# COMPUTING MINIMAL GENERATING SYSTEMS
# FOR SOME SPECIAL TORIC IDEALS

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ABSTRACT. Let $X_P$ be the projective toric surface associated to a lattice polytope $P$. If the number of lattice points lying on the boundary of $P$ is at least 4, it is known that $X_P$ is embeddable into a suitable projective space as zero set of finitely many quadrics. In this case, the determination of a minimal generating system of the toric ideal defining $X_P$ is reduced to a simple Gaussian elimination.

## 1. INTRODUCTION

Let $P \subset \mathbb{R}^2$ be a *lattice polygon,* i.e., a (convex, 2-dimensional) polygon, all of whose vertices belong to $\mathbb{Z}^2$. $P$ is known to be *normal* and *very ample*, and to have a canonical presentation

$$P = \left\{ \mathbf{x} \in \mathbb{R}^2 \, | \, \langle \mathbf{x}, \mathbf{u}_F \rangle \geq -a_F \text{ for all } F \in \mathcal{F}(P) \right\},$$

where $\mathcal{F}(P)$ is the set of the facets (edges) of $P$, $a_F \in \mathbb{Z}$ and $\mathbf{u}_F \in \mathbb{Z}^2$ the inward-pointing facet normal, i.e., the minimal generator of the ray $\mathbb{R}_{\geq 0}\mathbf{u}_F$. The corresponding compact complex toric surface $X_P$ is therefore normal and projective, and

$$D_P := \sum_{F \in \mathcal{F}(P)} a_F \, \overline{\mathrm{orb}(\mathbb{R}_{\geq 0}\mathbf{u}_F)}$$

is a very ample Cartier divisor on $X_P$, where $\overline{\mathrm{orb}(\mathbb{R}_{\geq 0}\mathbf{u}_F)}$ is the Zariski closure of the orbit of the ray $\mathbb{R}_{\geq 0}\mathbf{u}_F$ w.r.t. the natural $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^2, \mathbb{C}^*)$-action. (See [4, Corollaries 2.2.13 and 2.2.19 (b), pp. 70-71, (4.2.6), p. 182, and Proposition 6.1.10 (c), p. 269 ].) Setting $\delta_P := \sharp(P \cap \mathbb{Z}^2) - 1$, the complete linear system $|D_P|$ induces the closed embedding $\Phi_{|D_P|}$,

$$\mathbb{T} \overset{\iota}{\lhook\joinrel\longrightarrow} X_P \overset{\Phi_{|D_P|}}{\lhook\joinrel\longrightarrow} \mathbb{P}_{\mathbb{C}}^{\delta_P}$$

with

$$\mathbb{T} \ni t \longmapsto (\Phi_{|D_P|} \circ \iota)(t) := [... : z_{(i,j)} : ...]_{(i,j) \in P \cap \mathbb{Z}^2} \in \mathbb{P}_{\mathbb{C}}^{\delta_P}, \ z_{(i,j)} := \chi^{(i,j)}(t),$$

where $\chi^{(i,j)} : \mathbb{T} \to \mathbb{C}^*$ is the character associated to the lattice point $(i,j)$ (with $\mathbb{T}$ denoting the algebraic torus $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^2, \mathbb{C}^*)$), for all $(i,j) \in P \cap \mathbb{Z}^2$. The image $\Phi_{|D_P|}(X_P)$ of $X_P$ under $\Phi_{|D_P|}$ is the Zariski closure of $\mathrm{Im}(\Phi_{|D_P|} \circ \iota)$ in $\mathbb{P}_{\mathbb{C}}^{\delta_P}$ and can be viewed as the projective variety $\mathrm{Proj}(S_P)$, where

$$S_P := \mathbb{C}[C(P) \cap \mathbb{Z}^3] = \bigoplus_{\kappa=0}^{\infty} \left( \bigoplus_{(i,j) \in (\kappa P) \cap \mathbb{Z}^2} \mathbb{C} \cdot \chi^{(i,j)} s^{\kappa} \right)$$

(with $C(P) := \{(\lambda y_1, \lambda y_2, \lambda)\,|\,\lambda \in \mathbb{R}_{\geq 0}$ and $(y_1, y_2) \in P\})$ is the semigroup algebra which is naturally graded by setting $\deg(\chi^{(i,j)} s^{\kappa}) := \kappa$. (For a detailed exposition see [4, Theorem 2.3.1, p. 75; Proposition 5.4.7, pp. 237-238; Theorem 5.4.8, pp. 239-240, and Theorem 7.1.13, pp. 325-326].) Equivalently, it can be viewed as the zero set $\mathbb{V}(I_{\mathcal{A}_P}) \subset \mathbb{P}_{\mathbb{C}}^{\delta_P}$ of the homogeneous ideal $I_{\mathcal{A}_P} := \mathrm{Ker}(\pi_P)$, where

$$\mathcal{A}_P := \left\{(i, j, 1)\,\big|\,(i, j) \in P \cap \mathbb{Z}^2\right\} \subset \mathbb{Z}^2 \times \{1\} \subset \mathbb{Z}^3,$$

and $\pi_P$ is the $\mathbb{C}$-algebra homomorphism

$$\mathbb{C}[... : z_{(i,j)} : ....]_{(i,j)\in P\cap\mathbb{Z}^2} \xrightarrow{\pi_P} \mathbb{C}[..., \chi^{(i,j,1)}, ....]_{(i,j,1)\in\mathcal{A}_P}, \quad z_{(i,j)} \longmapsto \chi^{(i,j,1)}.$$

