4/5, 1/5)^{\intercal}$. As $\mathcal{O}_{U_{\sigma_0}}$-ideal sheaf $\mathcal{I}$ can be taken the pullback (via ι) of the ideal sheaf $\mathcal{J} \cong J^{\sim}$ supporting the z_3-axis of $\mathbb{C}^3$, with $J := (z_1, z_2, z_1 z_3)$ (in $\mathbb{C}[z_1, z_2, z_3]$), because

$$\sigma_1 = \{\mathbf{y} \in \sigma_0 \mid \langle m_2 - m_1, \mathbf{y}\rangle \geq 0 \quad \text{and} \quad \langle m_3, \mathbf{y}\rangle \geq 0\},$$

$$\sigma_2 = \{\mathbf{y} \in \sigma_0 \mid \langle m_1 - m_2, \mathbf{y}\rangle \geq 0 \quad \text{and} \quad \langle m_1 + m_3 - m_2, \mathbf{y}\rangle \geq 0\}.$$

• Another characterization of $\mathbf{Bl}^{\mathcal{I}}_{\mathrm{orb}(\sigma_0)}(U_{\sigma_0})$ avoiding the embedding is provided by the following commutative diagram due to the extension of the group action on $\mathbf{Bl}^{\mathrm{red}}_{\mathbf{0}}(\mathbb{C}^2)$:

$$\begin{array}{lcl} \mathbf{Bl}^{\mathrm{red}}_{\mathbf{0}}(\mathbb{C}^2) & \twoheadrightarrow & \mathbf{Bl}^{\mathcal{I}}_{\mathrm{orb}(\sigma_0)}(U_{\sigma_0}) \cong \mathbf{Bl}^{\mathrm{red}}_{\mathbf{0}}(\mathbb{C}^2)/G \\ \downarrow & & \downarrow \\ \mathbb{C}^2 & \twoheadrightarrow & U_{\sigma_0} = \mathbb{C}^2/G \end{array}$$

Clearly, our blown up space is isomorphic to the quotient of the blown up $\mathbb{C}^2$ at the origin divided by G. For h_2 we proceed analogously by letting σ_2 play the role of σ_0. Obviously, the same is valid for h_3 and h_4. (Comparing the above two factorizations, we see that the first one is "double speedy".)

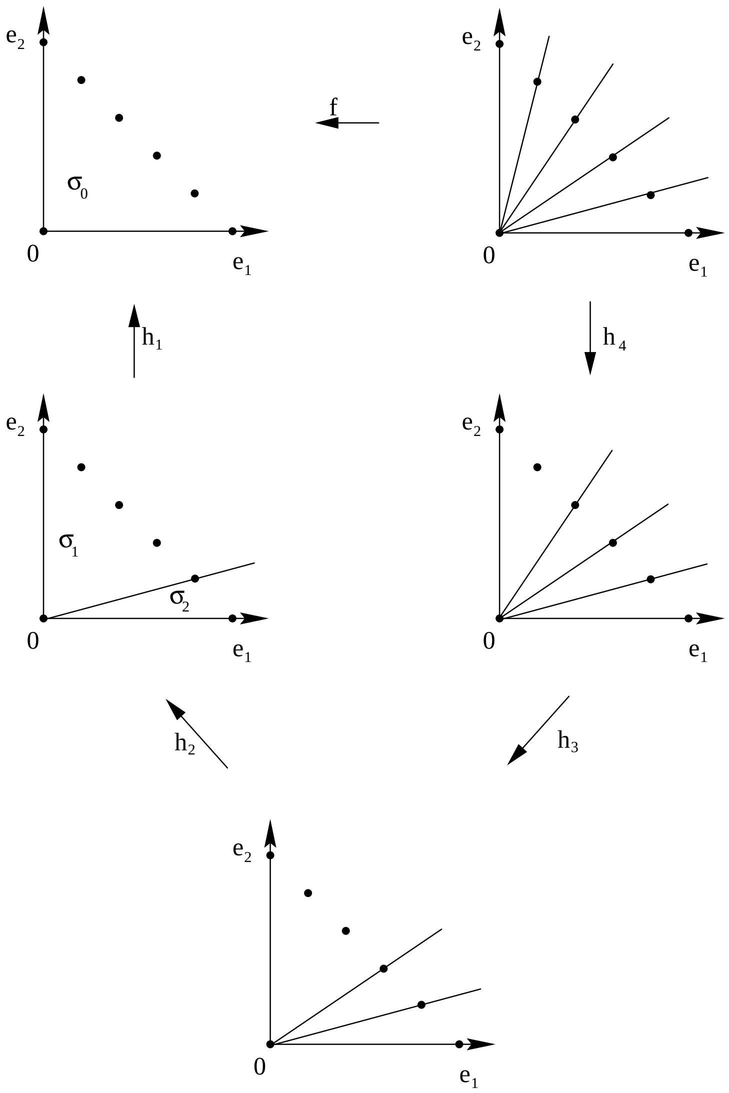


Fig. 5

(iii) Finally, let us point out that also $f = f_{\mathcal{T}}$ itself can be similarly regarded as a blow-up morphism

$$f : \mathbf{Bl}^{\mathcal{I}}_{\mathrm{orb}(\sigma_0)}(U_{\sigma_0}) \longrightarrow U_{\sigma_0} \ .$$

As $\mathcal{I}$ one may choose the pullback of $\mathcal{J} \cong J^{\sim}$ with J the ideal

$$J := \left(z_1^4, z_1^3\, z_2, z_1^2\, z_2^3, z_1^2\, z_2\, z_3, z_1^2\, z_3^2\right) \quad \text{in} \quad \mathbb{C}[z_1, z_2, z_3]\,,$$

because

$$\left\{ \mathbf{y} \in \sigma_0 \;\middle|\; \begin{array}{l} \langle m_2 - m_1, \mathbf{y}\rangle \geq 0, \\ \langle 3\, m_2 - 2\, m_1, \mathbf{y}\rangle \geq 0, \\ \langle m_2 + m_3 - 2\, m_1, \mathbf{y}\rangle \geq 0, \\ \langle 2\, m_3 - 2\, m_1, \mathbf{y}\rangle \geq 0 \end{array} \right\} = \mathrm{pos}\left(\left\{ e_2,\ \left(\tfrac{1}{5}, \tfrac{4}{5}\right)^{\intercal} \right\}\right),$$

$$\left\{ \mathbf{y} \in \sigma_0 \;\middle|\; \begin{array}{l} \langle m_1 - m_2, \mathbf{y}\rangle \geq 0, \\ \langle 2\, m_2 - m_1, \mathbf{y}\rangle \geq 0, \\ \langle m_3 - m_1, \mathbf{y}\rangle \geq 0, \\ \langle 2\, m_3 - m_1 - m_2, \mathbf{y}\rangle \geq 0 \end{array} \right\} = \mathrm{pos}\left(\left\{ \left(\tfrac{1}{5}, \tfrac{4}{5}\right)^{\intercal}, \left(\tfrac{2}{5}, \tfrac{3}{5}\right)^{\intercal} \right\}\right),$$

and

$$\left\{ \mathbf{y} \in \sigma_0 \;\middle|\; \begin{array}{l} \langle 2\, m_1 - 3\, m_2, \mathbf{y}\rangle \geq 0, \\ \langle m_1 - 2\, m_2, \mathbf{y}\rangle \geq 0, \\ \langle m_3 - 2\, m_2, \mathbf{y}\rangle \geq 0, \\ \langle 2\, m_3 - 3\, m_2, \mathbf{y}\rangle \geq 0 \end{array} \right\} = \mathrm{pos}\left(\left\{ \left(\tfrac{2}{5}, \tfrac{3}{5}\right)^{\intercal}, \left(\tfrac{3}{5}, \tfrac{2}{5}\right)^{\intercal} \right\}\right),$$

$$\left\{ \mathbf{y} \in \sigma_0 \;\middle|\; \begin{array}{l} \langle 2\, m_1 - m_2 - m_3, \mathbf{y}\rangle \geq 0 \\ \langle m_1 - m_3, \mathbf{y}\rangle \geq 0, \\ \langle 2\, m_2 - m_3, \mathbf{y}\rangle \geq 0, \\ \langle m_3 - m_2, \mathbf{y}\rangle \geq 0 \end{array} \right\} = \mathrm{pos}\left(\left\{ \left(\tfrac{3}{5}, \tfrac{2}{5}\right)^{\intercal}, \left(\tfrac{4}{5}, \tfrac{1}{5}\right)^{\intercal} \right\}\right),$$

and finally

$$\left\{ \mathbf{y} \in \sigma_0 \;\middle|\; \begin{array}{l} \langle 2\, m_1 - 2\, m_3, \mathbf{y}\rangle \geq 0, \\ \langle m_1 + m_2 - 2\, m_3, \mathbf{y}\rangle \geq 0, \\ \langle 3\, m_2 - 2\, m_3, \mathbf{y}\rangle \geq 0, \\ \langle m_2 - m_3, \mathbf{y}\rangle \geq 0 \end{array} \right\} = \mathrm{pos}\left(\left\{ \left(\frac{4}{5}, \frac{1}{5}\right)^{\intercal}, e_1 \right\}\right).$$

Exercise 7.3. To represent f as the restriction (over U_{σ_0}) of a proper birational morphism which comes from a single blow-up of a torus-invariant 0-*dimensional* ideal of $\mathbb{C}[z_1, z_2, z_3]$ supporting only $\mathbf{0} \in U_{\sigma_0} \subset \mathbb{C}^3$ (cf. thm. 3.10), it is enough (instead of the the above J) to consider

$$J = \left(z_1^6,\ z_1^5\, z_2,\ z_1^4\, z_2^3,\ z_1^4\, z_2\, z_3,\ z_1^4\, z_3^2,\ z_1^2\, z_3,\ z_1\, z_2\, z_3,\ z_2^3\, z_3,\ z_1\, z_3^2,\ z_3^3\right) \ .$$

(From the point of view of toric geometry, this means that one has to find among the defining monomials suitable monomials involving powers of the variables z_1 and z_3 separately, such that the corresponding order function becomes again linear precisely on the above five maximal cones.)

(b) We now focus our attention to the new phenomena which arise in dimension three.

Theorem 7.4 (What happens in the "intermediate" dimension 3 ?). *Let* $\left(\mathbb{C}^3/G, [\mathbf{0}]\right) = \left(X\left(N_G, \Delta_G\right), \mathrm{orb}\left(\sigma_0\right)\right)$ *a 3-dimensional Gorenstein abelian quotient msc-singularity and* $\mathcal{T}$ *a maximal (and therefore basic) triangulation of the junior simplex* $\mathfrak{s}_G$ *inducing a crepant, full resolution*

$$f : X\left(N_G,\ \widehat{\Delta_G}\left(\mathcal{T}\right)\right) \longrightarrow X\left(N_G, \Delta_G\right) \ .$$

For any $n \in \mathrm{vert}(\mathcal{T})$*, let* D_n *denote the closure* $V\left(\mathrm{pos}\left(\{n\}\right)\right)$*.*

(i) *If* $n \in \mathrm{int}(\mathfrak{s}_G) \cap N_G$*, then* D_n *is a rational compact surface coming from (usual) blow-ups either of* $\mathbb{P}^2_{\mathbb{C}}$ *or of a Hirzebruch surface* $\mathbb{F}_\lambda$ *at finitely many* $T_{N_G(\mathrm{pos}(\{n\}))}$*-fixed points.*

(ii) *If* $\partial\mathfrak{s}_G \cap \left(N_G \smallsetminus \{e_1, e_2, e_3\}\right) \neq \varnothing$*, and* $n \in \mathrm{conv}(e_{i_1}, e_{i_2}) \smallsetminus \{e_{i_1}, e_{i_2}\}$*, with* $\{i_1, i_2\} \subset \{1,2,3\}$*,* $i_1 \neq i_2$*, and* $\{i_3\} = \{1,2,3\} \smallsetminus \{i_1, i_2\}$*, then* D_n *is the total space of a ruled fibration over the "*i_3*-axis" of* $\mathbb{C}^3$*. Its fibers over the punctured* i_3*-axis are isomorphic to* $\mathbb{P}^1_{\mathbb{C}}$*.*

(iii) *For three distinct vertices* n, n', n'' *of* $\mathcal{T}$*, we have*

$$\left(D_n \cdot D_{n'} \cdot D_{n''}\right) = \begin{cases} 1 \ , & \text{if } \ \mathrm{conv}\left(\{n, n', n''\}\right) \text{ is a 2-simplex of } \mathcal{T} \\ 0 \ , & \text{otherwise} \end{cases}$$

(iv) *If* $n, n' \in \mathrm{vert}(\mathcal{T})$*,* $\mathrm{conv}(\{n, n'\})$ *is a 1-simplex of* $\mathcal{T}$*, but no both* n *and* n' *belong the same face of* $\partial\mathfrak{s}_G$*, then there exist exactly two vertices* $\mathfrak{y}, \mathfrak{y}'$ *of* $\mathcal{T}$*, such that* $\mathrm{conv}(\{n, n', \mathfrak{y}\})$*,* $\mathrm{conv}(\{n, n', \mathfrak{y}'\})$ *are 2-simplices of* $\mathcal{T}$ *satisfying a* $\mathbb{Z}$*-linear dependency equation of the form*

$$\kappa\, n + \kappa'\, n' + \mathfrak{y} + \mathfrak{y}' = 0, \ \text{ for some unique } \kappa, \kappa' \in \mathbb{Z} \text{ with } \kappa + \kappa' = -2 \tag{7.1}$$

and

$$\kappa = \left(D_n^2 \cdot D_{n'}\right), \quad \kappa' = \left(D_n \cdot D_{n'}^2\right) \tag{7.2}$$

Furthermore, the normal bundle of the rational intersection curve

$$C = V\left(\mathrm{pos}\left(\{n, n'\}\right)\right)$$

splits into the direct sum:

$$\mathcal{N}_{C\,/\,X\left(N_G,\ \widehat{\Delta_G}(\mathcal{T})\right)} \cong \mathcal{O}_C\left(\kappa\right) \ \oplus \mathcal{O}_C\left(\kappa'\right) \tag{7.3}$$

(v) *If* $n \in \operatorname{int}(\mathfrak{s}_G) \cap N_G$, *then* D_n *has self-intersection number*

$$D_n^3 = 12 - \# \left(\mathrm{Gen}\left(\mathrm{Star}\left(\mathrm{pos}\left(\{n\}\right), \Delta_G\right)\right)\right) \tag{7.4}$$