**Theorem 1.1** (Koelman [7]). *If $\sharp(\partial P \cap \mathbb{Z}^2) \geq 4$, then $I_{\mathcal{A}_P}$ is generated by all possible quadratic binomials, i.e.,*

$$I_{\mathcal{A}_P} = \left\langle \left\{z_{(i_1,j_1)}z_{(i_2,j_2)} - z_{(i'_1,j'_1)}z_{(i'_2,j'_2)} \,\middle|\, \begin{array}{l} (i_1,j_1),(i_2,j_2),(i'_1,j'_1),(i'_2,j'_2) \in P \cap \mathbb{Z}^2, \\ \text{with } (i_1,j_1)+(i_2,j_2)=(i'_1,j'_1)+(i'_2,j'_2) \end{array} \right\} \right\rangle.$$

**Corollary 1.2** (Castryck & Cools [2, §2]). *If $\sharp(\partial P \cap \mathbb{Z}^2) \geq 4$, and if we denote by $\beta_P$ the cardinality of any minimal system of quadrics generating the ideal $I_{\mathcal{A}_P}$, then*

$$\beta_P = \binom{\delta_P+2}{2} - \sharp(2P \cap \mathbb{Z}^2). \tag{1.1}$$

*Proof.* If $\mathrm{HP}_2(\mathbb{P}_{\mathbb{C}}^{\delta_P}) := \{\text{homogeneous polynomials (in } \delta_P + 1 \text{ variables) of degree 2}\}$, then the $\mathbb{C}$-vector space homomorphism

$$f : \mathrm{HP}_2(\mathbb{P}_{\mathbb{C}}^{\delta_P}) \longrightarrow \mathbb{C}\left[x^{\pm 1}, y^{\pm 1}\right], \text{ mapping } z_{(i_1,j_1)}z_{(i_2,j_2)} \text{ onto } x^{i_1+i_2}y^{j_1+j_2},$$

has as kernel $\mathrm{Ker}(f)$ the $\mathbb{C}$-vector space of homogeneous polynomials of degree 2 which belong to $I_{\mathcal{A}_P}$ and as image $\mathrm{Im}(f)$ the linear span of $\{x^iy^j\,|\,(i,j) \in 2P\cap\mathbb{Z}^2\}$ (because every lattice point in $2P$ is the sum of two lattice points of $P$, cf. [4, Theorem 2.2.12, pp. 68-69]). Taking into account Koelman's Theorem 1.1, [8, Lemma 4.1, p. 31], and the fact that $\mathbb{V}(I_{\mathcal{A}_P})$ is not contained in any hyperplane of $\mathbb{P}_{\mathbb{C}}^{\delta_P}$, the equality $\dim_{\mathbb{C}}(\mathrm{Ker}(f)) = \dim_{\mathbb{C}}(\mathrm{HP}_2(\mathbb{P}_{\mathbb{C}}^{\delta_P})) - \dim_{\mathbb{C}}(\mathrm{Im}(f))$ gives (1.1). $\square$

**Examples 1.3.** (i) If $a, b$ are two positive integers, then the projective toric surface $X_{P_{a,b}} \cong \mathbb{V}(I_{\mathcal{A}_{P_{a,b}}}) \subset \mathbb{P}_{\mathbb{C}}^{\delta_P}$ which is associated to the lattice quadrilateral

$$P_{a,b} := \mathrm{conv}(\{(0,0),(a,0),(b,1),(0,1)\})$$

(where "conv" stands for *convex hull*, and $\delta_{P_{a,b}} = a+b+1$), is isomorphic to the intersection of

$$\begin{aligned} \beta_{P_{a,b}} &= \binom{\delta_{P_{a,b}}+2}{2} - \sharp(2P_{a,b}\cap\mathbb{Z}^2) \\ &= \frac{(a+b+2)(a+b+3)}{2} - 3(a+b+1) = \tfrac{1}{2}(a+b-1)(a+b) \end{aligned}$$

quadrics, i.e., to the *rational normal scroll of type* $(a, b)$ w.r.t. the homogeneous coordinates $[... : z_{(i,j)} : ....]_{(i,j)\in P_{a,b}\cap\mathbb{Z}^2}$ satisfying the "2-minors condition"

$$\mathrm{rank}\begin{pmatrix} z_{(0,0)} & z_{(1,0)} & \cdots & z_{(a-1,0)} & z_{(0,1)} & z_{(1,1)} & \cdots & z_{(b-1,1)} \\ z_{(1,0)} & z_{(2,0)} & \cdots & z_{(a,0)} & z_{(1,1)} & z_{(2,1)} & \cdots & z_{(b,1)} \end{pmatrix} \leq 1.$$