(vi) *For any other maximal triangulation* $\mathcal{T}'$ *of* $\mathfrak{s}_G$, *there exists a birational morphism (isomorphism in codimension* 1)

$$X\left(N_G,\ \widehat{\Delta_G}\left(\mathcal{T}\right)\right) \to X\left(N_G,\ \widehat{\Delta_G}\left(\mathcal{T}'\right)\right)$$

which is a composite of finitely many elementary transformations (= simple flops) *w.r.t. rational smooth curves.*

Proof. (i) follows from Oda's classification of smooth compact toric surfaces ([54], thm. 1.28), (ii) is clear by construction, and (iii) by (2.3). The vectorial $\mathbb{Z}$-linear dependency equation (7.1) of (iv) with $\kappa + \kappa' = -2$ is obvious because n, n', $\mathfrak{y}$, $\mathfrak{y}'$ are "junior" elements; (7.2) follows from (2.4) and (7.3) from the splitting principle of holomorphic vector bundles over $C \cong \mathbb{P}^1_{\mathbb{C}}$, the normal bundle exact sequence, and the triviality of the dualizing sheaf of $X\left(N_G,\ \widehat{\Delta_G}\left(\mathcal{T}\right)\right)$. (7.4) in (v) can be proved by making use of adjunction formula, combined with Noether's formula, and $\chi\left(D_n, \mathcal{O}_{D_n}\right) = 1$. For the proof of (vi) we refer to Danilov [13], prop. 2, or Oda [54], prop. 1.30 (ii). $\square$

Remark 7.5. In dimension three one has $\mathbf{QP\text{-}CDES}\left(X\left(N_G, \Delta_G\right)\right) \neq \varnothing$ because

$$\mathbf{Coh\text{-}LTR}^{\mathrm{basic}}_{N_G}\left(\mathfrak{s}_G\right) \quad = \quad \mathbf{Coh\text{-}LTR}^{\max}_{N_G}\left(\mathfrak{s}_G\right) \neq \varnothing$$

by prop. 6.6 and thm. 6.9. Nevertheless, in contrast to what takes place in dimension 2, there exist lots of examples of finite abelian subgroups G of $\mathrm{SL}(3,\mathbb{C})$ acting linearly on $\mathbb{C}^3$ whose junior simplex $\mathfrak{s}_G$ admits basic, non-coherent triangulations $\mathcal{T}$, which in turn induce crepant, full, non-projective desingularizations of $X\left(N_G, \Delta_G\right)$. A simple example of this kind comes into being by taking $G \cong (\mathbb{Z}\,/\,4\mathbb{Z}) \times (\mathbb{Z}\,/\,4\mathbb{Z})$ to be defined as the abelian subroup of $\mathrm{SL}(3,\mathbb{C})$ generated by the diagonal elements $\mathrm{diag}\left(\zeta_4, \zeta_4^3, 1\right)$ and $\mathrm{diag}\left(1, \zeta_4, \zeta_4^3\right)$, and $\mathcal{T}$ the triangulation of figure **6**. $\mathcal{T}$ suffers from the "*whirlpool-syndrome*" which makes the application of patching lemma 6.1 impossible, though strictly upper convex support functions can be defined on each of its 2-simplices. The incompatibility of these local strictly upper convex support functions along the intersection loci of 1-simplices can be also explained by means of their "heights" (cf. [73], p. 64); the assertion of the existence of a global upper convex support function on $|\mathcal{T}|$ would lead to a system of a finite number of inconsistent integer linear inequalities, and hence to a contradiction.

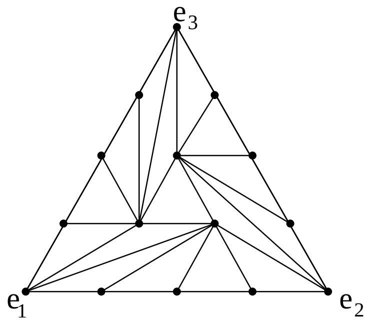


Fig. 6

Proposition 7.6 (Cohomology dimensions).
Let $(\mathbb{C}^3/G, [\mathbf{0}]) = (X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ *denote a* 3*-dimensional Gorenstein cyclic quotient singularity of type* $1/l\ (\alpha_1, \alpha_2, \alpha_3)$. *Then the dimensions of the non-trivial cohomology groups of the overlying spaces* $X\left(N_G,\ \widehat{\Delta_G}\right)$ *of any* T_{N_G}*-equivariant crepant, full desingularization of* $X(N_G, \Delta_G)$ *are given by the formulae*:

$$\dim_{\mathbb{Q}} H^{2i}\left(X\left(N_G,\ \widehat{\Delta_G}\right); \mathbb{Q}\right) = \begin{cases} 1 & , \text{ if } \quad i = 0 \\ \frac{1}{2}\left(l + \sum_{j=1}^{3} \gcd(\alpha_j, l)\right) - 2 & , \text{ if } \quad i = 1 \\ \frac{1}{2}\left(l - \sum_{j=1}^{3} \gcd(\alpha_j, l)\right) + 1 & , \text{ if } \quad i = 2 \end{cases}$$

Proof. Since $\#\ (\partial(\mathfrak{s}_G) \cap N_G) = \sum_{j=1}^{3} \gcd(\alpha_j, l) - 3$, the lattice points representing the inverses of the remaining junior group elements (w.r.t. $\mathrm{int}(\mathfrak{s}_G)$) belong to $2\,\mathfrak{s}_G$, and

$$\#\ (\mathfrak{s}_G \cap N_G) + \#\ ((2\,\mathfrak{s}_G) \cap \mathbf{Par}(\sigma_0) \cap N_G) = l - 1\ ,$$

we have

$$\#\ (\mathrm{int}(\mathfrak{s}_G) \cap N_G) = \#\ ((2\,\mathfrak{s}_G) \cap \mathbf{Par}(\sigma_0) \cap N_G) = \frac{1}{2}\left(l - \sum_{j=1}^{3} \gcd(\alpha_j, l)\right) + 1$$

and the above formulae follow from (6.2), (6.4). □

Exercise 7.7. Generalize prop. 7.6 for arbitrary abelian acting groups. [Hint. Fix a splitting of G into cyclic groups. Use denumerants of weighted partitions instead of gcd's.]

Proposition 7.8 (Uniqueness criterion in dimension 3). *Up to isomorphism, the 3-dimensional Gorenstein abelian quotient msc-singularities* $\left(\mathbb{C}^3/G, [\mathbf{0}]\right) = (X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ *which admit a unique (full) resolution, are cyclic of type* either

(i) $\dfrac{1}{l}\,(1,1,l-2)\,,\ l \geq 3,$ or

(ii) $\dfrac{1}{7}\,(1,2,4)$

• *In case* (i) *there are* $\lfloor \frac{l}{2} \rfloor$ *exceptional prime divisors*

$$D_j := V\left(\mathrm{pos}\left(\left\{n^{(j)}\right\}\right)\right), \quad n^{(j)} := \frac{1}{l}\,(j,j,l-2j)\,, \quad 1 \leq j \leq \lfloor \tfrac{l}{2} \rfloor,$$

all of whose are compact up to the last one for l *even* (see fig. **7** and **8**). *In particular, one has*

$$D_j \cong \begin{cases} \mathbb{F}_{l-2\,j} & ,\ \text{if} \quad 1 \leq j \leq \lfloor \frac{l}{2} \rfloor - 1,\ l \geq 4 \\ \mathbb{P}^2_{\mathbb{C}} & ,\ \text{if} \quad j = \frac{l-1}{2},\ l \text{ odd} \geq 3 \\ \mathbb{P}^1_{\mathbb{C}} \times \mathbb{C} & ,\ \text{if} \quad j = \frac{l}{2},\ l \text{ even } \geq 4 \end{cases}$$

The "3-dimensional" $\lfloor \frac{l}{2} \rfloor \times \lfloor \frac{l}{2} \rfloor \times \lfloor \frac{l}{2} \rfloor$ *intersection-number-matrix is determined by*

$$(D_i \cdot D_j \cdot D_k) = \begin{cases} 0 & ,\ \text{if} \quad i \neq j,\ j \neq k, \\ l - 2\,(i+1) & ,\ \text{if} \quad i = j = k-1 \\ 0 & ,\ \text{if} \quad i = j,\ k - j \geq 2 \\ 2\,i - l & ,\ \text{if} \quad i + 1 = j = k \\ 0 & ,\ \text{if} \quad j = k,\ j - i \geq 2 \\ 8 & ,\ \text{if} \quad i = j = k \neq (l-1)\,/\,2,\ l \text{ odd} \\ 9 & ,\ \text{if} \quad i = j = k = (l-1)\,/\,2,\ l \text{ odd} \\ 8 & ,\ \text{if} \quad i = j = k \neq l\,/\,2,\ l \text{ even} \\ -2 & ,\ \text{if} \quad i = j = k = l\,/\,2,\ l \text{ even} \end{cases}$$

for all $1 \leq i \leq j \leq k \leq \lfloor \frac{l}{2} \rfloor$ *(and by the usual symmetric property of intersection numbers). Moreover, the non-trivial cohomology dimensions of the desingularizing space equal* $1, \lfloor \frac{l}{2} \rfloor$ *and* $\lfloor \frac{l-1}{2} \rfloor$*, respectively (by prop.* 7.6*).*

• *In case* (ii), *there are three exceptional prime divisors, namely*

$$D_n = V\left(\mathrm{pos}\left(\{n\}\right)\right), \; D_{n'} = V\left(\mathrm{pos}\left(\{n'\}\right)\right), \; D_{n''} = V\left(\mathrm{pos}\left(\{n''\}\right)\right),$$

with

$$n = \frac{1}{7}(1,2,4)^{\intercal}, \quad n' = \frac{1}{7}(2,4,1)^{\intercal}, \quad n'' = \frac{1}{7}(4,1,2)^{\intercal},$$

each of which is isomorphic to $\mathbb{F}_2$. *They intersect each other paiwise along three rational curves which play interchangeably the roles of the fibers and of the* 0*-sections of the three projectivized* $\mathbb{P}^1_{\mathbb{C}}$*-bundles* $\mathbb{F}_2 \to \mathbb{P}^1_{\mathbb{C}}$ (see fig. **9**). *Obviously, in both cases* (i) *and* (ii) *the desingularizing birational morphism is projective* (*by prop.* 6.6).

Proof. The uniqueness (up to automorphisms of $\mathrm{aff}(\mathfrak{s}_G) \cap N_G$) of the triangulation $\mathcal{T}$ of $\mathfrak{s}_G$ (inducing a unique crepant desingularization, up to isomorphism) means that for every 4-tuple $\{n_1, n_2, n_3, n_4\}$ of distinct elements of $\mathfrak{s}_G \cap N_G$,

$$\{n_1, n_2, n_3, n_4\} \subset \mathrm{conv}\left(\{n_{\nu_1}, n_{\nu_2}, n_{\nu_3}\}\right), \tag{7.5}$$

for all $\{\nu_1, \nu_2, \nu_3\} \subset \{1,2,3,4\}$ (so that $\mathrm{conv}(\{n_1, n_2, n_3, n_4\})$ cannot be a convex quadrilateral). Since

$$\mathrm{splcod}\left(\mathrm{orb}\left(\sigma_0\right); U_{\sigma_0}\right) = 3 \tag{7.6}$$

it is easy to prove that G cannot be abelian, *non-cyclic*. For cyclic G's acting on $\mathbb{C}^3$ by type $1/l\ (\alpha_1, \alpha_2, \alpha_3)$ the uniqueness condition will be examined in two different cases.

• If the cardinal number of $\mathfrak{s}_G \cap (N_G \smallsetminus \{e_1, e_2, e_3\})$ is ≥ 4, then (7.5) is equivalent to say that all points of it lie on a straight line going through precisely one of the vertices e_1, e_2, e_3 of $\mathfrak{s}_G$ (but, of course, $\#\left(\partial\mathfrak{s}_G \cap (N_G \smallsetminus \{e_1, e_2, e_3\})\right) \in \{0, 1\}$, because otherwise (7.6) would be violated). This occurs only in the case in which at least two of the weights $\alpha_1, \alpha_2, \alpha_3$ are equal; but then

$$(\alpha_1, \alpha_2, \alpha_3) \backsim (1, 1, l-2) \quad (\text{within } \Lambda(l; 3))$$

(check it or see lemma 8.1 below), and therefore $\left(\mathbb{C}^3/G, [\mathbf{0}]\right)$ must have type of the form (i).

• If $\#\left(\mathfrak{s}_G \cap (N_G \smallsetminus \{e_1, e_2, e_3\})\right) \leq 3$, then we get the inequality

$$\frac{1}{2}\left(l + \sum_{j=1}^{3} \gcd(\alpha_i, l)\right) - 2 \leq 3,$$

which is valid only for

$$\frac{1}{l}(\alpha_1, \alpha_2, \alpha_3) \in \left\{ \begin{array}{c} 1/3\ (1,1,1),\ 1/4\ (1,1,2),\ 1/5\ (1,1,3), \\ 1/5\ (1,2,2),\ 1/6\ (1,1,4),\ 1/7\ (1,1,5),\ 1/7\ (1,2,4) \end{array} \right\}.$$

(It suffices to assume $\alpha_1+\alpha_2+\alpha_3 = l$, because $(\alpha_1, \alpha_2, \alpha_3) \backsim (l-\alpha_1, l-\alpha_2, l-\alpha_3)$ within $\Lambda\,(l;3)$). Since $(1,2,2) \backsim (1,1,3)$ within $\Lambda\,(5;3)$, we see that from the above 7 possible types only (ii) $1/7\ (1,2,4)$ is inequivalent (w.r.t. "$\backsim$") to all those of the form (i). The non-vertex lattice points of $\mathfrak{s}_G$ for $(\mathbb{C}^3/G, [\mathbf{0}])$ being of this "new" type (ii) satisfy obviously (7.5) and the proof is completed.

• That the $\lfloor \frac{l}{2} \rfloor$ exceptional prime divisors D_j have the structure given above is an immediate consequence of the more general thm. 8.4 of the next section which will be proved for all dimensions. The intersection numbers $(D_i \cdot D_j \cdot D_k)$ are computable by (7.2), (7.4).

• Since

$$n + e_1 = 2\,n'', \quad n' + e_3 = 2\,n, \quad n'' + e_2 = 2\,n',$$

the above assertion for the structure of the exceptional prime divisors $D_n, D_{n'}, D_{n''}$ is obvious (e.g. by 4.5). For another proof, see Roan-Yau [69], pp. 272-273. $\square$

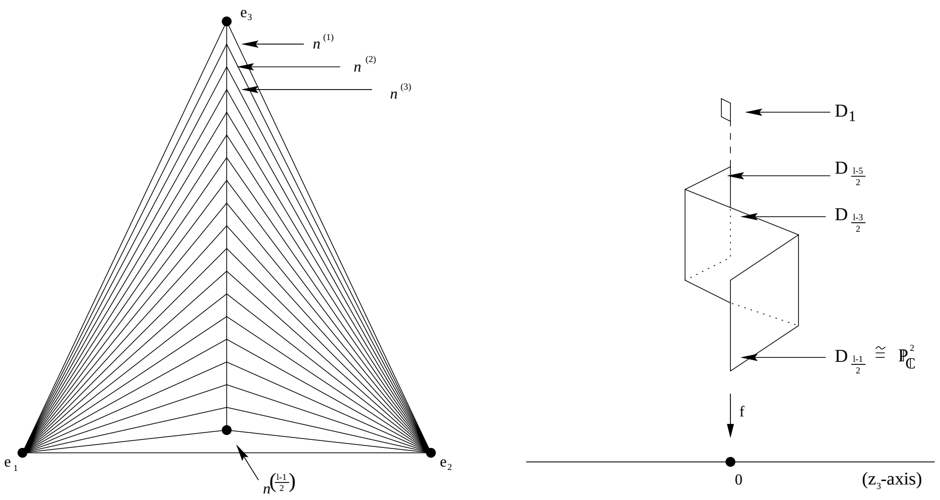


Fig. 7 (l odd)

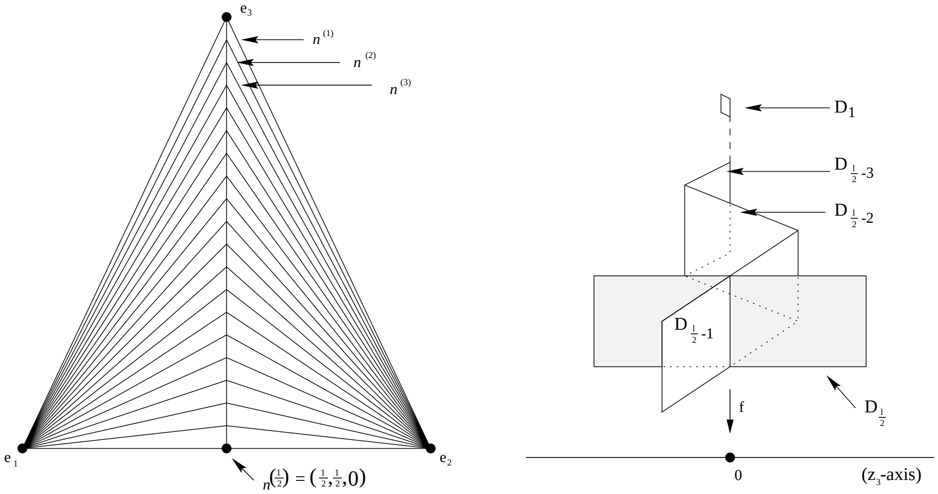


Fig. 8 (l even)

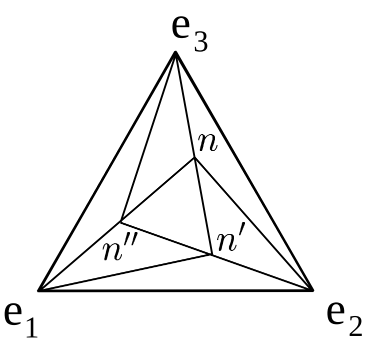


Fig. 9

8. On the monoparametrized singularity-series $\frac{1}{l}(1,\ldots,1,l-(r-1))$

This section contains our main results. Motivated by 7.1, the uniqueness criterion 7.8 (i) in dimension 3, and Reid's remark [63], 5.4, (concerning dimension 4), we study the monoparametrized singularity-series of *arbitrary* dimension with the simplest possible "lattice-geometry", i.e. those Gorenstein cyclic quotient singularities whose junior simplex encloses only lattice points lying on a single straight line.

Lemma 8.1. *Let* $(\mathbb{C}^r/G,[\mathbf{0}]) = (X(N_G,\Delta_G), \mathrm{orb}(\sigma_0))$ *be the Gorenstein cyclic quotient msc-singularity of type* $\frac{1}{l}(\alpha_1,\ldots,\alpha_r)$ *(with* $l=|G|\geq r\geq 2$*) for which at least* $r-1$ *of its defining weights are equal. Then (in the notation of § 5, (c))*

$$(\alpha_1,\ldots,\alpha_r) \backsim (1,\ldots,1,l-(r-1)) \quad (\text{within } \Lambda(l;r)) .$$

Proof. Using a permutation sending the $r-1$ equal defining weights of the above singularity to the first $r-1$ positions, we have

$$(\alpha_1,\ldots,\alpha_r) \backsim (\mu,\ldots,\mu,\alpha_r)$$

for an integer μ, $1\leq\mu\leq l-1$. Since G contains no pseudoreflections, $\gcd(\mu,l)=1$, i.e. there exists an integer ν, $1\leq\nu\leq l-1$, with $\gcd(\nu,l)=1$, and $\lambda\in\mathbb{Z}$, such that

$$\nu\,\mu+\lambda\,l=1 \tag{8.1}$$

Moreover, since $(\mathbb{C}^r/G,[\mathbf{0}])$ is Gorenstein, there must be an integer κ, $1\leq\kappa\leq r-1$, such that

$$(r-1)\,\mu+\alpha_r=\kappa\,l \tag{8.2}$$

Equalities (8.1) and (8.2) imply

$$\nu\,\alpha_r=(\kappa\,\nu+(r-1)\,\lambda-1)\,l+(l-(r-1)) \ .$$

Hence,

$$(\mu,\ldots,\mu,\alpha_r) \backsim ([\nu\,\mu]_l,\ldots,[\nu\,\mu]_l,[\nu\,\alpha_r]_l) = (1,\ldots,1,\,l-(r-1))$$

and we are done. $\square$

By the above lemma and cor. 5.5 we can obviously restrict ourselves to the study of singularities of type (8.3).

Theorem 8.2 (Resolution by a unique projective, crepant morphism). *Let* $(\mathbb{C}^r/G,[\mathbf{0}]) = (X(N_G,\Delta_G), \mathrm{orb}(\sigma_0))$ *be the Gorenstein cyclic quotient singularity of type*

$$\boxed{\frac{1}{l}\left(\underbrace{1,1,\ldots,1,1}_{(r-1)\text{-times}},\ l-(r-1)\right)} \tag{8.3}$$

with $l = |G| \geq r \geq 3$*. Then we have :*

(i) *This msc-singularity is isolated if and only if* $\gcd(l, r-1) = 1$.

(ii) *Up to affine integral transformation, there exists a* unique *triangulation*

$$\mathcal{T} \in \mathbf{LTR}^{\max}_{N_G}\ (\mathfrak{s}_G)$$

inducing a unique (isomorphism class of) crepant T_N*-equivariant partial resolution-morphism*

$$f : X\left(N_G, \widehat{\Delta}_G(\mathcal{T})\right) \to X(N_G, \Delta_G)$$

of $X(N_G, \Delta_G)$ *(with overlying space* $\mathbb{Q}$*-factorial, and maximal with respect to non-discrepancy).*

(iii) $\mathcal{T} \in \mathbf{Coh\text{-}LTR}^{\max}_{N_G}\ (\mathfrak{s}_G)$*, i.e., f is projective.*

(iv) $\mathcal{T} \in \mathbf{Coh\text{-}LTR}^{\text{basic}}_{N_G}\ (\mathfrak{s}_G)$ *(in other words, f gives rise to a* full *projective, crepant desingularization) if and only if*

$$\boxed{[\ l\ ]_{\ r-1} \in \{0,1\}} \tag{8.4}$$

(i.e., iff either $l \equiv 0 \bmod (r-1)$ or $l \equiv 1 \bmod (r-1)$.)

(v) *For l satisfying condition* (8.4)*, the dimensions of the non-trivial cohomology groups of the resolving space* $X\left(N_G, \widehat{\Delta}_G\right)$ *are given by the formulae :*

$$\boxed{\dim_{\mathbb{Q}} H^{2i}\left(X\left(N_G, \widehat{\Delta}_G\right); \mathbb{Q}\right) = \begin{cases} 1 & , \text{ for } \quad i = 0 \\ \left\lfloor \frac{l}{r-1} \right\rfloor & , \text{ for } \quad i \in \{1,2,\ldots,r-2\} \\ \left\lfloor \frac{l-1}{r-1} \right\rfloor & , \text{ for } \quad i = r-1 \end{cases}} \tag{8.5}$$

Proof. (i) This follows directly from 5.4. Note that if $\gcd(l, r-1) \geq 2$, then

$$\mathrm{Sing}\,(X(N_G, \Delta_G)) = U_{\sigma_0}\left(\mathrm{pos}\left(\{e_1, e_2, \ldots, e_{r-2}, e_{r-1}\}\right)\right),$$

i.e. the singular locus of $X(N_G, \Delta_G) = \mathbb{C}^r/G$ is the entire "z_r-axis" of $\mathbb{C}^r$.

(ii) Let us first introduce some notation and make certain preparatory remarks.

Define the vectors:

$$n^{(j)} := \begin{cases} e_r & , \quad \text{if} \quad j = 0 \\ \frac{1}{l}\left(\underbrace{j, j, \ldots, j, j}_{(r-1)\text{-times}}, [j \cdot (l - (r-1))]_l\right)^{\intercal} & , \quad \text{if} \quad j \in \{1, 2, \ldots, l-1\} \end{cases}$$

and denote by $n_i^{(j)}, 1 \leq i \leq r$, the i-th coordinate of each $n^{(j)}$, $0 \leq j \leq l-1$, within $(N_G)_{\mathbb{R}} \cong \mathbb{R}^r$. We have

$$\sum_{i=1}^{r} n_i^{(j+1)} \geq \sum_{i=1}^{r} n_i^{(j)} + \frac{1}{l}(r-1) - 1, \quad \forall j, \quad 1 \leq j \leq l-1.$$

If for some fixed $j = j_\bullet \in \{1, \ldots, l-1\}$, $\sum_{i=1}^{r} n_i^{(j_\bullet)} \geq 2$, then obviously

$$\sum_{i=1}^{r} n_i^{(k)} \geq 2, \quad \forall k, \quad 1 \leq j_\bullet \leq k \leq l-1.$$

Since

$$\sum_{i=1}^{r} n_i^{(j)} = \frac{1}{l}\left((r-1) \cdot j + [j \cdot (l-(r-1))]_l\right)$$

we obtain the inclusion

$$\mathfrak{s}_G \cap N_G \subset \{e_1, \ldots, e_r\} \cup \left\{ n^{(j)} \;\middle|\; 1 \leq j \leq \left\lfloor \frac{l}{r-1} \right\rfloor \right\} .$$

And conversely, for all j, $1 \leq j \leq \lfloor \frac{l}{r-1} \rfloor$, we have $[j \cdot (l-(r-1))]_l = l - j \cdot (r-1)$, which means that $\sum_{i=1}^{r} n_i^{(j)} = 1$. Thus, we get the equality

$$\mathfrak{s}_G \cap N_G = \{e_1, \ldots, e_r\} \cup \left\{ n^{(j)} \;\middle|\; 1 \leq j \leq \left\lfloor \frac{l}{r-1} \right\rfloor \right\} \tag{8.6}$$

• *Construction of $\mathcal{T}$.* At first define

$$\Xi_r := \left\{ (\xi_1, \xi_2, \ldots, \xi_{r-2}) \in (\{1, 2, \ldots, r-2, r-1\})^{r-2} \mid \xi_1 < \xi_2 < \cdots < \xi_{r-2} \right\},$$

and

$$B(j; \xi_1, \xi_2, \ldots, \xi_{r-2}) := \left\{ n^{(j-1)}, n^{(j)}, \; e_{\xi_1}, e_{\xi_2}, \ldots, e_{\xi_{r-2}} \right\},$$

as well as

$$\mathbf{s}(j; \xi_1, \xi_2, \ldots, \xi_{r-2}) := \operatorname{conv}\left(B(j; \xi_1, \xi_2, \ldots, \xi_{r-2})\right),$$

and

$$\sigma\left(j;\xi_1,\xi_2,\ldots,\xi_{r-2}\right) := \sigma_{\mathbf{s}\left(j;\xi_1,\xi_2,\ldots,\xi_{r-2}\right)} = \mathrm{pos}\left(B\left(j;\xi_1,\xi_2,\ldots,\xi_{r-2}\right)\right),$$

for all j,

$1 \leq j \leq \left\lfloor \dfrac{l}{r-1} \right\rfloor$, and all $r-1$ possible $(r-2)$-tuples $\left(\xi_1,\xi_2,\ldots,\xi_{r-2}\right) \in \Xi_r$.

All lattice points $n^{(j)}$, $0 \leq j \leq \left\lfloor \frac{l}{r-1} \right\rfloor$, lie on a *straight line* of $(N_G)_{\mathbb{R}} \cong \mathbb{R}^r$, namely on

$$\mathrm{conv}\left(\left\{n^{(0)}, n^{\left(\lfloor \frac{l}{r-1} \rfloor\right)}\right\}\right)$$

and they can be ordered canonically

$$n^{(0)}\ ,\ n^{(1)},\ \ldots\ ,\ n^{\left(\lfloor \frac{l}{r-1} \rfloor - 1\right)}\ ,\ n^{\left(\lfloor \frac{l}{r-1} \rfloor\right)}$$

via the increasing ordering of the enumerator-superscripts (j).
Next define the simplicial subdivision $\mathcal{T}$ (having support $|\mathcal{T}| \subset \mathfrak{s}_G$) as follows :

$$\mathcal{T} := \begin{cases} \mathfrak{E} & , \ \text{if } \ l \equiv 0 \ (\mathrm{mod}\ \ (r-1)) \\ \mathfrak{E} \ \cup \left\{ \begin{array}{c} \text{all faces of} \\ \mathrm{conv}\left(B_{\mathrm{last}}\right) \end{array} \right\} & , \ \text{otherwise} \end{cases}$$

where

$$B_{\mathrm{last}} := \left\{n^{\left(\lfloor \frac{l-1}{r-1} \rfloor\right)}, e_1, e_2, \ldots, e_{r-1}\right\},$$

and

$$\mathfrak{E} := \left\{ \begin{array}{c} \text{all faces of the simplices} \\ \mathbf{s}\left(j;\xi_1,\xi_2,\ldots,\xi_{r-2}\right) \end{array} \ \middle| \ \begin{array}{l} \text{for all } j,\ 1 \leq j \leq \left\lfloor \frac{l}{r-1} \right\rfloor, \text{ and} \\ \text{all } \ \left(\xi_1,\xi_2,\ldots,\xi_{r-2}\right) \in \Xi_r \end{array} \right\}.$$

• $\mathcal{T}$ is a triangulation. The proof of the assertion that the intersection of two arbitrary simplices of $\mathcal{T}$ is either a face of both or the empty set is left as an easy exercise to the reader.

• $|\mathcal{T}| = \mathfrak{s}_G$, i.e., its support covers the entire junior simplex. It is straightforward to show that

$$\begin{aligned} &\mathrm{Vol}(\mathrm{conv}\left(\{\mathbf{0}\}, \mathbf{s}\left(j;\xi_1,\xi_2,\ldots,\xi_{r-2}\right)\right)) = \\ &\frac{1}{r!}\left|\det\left(n^{(j-1)}, n^{(j)}, e_{\xi_1}, e_{\xi_2}, \ldots, e_{\xi_{r-2}}\right)\right| = \\ &\frac{1}{r!}\frac{1}{l^2}\ \left|(j-1)\cdot(l-j\,(r-1)) - j\cdot(l-(j-1)\,(r-1))\right| = \frac{1}{lr!} \qquad (8.7) \end{aligned}$$

and

$$\mathrm{Vol}(\mathrm{conv}\left(\{\mathbf{0}\}, B_{\mathrm{last}}\right)) = \frac{[l]_{r-1}}{lr!}. \tag{8.8}$$

Thus the assertion is an immediate consequence of the equality :

$$\mathrm{Vol}\left(\mathrm{conv}\left(\{\mathbf{0}\}, \bigcup \{\mathbf{s} \mid \mathbf{s} \text{ simplex of } \mathcal{T}\}\right)\right) =$$

$$= \frac{1}{r!}\frac{1}{l}\left(\left\lfloor \frac{l}{r-1} \right\rfloor \cdot (r-1) + [\ l\ ]_{\ r-1}\right) = \frac{1}{r!} = \mathrm{Vol}\left(\mathrm{conv}\left(\{\mathbf{0}\}, \mathfrak{s}_G\right)\right)\ .$$

• $\mathcal{T}$ is a maximal triangulation. This is obvious because $\mathfrak{s}_G \cap N_G = \mathrm{vert}(\mathcal{T})$ by the construction of $\mathcal{T}$ and (8.6).

• $\mathcal{T}$ is unique. Suppose $\mathbf{s}$ is an arbitrary elementary $(r-1)$-simplex, such that $\mathrm{vert}(\mathbf{s}) \subset \mathfrak{s}_G \cap N_G$. Clearly, $\mathbf{s} \subsetneqq \mathfrak{s}_G$ because $\mathfrak{s}_G$ itself is non-elementary. This implies

$$\mathrm{vert}\,(\mathbf{s}) \cap (\mathfrak{s}_G \cap N_G \smallsetminus \{e_1, \ldots, e_r\}) \neq \varnothing\ .$$

Since $n^{(j)}$'s are collinear and $\mathbf{s}$ elementary simplex, $\mathrm{vert}(\mathbf{s})$ contains at most 2 consecutive lattice points among them. Hence,

$$\#\ (\mathrm{vert}\,(\mathbf{s}) \cap (\mathfrak{s}_G \cap N_G \smallsetminus \{e_1, \ldots, e_r\})) \in \{1, 2\}\ .$$

If this number equals 2, then $\mathbf{s}$ has to be of the form $\mathbf{s}\left(j; \xi_1, \ldots, \xi_{r-2}\right)$, for some $j \in \left\{0, \ldots, \lfloor \frac{l}{r-1} \rfloor\right\}$ and $\left(\xi_1, \ldots, \xi_{r-2}\right) \in \Xi_r$ (by definition). Otherwise, $\mathbf{s}$ possesses only one vertex belonging to the relative interior of $\mathfrak{s}_G$. However, since the $n^{(j)}$'s are collinear and $n^{(0)} = e_r$ we must have $l \not\equiv 0 \bmod (r-1)$ and $\mathbf{s} = B_{\mathrm{last}}$. Hence, $\mathcal{T}$ is uniquely determined.

(iii) The coherence of $\mathcal{T}$ will be proved by induction on the number $\lfloor \frac{l}{r-1} \rfloor$. The case in which this equals 1 is trivial. Suppose $\lfloor \frac{l}{r-1} \rfloor > 1$. It is easy to check that the simplex

$$\widetilde{\mathbf{s}} := \mathrm{conv}\left(\left\{n^{(1)}, e_1, \ldots, e_{r-1}\right\}\right)$$

can be mapped by an affine integral transformation onto the junior simplex of an equidimensional Gorenstein cyclic quotient singularity of type

$$\frac{1}{l-(r-1)}\ (1, \ldots, 1, l-2\,(r-1))$$

with

$$\left\lfloor \frac{l-(r-1)}{r-1} \right\rfloor = \left\lfloor \frac{l}{r-1} - 1 \right\rfloor = \left\lfloor \frac{l}{r-1} \right\rfloor - 1 < \left\lfloor \frac{l}{r-1} \right\rfloor\ .$$

By induction hypothesis and by construction we may therefore assume that the restriction $\mathcal{T}\,|_{\widetilde{\mathbf{s}}}$ of $\mathcal{T}$ on $\widetilde{\mathbf{s}}$ is coherent. Choose a $\widetilde{\psi} \in \mathrm{SUCSF}_{\mathbb{R}}\,(\mathcal{T}\,|_{\widetilde{\mathbf{s}}})$ and use the abbreviation

$$\mathbf{s}_\iota := \begin{cases} \mathbf{s}\,(1;\,1, 2, \ldots, r-3,\, r-2) & , \text{ for } \iota = 1 \\ \mathbf{s}\,(1;\,1, \ldots, \iota-1, \iota+1, \ldots, r-1) & , \ \forall \iota,\ 2 \leq \iota \leq r-2 \\ \mathbf{s}\,(1;\,2,\,3, \ldots, r-2,\, r-1) & , \text{ for } \iota = r-1 \end{cases}$$

Note that for each $\mathbf{y} \in \mathbf{s}_\iota$ and ι, $1 \leq \iota \leq r-1$, there exists an $\mathbf{y}' \in \widetilde{\mathbf{s}}$, such that

$$\mathbf{y}' \in \ \mathrm{lin}\,(\mathrm{conv}\,(\{e_r, \mathbf{y}\})) \cap \widetilde{\mathbf{s}} \quad \& \quad \mathbf{y} = (1-t)\ \mathbf{y}' + t\, e_r, \quad \text{for a} \quad t \in [0,1]\ ,$$

because of the convexity of $\widetilde{\mathbf{s}} \subset \mathfrak{s}_G$. Now define support functions $\psi_\iota : |\mathcal{T}\,|_{\mathbf{s}_\iota}| \to \mathbb{R}$ by

$$\mathbf{s}_\iota \ni \mathbf{y} \longmapsto \psi_\iota\,(\mathbf{y}) := \ (1-t)\ \widetilde{\psi}\,(\mathbf{y}') + t\ \in \mathbb{R}\,.$$

One verifies easily that $\psi_\iota \in \mathrm{SUCSF}_{\mathbb{R}}\,(\mathcal{T}\,|_{\mathbf{s}_\iota})$ and

$$\psi_\iota\ |_{\widetilde{\mathbf{s}} \cap \mathbf{s}_\iota} = \widetilde{\psi}\ |_{\widetilde{\mathbf{s}} \cap \mathbf{s}_\iota}$$

for all ι, $1 \leq \iota \leq r-1$. Applying patching lemma 6.1, we get $\mathrm{SUCSF}_{\mathbb{R}}\,(\mathcal{T}) \neq \varnothing$ and we are done.

• An alternative (but not directly constructive) method for showing the coherence of $\mathcal{T}$, is to combine the uniqueness of $\mathcal{T}$ with prop. 6.6.

(iv) Since $\mathcal{T}$ is uniquely determined the volume formulae in (8.7) and (8.8) show that f is a full, crepant torus-equivariant resolution-morphism if and only if $[l]_{r-1} \in \{0,1\}$. Another, more direct way, to verify that condition (8.4) is necessary is the following:
If we assume that $[\ l\ ]_{\ r-1} \notin \{0,1\}$, the group order l can be written as

$$l = \left\lfloor \frac{l}{r-1} \right\rfloor \cdot (r-1) + [\ l\ ]_{\ r-1}\,, \quad \text{with} \quad [\ l\ ]_{\ r-1} \in \{2, \ldots, r-2\}\,. \tag{8.9}$$

Note that

$$n_r^{(j)} = 1 - \frac{j \cdot (r-1)}{l}\,, \quad \forall j, \quad 1 \leq j \leq \left\lfloor \frac{l}{r-1} \right\rfloor\,,$$

and that the decreasing sequence

$$n_r^{(1)} > n_r^{(2)} > \cdots > n_r^{\left(\left\lfloor \frac{l}{r-1} \right\rfloor\right)} \tag{8.10}$$

has minimum element

$$n_r^{\left(\left\lfloor \frac{l}{r-1} \right\rfloor\right)} = \frac{[\ l\ ]_{\ r-1}}{l}\ \left(\geq \frac{2}{l}\right)\ .$$

We shall discuss the two possible cases w.r.t. the values taken by $\gcd(l, r-1)$ separately.

I) $\gcd(l, r-1) = 1$. Then there exists a $\kappa \in \{1, \ldots, l-1\}$, such that

$$\kappa\ (l - (r-1)) \equiv 1\ (\mathrm{mod}\ l)\,.$$

Clearly,

$$n_r^{(\kappa)} = \frac{1}{l}\ [\kappa\ (l-(r-1))]_l = \frac{1}{l} < n_r^{\left(\left\lfloor \frac{l}{r-1} \right\rfloor\right)} \tag{8.11}$$

and consequently,

$$\kappa \in \left\{ \left\lfloor \frac{l}{r-1} \right\rfloor + 1, \ldots, l-1 \right\} \Rightarrow n^{(\kappa)} \notin \mathfrak{s}_G \cap N_G \ ,$$

by (8.6) and (8.10). If there were a basic triangulation of $\mathfrak{s}_G$, then $n^{(\kappa)}$ would not belong to $\mathbf{Hlb}_{N_G}(\sigma_0)$ (by the equality (6.5) of thm. 6.15), but it would be representable as a linear combination of (at least two) elements of the Hilbert basis $\mathbf{Hlb}_{N_G}(\sigma_0)$ with positive integer coefficients. This would obviously contradict inequality (8.11) because of (8.10).

II) $\gcd(l, r-1) \geq 2$. In this case, define $\kappa := l \,/\, \gcd(l, r-1)$. Then

$$\kappa \ (l - (r-1)) \equiv 0 \ (\mathrm{mod}\ l) \,,$$

and repeating the same argumentation for this new $n_r^{(\kappa)} = 0$ as in I), we arrive again at a contradicting conclusion.

(v) Formula (8.5) follows from (6.4). In particular,

$$\dim_{\mathbb{Q}} H^{2i} \left(X \left(N_G, \widehat{\Delta}_G \right) ; \mathbb{Q} \right) = \# \left\{ j \in [0, l) \cap \mathbb{Z} \ \middle| \ \lfloor \tfrac{(i-1)\, l}{r-1} \rfloor + 1 \leq j \leq \lfloor \tfrac{i\, l}{r-1} \rfloor \right\}$$

and the proof of the theorem is completed. □

Remark 8.3. (i) To prove the necessity of condition (8.4) for $\mathcal{T}$ to be basic, we preferred to make use of "Hilbert-base technology" because it is generally applicable to *any* Gorenstein abelian quotient singularity. Alternative ad hoc methods (for the above special situation) are either a suitable direct manipulation of determinants or the use of normalized-volume-bound. According to the latter one, violation of (8.4) would imply for the topological Euler-Poincaré characteristic of the overlying space:

$$\begin{aligned}
&\chi \left(X \left(N_G, \widehat{\Delta}_G (\mathcal{T}) \right) \right) = \# \left(\widehat{\Delta}_G (\mathcal{T}) (r) \right) = \\
&= \# \left((r-1)\text{-simplices of } \mathcal{T} \right) = && \text{(by (2.7))} \\
&= \lfloor \tfrac{l}{r-1} \rfloor \cdot (\# \ (\Xi_r)) + 1 = \lfloor \tfrac{l}{r-1} \rfloor \cdot (r-1) + 1 = \\
&= l - [\ l\ ]_{\ r-1} + 1 < l = |G| && \text{(by (8.9))}
\end{aligned}$$

which would be impossible for basic $\mathcal{T}$ by (6.3).

(ii) In fact, if $[\ l\ ]_{\ r-1} \notin \{0, 1\}$ (which is possible only for $r \geq 4$), the only non-basic $(r-1)$-simplex of $\mathcal{T}$ is $\mathrm{conv}(B_{\mathrm{last}})$. If $\gcd(l, r-1) = 1$ then the toric variety

$X\left(N_G, \widehat{\Delta}_G(\mathcal{T})\right)$ has a Gorenstein terminal, isolated, cyclic quotient singularity of type

$$\frac{1}{[\,l\,]_{r-1}}\left(\underbrace{1,1,\ldots,1,1}_{(r-1)\text{-times}},\ [-(r-1)]_{[\,l\,]_{r-1}}\right)$$

lying on the affine piece $U_{\mathrm{pos}(B_{\mathrm{last}})}$. Otherwise, the singular locus is not a singleton; more precisely, it is 1-dimensional, i.e., the corresponding quotient singularity has splitting codimension $r-1$, and can be viewed as a 1-parameter "Schar" ($\approx$ crowd) of terminal singularities of type

$$\frac{1}{\left(\frac{r-1}{\gcd(l,r-1)}\right)}\left(\underbrace{1,1,\ldots,1,1}_{(r-1)\text{-times}}\right)$$

along (the sections of the normal sheaf of) the proper transform of the "z_r-axis" $\mathrm{STR}(U_{\sigma_0}(\mathrm{pos}(\{e_1,\ldots,e_{r-1}\})),\ f_{\mathcal{T}})$ lying on $X\left(N_G, \widehat{\Delta}_G(\mathcal{T})\right)$.

The triangulated junior terahedron of the simplest example $1/5\ (1,1,1,2)$ is drawn in fig. **10**. The "low" tetrahedron induces the classical involutional terminal singularity of type $1/2\ (1,1,1,1)$.

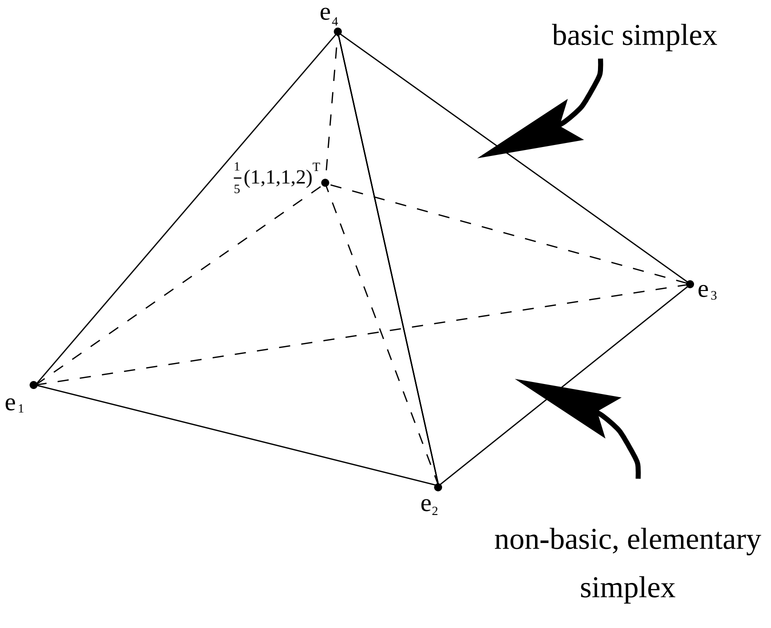


Fig. 10

Theorem 8.4 (Exceptional prime divisors and intersection numbers). *Let* $(\mathbb{C}^r/G, [\mathbf{0}]) = (X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ *denote the Gorenstein cyclic quotient singularity of type* (8.3). *If* l *satisfies condition* (8.4), *then :*

(i) *The exceptional locus of* $f : X\left(N_G, \widehat{\Delta}_G(\mathcal{T})\right) \to X(N_G, \Delta_G)$ *consists of* $\lfloor \frac{l}{r-1} \rfloor$ *prime divisors*

$$D_j := D_{n^{(j)}} = V\left(\tau^{(j)}\right) = X\left(N_G\left(\tau^{(j)}\right); \mathrm{Star}\left(\left(\tau^{(j)}; \widehat{\Delta}_G(\mathcal{T})\right)\right)\right), \; 1 \leq j \leq \lfloor \tfrac{l}{r-1} \rfloor,$$

on $X\left(N_G, \widehat{\Delta}_G(\mathcal{T})\right)$, *with* $\tau^{(j)} := \mathbb{R}_{\geq 0}\, n^{(j)}$, *having the following structure :*

$$\boxed{D_j \cong \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^{r-2}_{\mathbb{C}}} \oplus \mathcal{O}_{\mathbb{P}^{r-2}_{\mathbb{C}}}(l-(r-1)j)\right) \qquad (\text{as } \mathbb{P}^1_{\mathbb{C}}\text{-bundles over } \mathbb{P}^{r-2}_{\mathbb{C}})}$$

for all j, $j \in \left\{1, 2, \ldots, \lfloor \frac{l}{r-1} \rfloor - 1\right\}$, *and*

$$\boxed{D_{\lfloor \frac{l}{r-1} \rfloor} \cong \begin{cases} \mathbb{P}^{r-1}_{\mathbb{C}} & , \text{ if } \; l \equiv 1 \bmod (r-1) \\ \mathbb{P}^{r-2}_{\mathbb{C}} \times \mathbb{C} & , \text{ if } \; l \equiv 0 \bmod (r-1) \end{cases}}$$

(ii) *For the (highest) intersection numbers of two consecutive exceptional prime divisors we get :*

$$\boxed{\begin{aligned} \left(D_j^{r-1} \cdot D_{j+1}\right) &= (l-(r-1)(j+1))^{r-2} \\ \left(D_j \cdot D_{j+1}^{r-1}\right) &= ((r-1)j-l)^{r-2} \end{aligned}} \tag{8.12}$$

and for the self-intersection numbers :

$$\boxed{D_j^r = \sum_{i=0}^{r-2} \binom{r-1}{i} \; (-2)^{r-i-1} \left((l-(r-1)j)-r\right)^i (l-(r-1)j)^{r-i-2}} \tag{8.13}$$

for all j, $j \in \left\{1, 2, \ldots, \lfloor \frac{l}{r-1} \rfloor - 1\right\}$, *and*

$$\boxed{\left(D_{\lfloor \frac{l}{r-1} \rfloor}\right)^{r} = \begin{cases} (-r)^{\ r-1} & , \ \text{if} \quad l \equiv 1 \bmod (r-1) \\ (-(r-1))^{r-2} & , \ \text{if} \quad l \equiv 0 \bmod (r-1) \end{cases}} \tag{8.14}$$

Proof. (i) We distinguish three cases depending on the range of j and the divisibility condition on l.

First case. Let $j \in \left\{1, 2, \ldots, \lfloor \frac{l}{r-1} \rfloor - 1\right\}$. Obviously, there are exactly two primitive collections in $\mathrm{Gen}\Big(\mathrm{Star}\left(\tau^{(j)}; \widehat{\Delta}_G\left(\mathcal{T}\right)\right)\Big)$, namely

$$\{e_1 + (N_G)_{\tau^{(j)}}, \ldots, e_{r-1} + (N_G)_{\tau^{(j)}}\} \text{ and } \left\{n^{(j-1)} + (N_G)_{\tau^{(j)}}, n^{(j+1)} + (N_G)_{\tau^{(j)}}\right\}$$

having no common elements. Furthermore,

$$n^{(j-1)} + n^{(j+1)} = 2\, n^{(j)} \Longrightarrow \left(n^{(j-1)} + (N_G)_{\tau^{(j)}}\right) + \left(n^{(j+1)} + (N_G)_{\tau^{(j)}}\right) = \mathbf{0}_{N_G\left(\tau^{(j)}\right)}.$$

Hence, each D_j has to be the total space of a $\mathbb{P}^1_{\mathbb{C}}$-bundle over an $(r-2)$-dimensional smooth, compact toric variety, and since $\mathrm{Star}\left(\tau^{(j)}; \widehat{\Delta}_G\left(\mathcal{T}\right)\right)$ is a splitting fan, it will be, in addition, the total space of the projectivization of a decomposable bundle (by prop. 4.2 and thm. 4.4). On the other hand,

$$\#\left(\mathrm{Gen}\left(\mathrm{Star}\left(\tau^{(j)}; \widehat{\Delta}_G\left(\mathcal{T}\right)\right)\right)\right) = r + 1,$$

which means that D_j has Picard number 2, and has therefore to be isomorphic to the total space of the projectivization of a decomposable bundle of the form $\mathcal{O}_{\mathbb{P}^{r-2}_{\mathbb{C}}} \oplus \mathcal{O}_{\mathbb{P}^{r-2}_{\mathbb{C}}}(\lambda)$ over $\mathbb{P}^{r-2}_{\mathbb{C}}$ (by Kleinschmidt's classification theorem 4.6). For this reason, it suffices to determine this single twisting number λ by using lemma 4.5. It is easy to verify, that we have the $\mathbb{Z}$-linear dependence-relations

$$e_1 + e_2 + \cdots + e_{r-2} + e_{r-1} = \left((r-1)\,(j+1) - l\right)\ n^{(j)} + \left(l - (r-1)\,j\right)\ n^{(j+1)}$$

i.e.,

$$\sum_{\iota=1}^{r-1} \left(e_\iota + (N_G)_{\tau^{(j)}}\right) - \left(\left(l - (r-1)\,j\right)\ n^{(j+1)} + (N_G)_{\tau^{(j)}}\right) = \mathbf{0}_{N_G\left(\tau^{(j)}\right)}\ .$$

Consequently,

$$D_j \cong Y\left(r-1; l - (r-1)\,j\right)\ .$$

Second case. Let $l \equiv 1 \bmod (r-1)$ and $j = \lfloor \frac{l}{r-1} \rfloor = \frac{l-1}{r-1}$. Since

$$\left(e_1 + (N_G)_{\tau^{(j)}}\right) + \cdots + \left(e_{r-1} + (N_G)_{\tau^{(j)}}\right) + \left(n^{(j-1)} + (N_G)_{\tau^{(j)}}\right) = \mathbf{0}_{N_G\left(\tau^{(j)}\right)}\ ,$$

we have obviously $D_j \cong \mathbb{P}^{r-1}_{\mathbb{C}}$.

Third case. Let $l \equiv 0 \bmod (r-1)$ and $j = \lfloor \frac{l}{r-1} \rfloor = \frac{l}{r-1}$. Since $n_r^{(j)}$ equals 0, $n^{(j)}$ lies on the facet $\mathrm{conv}(\{e_1, \ldots, e_{r-1}\})$ of $\mathfrak{s}_G$. This means that the star of $\tau^{(j)}$ in $\widehat{\Delta}_G(\mathcal{T})$ can be written as a direct product of the form

$$\mathrm{Star}\left(\tau^{(j)}; \widehat{\Delta}_G(\mathcal{T})\right) = \mathrm{Star}\left(\tau^{(j)}; \widehat{\Delta}_G(\mathcal{T})(\mathrm{pos}(e_1, .., e_{r-1}))\right) \times (\text{a half-line in } \mathbb{R})$$

and D_j splits into

$$X\left(N_G(\mathrm{pos}(e_1, .., e_{r-1})); \mathrm{Star}\left(\left(\tau^{(j)}; \widehat{\Delta}_G(\mathcal{T})(\mathrm{pos}(e_1, .., e_{r-1}))\right)\right)\right) \times \mathbb{C}$$

where this first factor is isomorphic to $\mathbb{P}_{\mathbb{C}}^{r-2}$ because

$$e_1 + \cdots + e_{r-1} = (r-1) \cdot n^{(j)} .$$

(ii) Since

$$\left(D_j^{r-1} \cdot D_{j+1}\right) = D_j^{r-1} \mid_{D_{j+1}}$$

equals

$$\left(V\left(\tau^{(j)} + (N_G)_{\tau^{(j+1)}}\right)\right)^{r-1} \Big|_{X\left(N_G\left(\tau^{(j+1)}\right); \mathrm{Star}\left(\left(\tau^{(j+1)}; \widehat{\Delta}_G(\mathcal{T})\right)\right)\right)},$$

we obtain

$$\left(D_j^{r-1} \cdot D_{j+1}\right) = (l - (r-1)(j+1))^{r-2}$$

by (4.5). Similarly for $\left(D_j \cdot D_{j+1}^{r-1}\right)$; regarding D_j as the total space of a $\mathbb{P}_{\mathbb{C}}^1$-bundle over $\mathbb{P}_{\mathbb{C}}^{r-2}$, and making this time use of the "opposite" piece of the affine covering of its typical fiber $\cong \mathbb{P}_{\mathbb{C}}^1$, we get the second formula of (8.12) by restricting D_{j+1}^{r-1} on

$$D_j = X\left(N_G\left(\tau^{(j)}\right); \mathrm{Star}\left(\left(\tau^{(j)}; \widehat{\Delta}_G(\mathcal{T})\right)\right)\right)$$

and by (4.5) after sign-change (i.e., after having identified $V\left(\tau^{(j+1)} + (N_G)_{\tau^{(j)}}\right)$ with the divisor on D_j whose associated line bundle is $\mathcal{O}_{D_j}(-1)$).
On the other hand, $\omega_{X\left(N_G, \widehat{\Delta}_G(\mathcal{T})\right)} \cong \mathcal{O}_{X\left(N_G, \widehat{\Delta}_G(\mathcal{T})\right)}$, and the adjunction formula gives

$$\omega_{D_j} \cong \omega_{X\left(N_G, \widehat{\Delta}_G(\mathcal{T})\right)} \otimes \mathcal{O}_{X\left(N_G, \widehat{\Delta}_G(\mathcal{T})\right)}(D_j) \cong \mathcal{O}_{D_j}(D_j) \cong \mathcal{N}_{D_j / X\left(N_G, \widehat{\Delta}_G(\mathcal{T})\right)},$$

where $\mathcal{N}_{D_j / X\left(N_G, \widehat{\Delta}_G(\mathcal{T})\right)}$ denotes the normal sheaf of D_j in $X\left(N_G, \widehat{\Delta}_G(\mathcal{T})\right)$. Hence, evaluating the highest power of the first Chern class of this sheaf at the fundamental cycle $[D_j]$ of D_j, we obtain for the self-intersection number :

$$D_j^r = c_1^{r-1}\left(\mathcal{N}_{D_j / X\left(N_G, \widehat{\Delta}_G(\mathcal{T})\right)}\right)([D_j]) = c_1^{r-1}\left(\omega_{D_j}\right)([D_j]) = K_{D_j}^{r-1} \tag{8.15}$$

Formula (8.13) for D_j^r follows from (8.15) and (4.4); formula (8.14) is obvious. □

9. Breaking down the desingularizing morphism

The unique crepant resolution-morphism of the singularities discussed in the previous section can be factorized into (normalized) blow-ups in several ways. We give here two canonical decompositions of $f = f_{\mathcal{T}}$ of this kind.

(a) Maintaining the notation introduced in the proof of thm. 8.2 for the construction of the unique, basic, coherent triangulation $\mathcal{T}$, besides $\mathbf{s}\left(j;\xi_1,\ldots,\xi_{r-2}\right)$'s (and $\mathrm{conv}(B_{\mathrm{last}})$, for $l \equiv 1 \bmod (r-1)$), we define

$$B\left(j,j';\xi_1,\xi_2,\ldots,\xi_{r-2}\right) := \left\{n^{(j)}, n^{(j')},\ e_{\xi_1}, e_{\xi_2},\ldots,e_{\xi_{r-2}}\right\},$$

as well as

$$\mathbf{s}\left(j,j';\xi_1,\xi_2,\ldots,\xi_{r-2}\right) := \ \mathrm{conv}\left(B\left(j,j';\xi_1,\xi_2,\ldots,\xi_{r-2}\right)\right),$$

and

$$\sigma\left(j,j';\xi_1,\xi_2,\ldots,\xi_{r-2}\right) := \sigma_{\mathbf{s}\left(j,j';\xi_1,\xi_2,\ldots,\xi_{r-2}\right)} = \ \mathrm{pos}\left(B\left(j,j';\xi_1,\xi_2,\ldots,\xi_{r-2}\right)\right),$$

for all indices j, j' with

$$0 \leq j \leq j' \leq \left\lfloor \frac{l}{r-1} \right\rfloor,$$

and for all $r-1$ possible $(r-2)$-tuples $\left(\xi_1,\xi_2,\ldots,\xi_{r-2}\right) \in \Xi_r$. Obviously,

$$\mathbf{s}\left(j,j';\xi_1,\xi_2,\ldots,\xi_{r-2}\right) = \mathbf{s}\left(j';\xi_1,\xi_2,\ldots,\xi_{r-2}\right)$$

whenever $j' = j+1$. Let now κ denote the positive integer

$$\kappa := \begin{cases} \left\lfloor \frac{1}{2}\left(\frac{l-1}{r-1}+1\right)\right\rfloor & ,\ \text{if} \quad l \equiv 1 \bmod (r-1) \\[2ex] \left\lfloor \frac{1}{2}\left(\frac{l}{r-1}\right)\right\rfloor + 1 & ,\ \text{if} \quad l \equiv 0 \bmod (r-1) \end{cases}$$

• For $l \equiv 1 \bmod (r-1)$ we introduce the following simplicial subdivisions of the junior simplex $\mathfrak{s}_G$:

$$\mathfrak{T}_1 := \left\{ \begin{array}{c} \mathbf{s}\left(1;\xi_1,\xi_2,\ldots,\xi_{r-2}\right),\ \mathbf{s}\left(1,\frac{l-1}{r-1};\xi_1,\xi_2,\ldots,\xi_{r-2}\right), \\ \mathrm{conv}\left(B_{\mathrm{last}}\right) = \mathbf{s}\left(\frac{l-1}{r-1},\frac{l-1}{r-1};\xi_1,\xi_2,\ldots,\xi_{r-2}\right), \\ \text{for all } \left(\xi_1,\xi_2,\ldots,\xi_{r-2}\right) \in \Xi_r, \text{ together with all their faces} \end{array} \right\}$$

and

$$\mathfrak{T}_{i+1} := \left[\mathfrak{T}_i \smallsetminus \left\{ \begin{array}{c} \mathbf{s}\left(i, \frac{l-1}{r-1} - i + 1; \xi_1, \xi_2, \ldots, \xi_{r-2}\right), \\ \text{for all } \left(\xi_1, \xi_2, \ldots, \xi_{r-2}\right) \in \Xi_r, \\ \text{together with its faces} \end{array} \right\} \right] \cup$$

$$\cup \left\{ \begin{array}{c} \mathbf{s}\left(i, i+1; \xi_1, \ldots, \xi_{r-2}\right), \ \mathbf{s}\left(i+1, \frac{l-1}{r-1} - i; \xi_1, \ldots, \xi_{r-2}\right), \\ \mathbf{s}\left(\frac{l-1}{r-1} - i, \frac{l-1}{r-1} - i + 1; \xi_1, \ldots, \xi_{r-2}\right), \\ \text{for all } \left(\xi_1, \ldots, \xi_{r-2}\right) \in \Xi_r, \text{ together with all their faces} \end{array} \right\}$$

for $\kappa \geq 2$ and all i, $1 \leq i \leq \kappa - 1$.

• Analogously, for $l \equiv 0 \bmod (r-1)$, we define

$$\mathfrak{T}_1 := \left\{ \begin{array}{c} \mathbf{s}\left(1; \xi_1, \xi_2, \ldots, \xi_{r-2}\right), \ \mathbf{s}\left(1, \frac{l}{r-1} - 1; \xi_1, \xi_2, \ldots, \xi_{r-2}\right), \\ \mathbf{s}\left(\frac{l}{r-1} - 1, \frac{l}{r-1} - 1; \xi_1, \xi_2, \ldots, \xi_{r-2}\right), \\ \text{for all } \left(\xi_1, \xi_2, \ldots, \xi_{r-2}\right) \in \Xi_r, \text{ together with all their faces} \end{array} \right\}$$

and

$$\mathfrak{T}_{i+1} := \left[\mathfrak{T}_i \smallsetminus \left\{ \begin{array}{c} \mathbf{s}\left(i, \frac{l}{r-1} - i; \xi_1, \xi_2, \ldots, \xi_{r-2}\right), \\ \text{for all } \left(\xi_1, \xi_2, \ldots, \xi_{r-2}\right) \in \Xi_r, \\ \text{together with its faces} \end{array} \right\} \right] \cup$$

$$\cup \left\{ \begin{array}{c} \mathbf{s}\left(i, i+1; \xi_1, \ldots, \xi_{r-2}\right), \ \mathbf{s}\left(i+1, \frac{l}{r-1} - i - 1; \xi_1, \ldots, \xi_{r-2}\right), \\ \mathbf{s}\left(\frac{l}{r-1} - i - 1, \frac{l}{r-1} - i; \xi_1, \ldots, \xi_{r-2}\right), \\ \text{for all } \left(\xi_1, \ldots, \xi_{r-2}\right) \in \Xi_r, \text{ together with all their faces} \end{array} \right\}$$

for $\kappa \geq 3$ and all i, $1 \leq i \leq \kappa - 2$, and finally

$$\mathfrak{T}_\kappa := \mathfrak{T}_{\kappa-1} \cup \left\{ \begin{array}{c} \mathbf{s}\left(\frac{l}{r-1}; \xi_1, \xi_2, \ldots, \xi_{r-2}\right), \\ \text{for all } \left(\xi_1, \xi_2, \ldots, \xi_{r-2}\right) \in \Xi_r, \\ \text{together with its faces} \end{array} \right\} .$$

Next lemma is obvious by construction.

Lemma 9.1. *All the above defined simplicial subdivisions $\mathfrak{T}_1, \ldots, \mathfrak{T}_\kappa$ are triangulations and cover the entire $\mathfrak{s}_G$.*

Proposition 9.2 (First, speedy factorization).
Let $(\mathbb{C}^r/G, [\mathbf{0}]) = (X(N_G, \Delta_G), \operatorname{orb}(\sigma_0))$ denote the Gorenstein cyclic quotient

singularity of type (8.3) *with* l *satisfying condition* (8.4). *Then the birational resolution-morphism* $f = f_{\mathcal{T}}$ *is the composite of the* κ *toric morphisms*

$$X_0 := X(N_G, \Delta_G) \overset{g_1}{\longleftarrow} X_1 \overset{g_2}{\longleftarrow} X_2 \longleftarrow \cdots \overset{g_{\kappa-1}}{\longleftarrow} X_{\kappa-1} \overset{g_\kappa}{\longleftarrow} X_\kappa := X\left(N_G, \widehat{\Delta}_G(\mathcal{T})\right)$$

with

$$X_i := X\left(N_G, \widehat{\Delta}_G(\mathfrak{T}_i)\right), \quad \forall i, \ \ 1 \leq i \leq \kappa - 1, \ \ \text{and } \mathcal{T} = \mathfrak{T}_\kappa \ .$$

In particular, in algebraic-geometric terms, one has

$$X_{i+1} \cong \text{Norm}\left[\mathbf{Bl}^{\text{red}}_{Z_i}(X_i)\right], \quad \forall i, \ \ 0 \leq i \leq \kappa - 1 \ ,$$

with centers

$$Z_i = \begin{cases} \text{orb}(\sigma_0) & , \text{ if } \ i = 0 \\ V\left(\text{pos}\left(\left\{n^{(i)}, n^{\left(\frac{l-1}{r-1} - i + 1\right)}\right\}\right)\right) & , \text{ if } \ \begin{cases} \kappa \geq 2, \ 1 \leq i \leq \kappa - 1, \\ \text{and } \ l \equiv 1 \bmod (r-1) \end{cases} \\ V\left(\text{pos}\left(\left\{n^{(i)}, n^{\left(\frac{l}{r-1} - i\right)}\right\}\right)\right) & , \text{ if } \ \begin{cases} \kappa \geq 3, \ 1 \leq i \leq \kappa - 2, \\ \text{and } \ l \equiv 0 \bmod (r-1) \end{cases} \\ V(\text{pos}(\{e_1, e_2, \ldots, e_{r-1}\})) & , \text{ if } \ i = \kappa - 1 \ \ \& \ \ l \equiv 0 \bmod (r-1) \end{cases}$$

Sketch of proof. Since N_G is a "skew" lattice, it is not so convenient to work directly with it. For this reason we consider the linear transformation

$$\Phi : \mathbb{R}^r \longrightarrow \mathbb{R}^r, \ \ \mathbf{y} \longmapsto \Phi(\mathbf{y}) := \mathcal{M} \cdot \mathbf{y},$$

with

$$\mathcal{M} := \begin{pmatrix} l & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & & 0 \\ -1 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & & 0 \\ -(l-(r-1)) & 0 & 0 & \cdots & 1 \end{pmatrix} \in \text{GL}(r, \mathbb{Q}) \ .$$

The image of the lattice N_G via Φ is the standard lattice $\mathbb{Z}^r = \sum_{i=1}^r \mathbb{Z}\, e_i$. In particular,

$$N_G = \mathbb{Z}^r + \mathbb{Z}\ \frac{1}{l}\ (1, 1, \ldots, 1, l-(r-1))^\intercal = \mathbb{Z}\ \frac{1}{l}\ (1, 1, \ldots, 1, l-(r-1))^\intercal + \sum_{i=2}^r \mathbb{Z}\, e_i,$$

and Φ maps this $\mathbb{Z}$-basis of N_G onto

$$\Phi\left(\frac{1}{l}\ (1, 1, \ldots, 1, l-(r-1))^\intercal\right) = e_1, \ \ \Phi(e_2) = e_2, \ \ \ldots \ , \ \ \Phi(e_r) = e_r,$$

the positive orthant σ_0 onto the cone

$$\overline{\sigma_0} := \Phi(\sigma_0) = \mathrm{pos}(\{\mathfrak{v}, e_2, e_3, \ldots, e_r\}),$$

with

$$\mathfrak{v} := \Phi(e_1) = \left(l, \underbrace{-1, -1, \ldots, -1}_{(r-2)\text{-times}}, -(l-(r-1)) \right)^{\intercal},$$

and the Hilbert basis $\mathbf{Hlb}_N(\sigma_0)$ of σ_0 w.r.t. N (cf. (6.5), (8.6)) onto

$$\begin{aligned} \Phi(\mathbf{Hlb}_N(\sigma_0)) &= \mathbf{Hlb}_{\mathbb{Z}^r}(\overline{\sigma_0}) = \mathrm{conv}(\{\mathfrak{v}, e_2, e_3, \ldots, e_r\}) \cap \mathbb{Z}^r = \\ &= \{\mathfrak{v}, e_2, e_3, \ldots, e_r\} \cup \left\{ \mathfrak{y}^{(1)}, \mathfrak{y}^{(2)}, \ldots, \mathfrak{y}^{(\nu)} \right\}, \end{aligned}$$

where $\nu := \lfloor \frac{l}{r-1} \rfloor$, and

$$\mathfrak{y}^{(j)} := \Phi\left(n^{(j)}\right) = \left(j, \underbrace{0, 0, \ldots, 0, 0}_{(r-2)\text{-times}}, -j+1 \right)^{\intercal}, \quad \forall j, \quad 1 \leq j \leq \nu.$$

(For $j = 1$, $\mathfrak{y}^{(1)} = e_1$). The dual cone of $\overline{\sigma_0}$ equals

$$(\overline{\sigma_0})^{\vee} = \mathrm{pos}\left(\left\{ e_1^{\vee}, e_1^{\vee} + l\, e_2^{\vee}, \ldots, e_1^{\vee} + l\, e_{r-1}^{\vee}, (l-(r-1))\, e_1^{\vee} + l\, e_r^{\vee} \right\}\right)$$

(with $\{e_1^{\vee}, \ldots, e_r^{\vee}\}$ denoting the dual of $\{e_1, \ldots, e_r\}$). For every m belonging to the Hilbert basis $\mathbf{Hlb}_{(\mathbb{Z}^r)^{\vee}}\left((\overline{\sigma_0})^{\vee}\right)$ (w.r.t. the dual lattice $(\mathbb{Z}^r)^{\vee}$) define

$$\tau[m] := \left\{ \mathbf{y} \in \overline{\sigma_0} \;\middle|\; \langle m, \mathbf{y} \rangle \leq \langle m', \mathbf{y} \rangle,\ \forall m',\ m' \in \mathbf{Hlb}_{(\mathbb{Z}^r)^{\vee}}\left((\overline{\sigma_0})^{\vee}\right) \right\}.$$

• Suppose $l \equiv 1 \bmod (r-1)$. At first we shall show that for any fixed $(r-2)$-tuple $(\xi_1, \xi_2, \ldots, \xi_{r-2}) \in \Xi_r$, with $\{q\} = \{1, 2, \ldots, r-1\} \smallsetminus \{\xi_1, \xi_2, \ldots, \xi_{r-2}\}$, we have

$$\Phi(\sigma(1; \xi_1, \ldots, \xi_{r-2})) = \begin{cases} \tau[e_1^{\vee}] & \text{if} \quad q = 1 \\ \tau\left[e_1^{\vee} + l\, e_q^{\vee}\right] & \text{if} \quad q \in \{2, \ldots, r-1\} \end{cases} \tag{9.1}$$

• Moreover, for $l > r$,

$$\Phi\left(\sigma\left(1, \tfrac{l-1}{r-1}; \xi_1, .., \xi_{r-2}\right)\right) = \begin{cases} \tau\left[e_1^{\vee} + e_r^{\vee}\right] & \text{if } q = 1 \\ \tau\left[e_1^{\vee} + e_r^{\vee} + (r-1)\, e_q^{\vee}\right] & q = 2, .., r-1 \end{cases} \tag{9.2}$$

and finally

$$\Phi(\mathrm{pos}(B_{\mathrm{last}})) = \tau\left[(l-(r-1))\, e_1^{\vee} + l\, e_r^{\vee}\right] \tag{9.3}$$

• *Proof of* (9.1) : Suppose first $q=1$. Obviously, $e_1^\vee \in \mathbf{Hlb}_{(\mathbb{Z}^r)^\vee}\left((\overline{\sigma_0})^\vee\right)$ and for all $m \in \mathbf{Hlb}_{(\mathbb{Z}^r)^\vee}\left((\overline{\sigma_0})^\vee\right)$ we get

$$\langle m, e_1\rangle \geq \langle e_1^\vee, e_1\rangle = 1 \Longrightarrow e_1 \in \tau\,[e_1^\vee]$$

and for every $j \in \{2,\ldots,r\}$,

$$\langle m, e_1\rangle \geq \langle e_1^\vee, e_j\rangle = 0 \Longrightarrow e_j \in \tau\,[e_1^\vee]\ .$$

Conversely, let $\mathbf{y}$ denote an arbitrary element of $\tau\,[e_1^\vee]$. Write

$$\mathbf{y} = \sum_{i=1}^{r} \mu_i\, e_i$$

as an $\mathbb{R}$-linear combination w.r.t. the basis $\{e_1,\ldots,e_r\}$. Since $e_1^\vee$, $e_1^\vee + l\,e_j^\vee$, $2 \leq j \leq r-1$, and $e_1^\vee + e_r^\vee$ belong to $(\overline{\sigma_0})^\vee$ we obtain

$$\langle e_1^\vee, \mathbf{y}\rangle \geq 0, \quad \left\langle e_1^\vee + l\,e_j^\vee, \mathbf{y}\right\rangle \geq \langle e_1^\vee, \mathbf{y}\rangle\,, \quad \langle e_1^\vee + e_r^\vee, \mathbf{y}\rangle \geq \langle e_1^\vee, \mathbf{y}\rangle\,,$$

i.e., $\mu_1,..,\mu_r \in \mathbb{R}_{\geq 0}$ and therefore $\mathbf{y} \in \Phi\left(\sigma\left(1;2,3,\ldots,r-1\right)\right) = \mathrm{pos}(\{e_1,..,e_r\})\,.$

• Suppose now $q \in \{2,\ldots,r-1\}$. Since

$$\left\langle e_1^\vee + l\,e_q^\vee, \mathfrak{v}\right\rangle = \left\langle e_1^\vee + l\,e_q^\vee, e_p\right\rangle = 0, \text{ for } p \in \{2,..,q-2,q-1,q+1,q+2,..,r\}\,,$$

and

$$\langle m, e_1\rangle \geq \left\langle e_1^\vee + l\,e_q^\vee, e_1\right\rangle = 1, \text{ for all } \ m \in \mathbf{Hlb}_{(\mathbb{Z}^r)^\vee}\left((\overline{\sigma_0})^\vee\right),$$

the inclusion "$\subset$" is obvious. Conversely, let $\mathbf{y}$ denote an arbitrary element of $\tau\left[e_1^\vee + l\,e_q^\vee\right]$ and write it as $\mathbb{R}$-linear combination

$$\mathbf{y} = \sum_{i \in \{1,2,\ldots,q-2,q-1,q+1,q+2,\ldots,r\}} \mu_i\, e_i + \mu_q\, \mathfrak{v}\,.$$

Since

$$\mu_1 = \left\langle e_1^\vee + l\,e_q^\vee, \mathbf{y}\right\rangle \geq 0, \quad \left\langle e_1^\vee + l\,e_q^\vee, \mathbf{y}\right\rangle \leq \langle e_1^\vee, \mathbf{y}\rangle \Longleftrightarrow \mu_q \geq 0,$$

and

$$\left\langle e_1^\vee + l\,e_q^\vee, \mathbf{y}\right\rangle \leq \left\langle e_1^\vee + l\,e_p^\vee, \mathbf{y}\right\rangle \Longleftrightarrow \mu_p \geq 0,$$

for $\ p \in \{2,\ldots,q-2,q-1,q+1,q+2,\ldots,r\}$, we have

$$\mathbf{y} \in \Phi\left(\sigma\left(1;2,..,q-2,q-1,q+1,q+2,..,r\right)\right) = \mathrm{pos}\left(\{\mathfrak{v}, e_1,..,e_{q-1},e_{q+1},..,e_r\}\right).$$

• *Proof of* (9.2) : Suppose first $q=1$. The inclusion "$\subset$" can be easily checked as before. Let $\mathbf{y}$ be an element of $\tau\,[e_1^\vee + e_r^\vee]$ and write it as linear combination

$$\mathbf{y} = \sum_{i=1}^{r-1} \mu_i\, e_i + \mu_r\, \mathfrak{y}^{(\nu)}\,.$$

We have

$$\langle e_1^\vee + e_r^\vee, \mathbf{y}\rangle \le \langle e_1^\vee, \mathbf{y}\rangle \Longleftrightarrow -\mu_r\nu + \mu_r \le 0 \Longleftrightarrow \mu_r \ge 0 \quad \left(\text{by } \nu = \tfrac{l-1}{r-1} > 1\right)$$

and

$$\langle e_1^\vee + e_r^\vee, \mathbf{y}\rangle \le \left\langle e_1^\vee + e_r^\vee + (r-1)\ e_j^\vee, \mathbf{y}\right\rangle \Longleftrightarrow \mu_j \ge 0,\ \forall j,\ 2 \le j \le r-1,$$

as well as

$$\langle e_1^\vee + e_r^\vee, \mathbf{y}\rangle \le \langle (l-(r-1))\ e_1^\vee + l\, e_r^\vee, \mathbf{y}\rangle$$

which is equivalent to

$$0 \le \langle (l-r)\, e_1^\vee + (l-1)\, e_r^\vee, \mathbf{y}\rangle = (l-r)\,(\mu_1 + \mu_r\nu) + (l-1)\,(\mu_r - \mu_r\nu) = (l-r)\,\mu_1,$$

i.e., $\mu_1 \ge 0$. Hence, $\mathbf{y} \in \Phi\left(\sigma\left(1, \frac{l-1}{r-1}; 2, 3, .., r-1\right)\right) = \mathrm{pos}\left(\left\{\mathfrak{y}^{(\nu)}, e_1, e_2, .., e_{r-1}\right\}\right)$.

• For $q \in \{2, \ldots, r-1\}$ the proof of the inclusion "$\subset$" is again easy. To prove "$\supset$" it is enough to consider an element $\mathbf{y} \in \tau\left[e_1^\vee + e_r^\vee + (r-1)\ e_q^\vee\right]$ and write it as linear combination

$$\mathbf{y} = \sum_{i \in \{1,2,\ldots,q-2,q-1,q+1,q+2,\ldots,r-1\}} \mu_i\, e_i + \mu_q\, \mathfrak{v} + \mu_r\, \mathfrak{y}^{(\nu)}.$$

By definition, $\mathbf{y}$ satisfies the three inequalities

$$\left\langle e_1^\vee + e_r^\vee + (r-1)\ e_q^\vee, \mathbf{y}\right\rangle \le \langle e_1^\vee + e_r^\vee, \mathbf{y}\rangle,$$

$$\left\langle e_1^\vee + e_r^\vee + (r-1)\ e_q^\vee, \mathbf{y}\right\rangle \le \langle (l-(r-1))\ e_1^\vee + l\, e_r^\vee, \mathbf{y}\rangle,$$

and

$$\left\langle e_1^\vee + e_r^\vee + (r-1)\ e_q^\vee, \mathbf{y}\right\rangle \le \left\langle e_1^\vee + e_r^\vee + (r-1)\ e_p^\vee, \mathbf{y}\right\rangle,$$

for all $p \in \{2, \ldots, q-2, q-1, q+1, q+2, \ldots, r\}$. Direct evaluation combined with

$$\langle (l-(r-1))\ e_1^\vee + l\, e_r^\vee, \mathfrak{v}\rangle = 0,\ \left\langle (l-(r-1))\ e_1^\vee + l\, e_r^\vee, \mathfrak{y}^{(\nu)}\right\rangle = l - \nu\ (r-1) = 1,$$

gives $\mu_i \in \mathbb{R}_{\ge 0}$, for all i, $1 \le i \le r$, i.e., $\mathbf{y}$ belongs to

$$\Phi\left(\sigma\left(1, \tfrac{l-1}{r-1}; 2, .., q-1, q+1, .., r\right)\right) = \mathrm{pos}\left(\left\{\mathfrak{v}, \mathfrak{y}^{(\nu)}, e_1, .., e_{q-1}, e_{q+1}, .., e_{r-1}\right\}\right).$$

• *Proof of* (9.3) : Since

$$\langle (l-(r-1))\ e_1^\vee + l\, e_r^\vee, \mathfrak{v}\rangle = \langle (l-(r-1))\ e_1^\vee + l\, e_r^\vee, e_j\rangle = 0,\ \forall j,\ 2 \le j \le r-1,$$

we have $\mathfrak{v}, e_2, \ldots, e_{r-1} \in \tau\left[(l-(r-1))\ e_1^\vee + l\, e_r^\vee\right]$. On the other hand, by the definition of $(\overline{\sigma_0})^\vee$,

$$\left\langle (l-(r-1))\ e_1^\vee + l\, e_r^\vee, \mathfrak{y}^{(\nu)}\right\rangle = 1 \le \left\langle m, \mathfrak{y}^{(\nu)}\right\rangle,\ \forall m,\ m \in (\overline{\sigma_0})^\vee \cap (\mathbb{Z}^r \smallsetminus \{\mathbf{0}\}).$$

Consequently,

$$\Phi\left(\mathrm{pos}\left(B_{\mathrm{last}}\right)\right)= \ \mathrm{pos}\left(\left\{\mathfrak{v},\mathfrak{y}^{(\nu)},e_2,e_3,\ldots,e_{r-1}\right\}\right)\subset\tau\left[\left(l-(r-1)\right)\,e_1^{\vee}+l\,e_r^{\vee}\right]\ .$$

To show the converse inclusion take again a $\mathbf{y}\in\tau\left[\left(l-(r-1)\right)\,e_1^{\vee}+l\,e_r^{\vee}\right]$, write it as linear combination

$$\mathbf{y}=\mu_1\,\mathfrak{v}+\sum_{j=2}^{r-1}\mu_j\,e_j+\mu_r\,\mathfrak{y}^{(\nu)},$$

and use the inequalities

$$\begin{array}{lll}0 & \leq & \left\langle\left(l-(r-1)\right)\,e_1^{\vee}+l\,e_r^{\vee},\mathbf{y}\right\rangle=\mu_r,\\ 0 & \leq & \left\langle\left(l-(r-1)\right)\,e_1^{\vee}+l\,e_r^{\vee},\mathbf{y}\right\rangle=\mu_r\leq\left\langle e_1^{\vee}+e_r^{\vee},\mathbf{y}\right\rangle=(r-1)\,\mu_1+\mu_r,\end{array}$$

and for all $j,\ 2\leq j\leq r-1$, the inequalities

$$\left\langle\left(l-(r-1)\right)\,e_1^{\vee}+l\,e_r^{\vee},\mathbf{y}\right\rangle=\mu_r\leq\left\langle e_1^{\vee}+e_r^{\vee}+(r-1)\,e_j^{\vee},\mathbf{y}\right\rangle=(r-1)\,\mu_j+\mu_r.$$

• By (9.1), (9.2), (9.3), and lemma 9.1 we obtain

$$\mathfrak{T}_1=\bigcup_{m\in\Phi(\mathbf{Hlb}_N(\sigma_0))}\left\{\begin{array}{c}\Phi^{-1}\left(\tau\left[m\right]\right)\ \text{together}\\ \text{with all their faces}\end{array}\right\}.$$

This means that $\widehat{\Delta}_G\left(\mathfrak{T}_1\right)=\left(\Delta_G\right)_{\mathbf{bl}}\left[\mathrm{orb}\left(\sigma_0\right)\right]$ by proposition 3.8, and therefore g_1 is indeed the proper birational morphism corresponding to the normalized, usual blow-up of $X\left(N_G,\Delta_G\right)$ at the closed point $\mathrm{orb}(\sigma_0)$.

• If $\kappa\geq 2$, in the second step we blow up (simultaneously) the $(r-2)$-dimensional common singular locus of all affine charts $U_{\sigma\left(1,\frac{l-1}{r-1};\xi_1,\xi_2,\ldots,\xi_{r-2}\right)}$. Note that locally a neighbourhood of such a singular point within $X\left(N_G,\widehat{\Delta}_G\left(\mathfrak{T}_1\right)\right)$ can be viewed like a

$$\left(\text{2-dimensional } A_{\frac{l-1}{r-1}-2}\text{-singularity}\right)\times\mathbb{C}^{r-2}\ .$$

To prove that the above defined triangulation $\mathfrak{T}_2$ induces the normalization of this blow-up, one applies theorem 3.6 and techniques similar to those used for $\mathfrak{T}_1$. The details are left as an exercise to the reader. Repeating the described procedure altogether $\kappa-1$ times we arrive at the entire basic triangulation $\mathcal{T}=\mathfrak{T}_\kappa$.

• The proof in the case in which $l\equiv 0\ \mathrm{mod}(r-1)$ can be done analogously and will be omitted. The only difference is that in the last step one blows up (once) the remaining 1-dimensional singular locus $V\left(\mathrm{pos}\left(\{e_1,e_2,\ldots,e_{r-1}\}\right)\right)$ inherited from the single non-basic facet of $\mathfrak{s}_G$. □

Figures **11** (a) and (b) show this speedy factorization of f for the singularities of type $1/10\ (1,1,8)$ and $1/11\ (1,1,9)$, respectively.

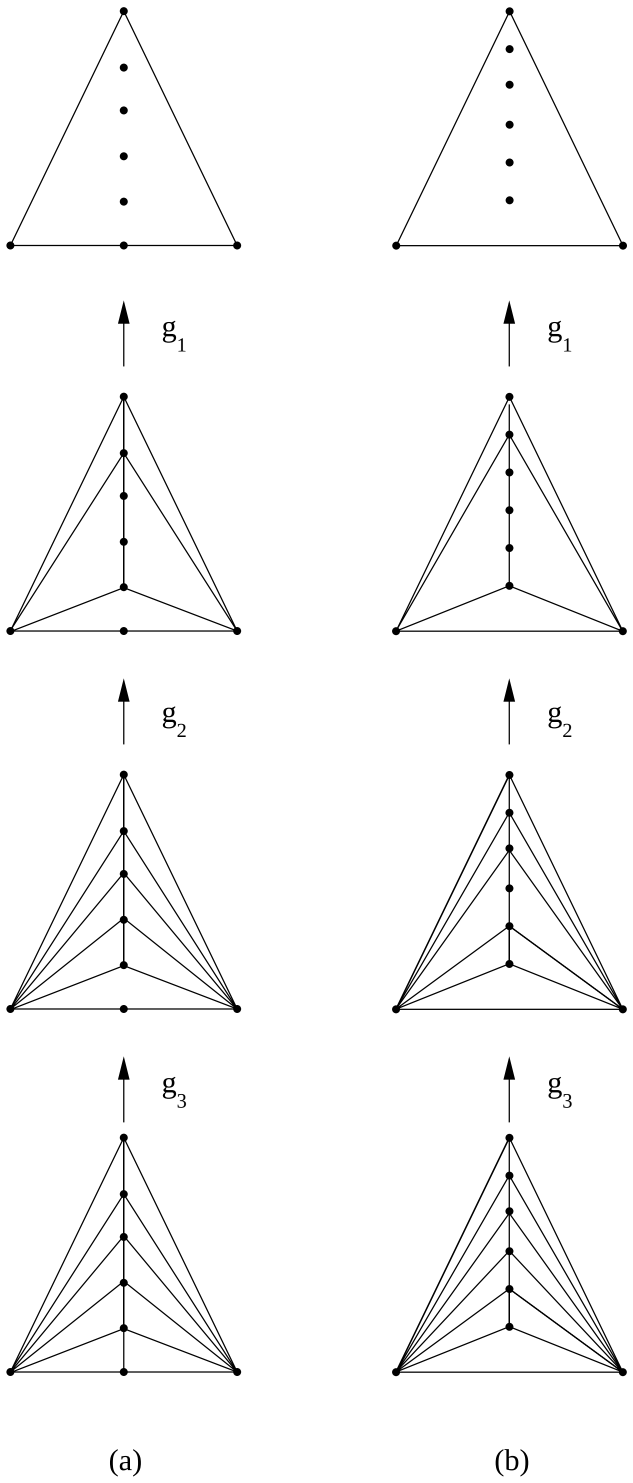


Fig. **11**

(b) A second canonical factorization of $f = f_{\mathcal{T}}$ is constructed by means of the following $\nu = \lfloor \frac{l}{r-1} \rfloor$ triangulations of the junior simplex:

$$\mathfrak{T}_1' := \left\{ \begin{array}{c} \mathbf{s}\left(1;\xi_1,\xi_2,\ldots,\xi_{r-2}\right),\ \mathrm{conv}\left(\left\{n^{(1)},e_{\xi_1},\ldots,e_{\xi_{r-2}}\right\}\right), \\ \text{for all } \left(\xi_1,\xi_2,\ldots,\xi_{r-2}\right) \in \Xi_r, \text{ together with all their faces} \end{array} \right\}$$

and

$$\mathfrak{T}_{i+1}' := \left[\mathfrak{T}_i' \smallsetminus \left\{ \begin{array}{c} \mathrm{conv}\left(\left\{n^{(i)},e_{\xi_1},\ldots,e_{\xi_{r-2}}\right\}\right), \\ \text{for all } \left(\xi_1,\xi_2,\ldots,\xi_{r-2}\right) \in \Xi_r, \\ \text{together with its faces} \end{array} \right\} \right] \cup$$

$$\cup \left\{ \begin{array}{c} \mathbf{s}\left(i+1;\xi_1,\ldots,\xi_{r-2}\right),\ \mathrm{conv}\left(\left\{n^{(i+1)},e_{\xi_1},\ldots,e_{\xi_{r-2}}\right\}\right), \\ \text{for all } \left(\xi_1,\ldots,\xi_{r-2}\right) \in \Xi_r, \text{ together with all their faces} \end{array} \right\}$$

for all i, $1 \leq i \leq \nu - 1$.

Proposition 9.3 (Second factorization).
Let $(\mathbb{C}^r/G,[\mathbf{0}]) = (X(N_G,\Delta_G), \mathrm{orb}(\sigma_0))$ be the Gorenstein cyclic quotient singularity of type (8.3) with l satisfying condition (8.4). Then the birational resolution-morphism $f = f_{\mathcal{T}}$ can be expressed also as the composite of $\nu = \lfloor \frac{l}{r-1} \rfloor$ toric morphisms

$$X_0 := X\left(N_G,\Delta_G\right) \xleftarrow{h_1} X_1 \xleftarrow{h_2} X_2 \longleftarrow \cdots \xleftarrow{h_{\nu-1}} X_{\nu-1} \xleftarrow{h_\nu} X_\nu := X\left(N_G, \widehat{\Delta}_G\left(\mathcal{T}\right)\right)$$

with

$X_i := X\left(N_G, \widehat{\Delta}_G\left(\mathfrak{T}_i'\right)\right), \forall i,\ 0 \leq i \leq \nu-1,$ and $\mathfrak{T}_0' =$ trival triangulation, $\mathfrak{T}_\nu' = \mathcal{T}$.

In particular,

$$X_{i+1} \cong \mathrm{Norm}\left[\mathbf{Bl}_{Z_i}^{\mathcal{I}_i}\left(X_i\right)\right], \quad \forall i,\ \ 0 \leq i \leq \nu - 1\ ,$$

for appropriate $\mathcal{O}_{X_i}$-ideal sheaves $\mathcal{I}_i$, such that $Z_i = \mathrm{supp}(\mathcal{O}_{X_i} / \mathcal{I}_i)$, where

$$Z_i = \left\{ \begin{array}{lll} \mathrm{orb}\left(\sigma_0\right) & , \text{ if} & i = 0 \\ \mathrm{orb}\left(\mathrm{pos}\left(\left\{n^{(i)},e_1,\ldots,e_{r-1}\right\}\right)\right) & , \text{ if} & \left\{ \begin{array}{l} \text{either } \nu \geq 2, 1 \leq i \leq \nu-1, \\ \text{and } l \equiv 1 \bmod (r-1) \\ \\ \text{or } \nu \geq 3, 1 \leq i \leq \nu - 2, \\ \text{and } l \equiv 0 \bmod (r-1) \end{array} \right. \\ V\left(\mathrm{pos}\left(\left\{e_1,e_2,\ldots,e_{r-1}\right\}\right)\right) & , \text{ if} & l \equiv 0 \bmod (r-1)\ \ \&\ \ i = \nu - 1 \end{array} \right.$$

(Up to the above last case and the case in which $\nu = 1$, all Z_i's are endowed with a non-reduced scheme structure).

Moreover, one has

$$X_{i+1} \cong \text{Norm}\left[\mathbf{Bl}^{\text{red}}_{\text{orb}(\theta_i)}\left(X\left(N_{[i]};\text{Star}\left(\theta_i;\widehat{\Delta}_G\left(\mathfrak{T}_i'\right)\right)\right)\right)/G_i\right],$$

with respect to the different lattice

$$N_{[i]} := \mathbb{Z}\,n^{(i)} \oplus \mathbb{Z}\,e_1 \oplus \cdots \oplus \mathbb{Z}\,e_{r-1}\ ,$$

and the star of the cone

$$\theta_i := \text{pos}\left(\left\{n^{(i)}, e_1, \ldots, e_{r-1}\right\}\right),\ \text{ for all } i \text{ with } \begin{cases} \text{either } 0 \leq i \leq \nu - 1, \\ \text{and } l \equiv 1 \bmod (r-1) \\ \\ \text{or } \nu \geq 2,\ 0 \leq i \leq \nu - 2, \\ \text{and } l \equiv 0 \bmod (r-1) \end{cases}$$

where $G_0 = G$ and $G_i = \langle g_i \rangle$ denotes the cyclic group of analytic automorphisms of

$$X\left(N_{[i]};\text{Star}\left(\theta_i;\widehat{\Delta}_G\left(\mathfrak{T}_i'\right)\right)\right) \cong \mathbb{C}^r$$

generated by

$$g_i : \mathbb{C}^r \ni (z_1, .., z_r) \longmapsto \left(\zeta_{l-i(r-1)} \cdot z_1, .., \zeta_{l-i(r-1)} \cdot z_{r-1}, \zeta_{l-i(r-1)}^{l-(i+1)(r-1)} \cdot z_r\right) \in \mathbb{C}^r .$$

The proof of 9.3 is an immediate generalization of that of the case in which $r = 2$ (see 7.2 (ii)), relies on a successive application of 3.6, and is left as an exercise to the reader. (The only difference is that whenever $r \geq 3$ and $l \equiv 0 \bmod (r-1)$, we also blow up the remaining 1-dimensional singular locus in the last step). Figures **12** (a) and (b) illustrate the triangulations inducing the factorization of f for the singularities of type $1/6\,(1,1,4)$ and $1/7\,(1,1,5)$, respectively.

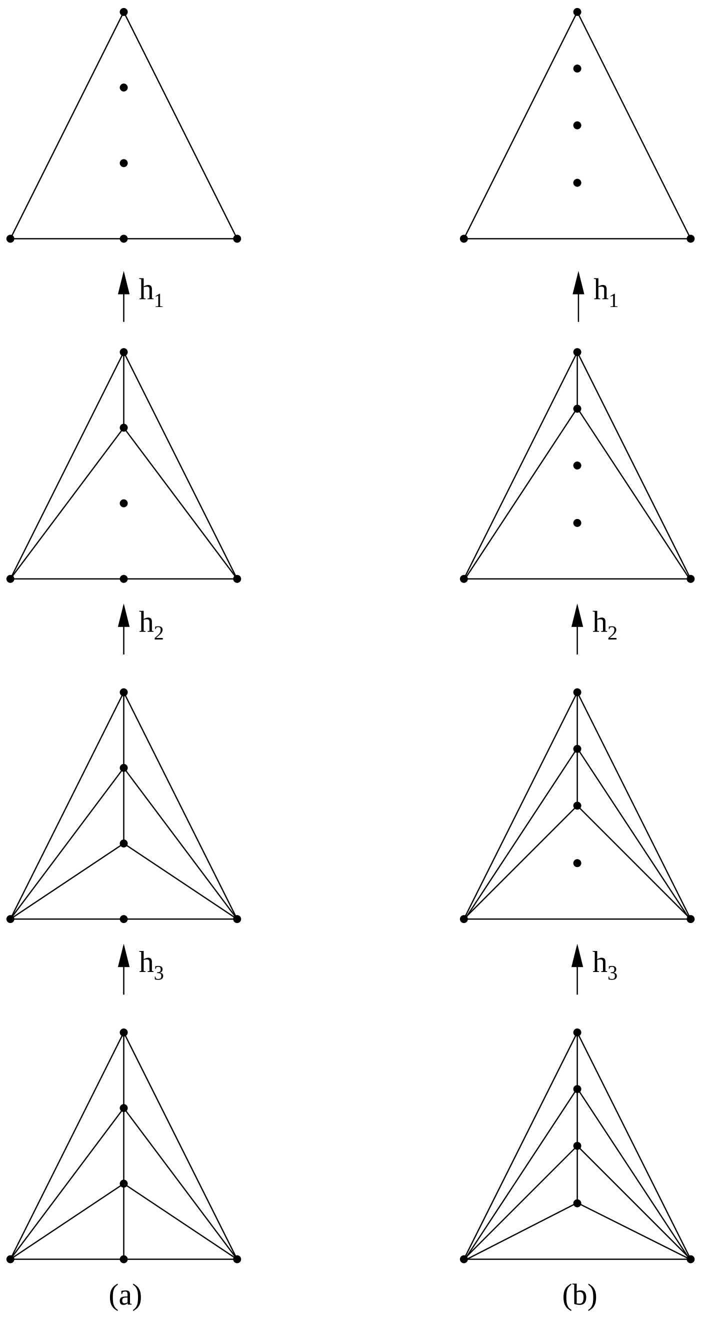


Fig. **12**

Remark 9.4. (i) Combining propositions 9.2 and 9.3 with corollary 3.4 one may obtain alternative proofs of the projectivity of $f = f_{\mathcal{T}}$.

(ii) The proofs of 9.2 and 9.3 work after minor modifications even if one omits the assumption for $\mathcal{T}$ to be basic.

Exercise 9.5. Under the assumption of prop. 9.2, determine a *single* $\mathcal{O}_{X(N_G,\Delta_G)}$-ideal sheaf $\mathcal{I}$, such that $\mathrm{Sing}(X(N_G,\Delta_G)) = \mathrm{supp}\big(\mathcal{O}_{X(N_G,\Delta_G)}/\mathcal{I}\big)$ and $f = f_{\mathcal{T}}$ itself is nothing but the normalized blow-up:

$$f : \mathrm{Norm}\left[\mathbf{Bl}^{\mathcal{I}}_{\mathrm{Sing}(X(N_G,\Delta_G))}\left(X\left(N_G,\Delta_G\left(\mathcal{T}\right)\right)\right)\right] \longrightarrow X\left(N_G,\Delta_G\right) = \mathbb{C}^r/G$$

of $X(N_G,\Delta_G)$ (cf. thm. 3.10 and rem. 7.2 (iii)).

10. Further remarks and a conjecture

As we already saw in 7.8 (ii), in dimension three, besides $\frac{1}{l}(1,1,l-2)$'s there is also another "new" Gorenstein, cyclic quotient singularity having a unique, projective, crepant resolution, namely $\frac{1}{7}(1,2,4)$. This can be generalized in arbitrary dimensions too!

Theorem 10.1. *The cyclic Gorenstein quotient singularity of type*

$$\boxed{\frac{1}{2^r-1}\left(1,2,2^2,2^3,\ldots,2^{r-2},2^{r-1}\right)}$$

can be fully resolved by a torus-equivariant projective crepant morphism in ***all*** *dimensions $r \geq 2$. Moreover, up to isomorphism, this resolution is* ***unique***.

The proof of theorem 10.1 will be given in [12]. As you guess, the required triangulation $\mathcal{T}$ will be the high-dimensional analogue of that of figure **9**. The details of the proof of the uniqueness of $\mathcal{T}$ and of the fact that it is indeed basic are somewhat lengthy, and involve binary representations, explicit Hilbert-basis determination and some tricks with determinants. The coherence of $\mathcal{T}$, on the other hand, can be shown directly by using tools from the theory of polytopes, i.e., by avoiding both patching lemma and factorization arguments.

Forgetting completely the uniqueness-property, we believe that this single singularity is again nothing but "the first member" of an infinite family of Gorenstein cyclic quotient singularities (10.1), called for simplicity r-dimensional *geometric progress singularity-series* of ratio k (in notation: $\mathrm{GPSS}(r;k)$), all of whose members admit the desired resolutions.

Conjecture 10.2 (GPSS($r;k$)-Conjecture). *All cyclic Gorenstein quotient singularities of type*

$$\boxed{\frac{1}{\left(\frac{k^r-1}{k-1}\right)}\ \left(1,k,k^2,k^3,\ldots,k^{r-2},k^{r-1}\right)} \tag{10.1}$$

admit torus-equivariant projective, crepant, full resolutions for ***all*** $r \geq 4$ *and* ***all*** $k \geq 2$.

Exercise 10.3. As a first approach to 10.2 (e.g. to GPSS($4;k$)-conjecture), consider the example $\frac{1}{40}\,(1,3,9,27)$ (with $k=3$) and normalize the blow-up of $X\,(N_G,\Delta_G)$ at $\mathrm{orb}(\sigma_0)$ (equipped with the reduced structure). What kind of triangulation of the junior tetrahedron $\mathfrak{s}_G$ will be induced by this procedure ? What would you expect as "next step" ? [Hint. Relate what you "see" with the singularities being studied in [11].]

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