In particular, for $a = b = 1$, $X_{P_{1,1}} \cong \mathbb{V}(z_{(0,0)}z_{(1,1)} - z_{(1,0)}z_{(0,1)}) \subset \mathbb{P}_{\mathbb{C}}^3$ can be viewed as the classical (smooth) *quadric hypersurface in* $\mathbb{P}_{\mathbb{C}}^3$ (which is isomorphic to $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ and birationally equivalent to $\mathbb{P}_{\mathbb{C}}^2$, cf. Figure 1).
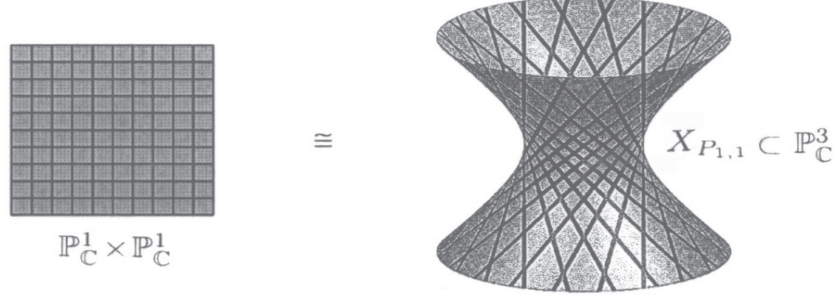
$$\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \qquad\qquad X_{P_{1,1}} \subset \mathbb{P}^3_{\mathbb{C}}$$

FIGURE 1.

(ii) Let $d$ be a positive integer. The $\mathbb{C}$-vector space

$$\mathbb{C}[\mathsf{X}_0, \mathsf{X}_1, \mathsf{X}_2]_d := \{F \in \mathbb{C}[\mathsf{X}_0, \mathsf{X}_1, \mathsf{X}_2] \,|\, F \text{ homogeneous of degree } d\} \cup \{\mathbf{0}\}$$

has the set $\{\mathsf{X}_0^{\alpha_0}\mathsf{X}_1^{\alpha_1}\mathsf{X}_2^{\alpha_2} \,|\, (\alpha_0, \alpha_1, \alpha_2) \in \mathcal{E}_{2,d}\}$ as one of its bases, where

$$\mathcal{E}_{2,d} := \left\{\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{Z}^3 \,|\, \alpha_0, \alpha_1, \alpha_2 \in [0, d] \text{ and } \alpha_0 + \alpha_1 + \alpha_2 = d\right\}.$$

For each $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \alpha_2) \in \mathcal{E}_{2,d}$ we write $\mathsf{X}^{\boldsymbol{\alpha}} := \mathsf{X}_0^{\alpha_0}\mathsf{X}_1^{\alpha_1}\mathsf{X}_2^{\alpha_2}$. Setting

$$\mathrm{Tr}_d := \mathrm{conv}(\{(0,0), (d,0), (0,d)\}) = \left\{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 \geq 0, x_2 \geq 0 \text{ and } x_1 + x_2 \leq d\right\}$$

we see that $\sharp(\partial \mathrm{Tr}_d \cap \mathbb{Z}^2) = 3d$ and $\sharp(\mathrm{Tr}_d \cap \mathbb{Z}^2) = \binom{d+2}{2}$, because

$$\mathrm{Tr}_d \cap \mathbb{Z}^2 \ni (m_1, m_2) \longmapsto (m_1+1, m_1+m_2+2) \in \left\{(\xi_1, \xi_2) \in \mathbb{Z}^2_{\geq 0} \,|\, 1 \leq \xi_1 < \xi_2 \leq d+2\right\}$$

is a bijective map with $\sharp\left\{(\xi_1, \xi_2) \in \mathbb{Z}^2_{\geq 0} \,|\, 1 \leq \xi_1 < \xi_2 \leq d+2\right\} = \binom{d+2}{2}$. On the other hand,

$$\sharp(\mathcal{E}_{2,d}) = \sharp\left\{(\alpha_0, \alpha_1, \alpha_2) \in \mathbb{Z}^3 \,|\, \alpha_0, \alpha_1, \alpha_2 \in [0, d] \text{ and } \alpha_0 + \alpha_1 + \alpha_2 \leq d\right\}$$

$$-\sharp\left\{(\alpha_0, \alpha_1, \alpha_2) \in \mathbb{Z}^3 \,|\, \alpha_0, \alpha_1, \alpha_2 \in [0, d-1] \text{ and } \alpha_0 + \alpha_1 + \alpha_2 \leq d-1\right\}$$

$$= \binom{d+3}{3} - \binom{d+2}{3} = \binom{d+2}{2} = \sharp(\mathrm{Tr}_d \cap \mathbb{Z}^2).$$

If $d \geq 2$, then using the homogeneous coordinates $[\ldots : z_{(i,j)} : \ldots]_{(i,j) \in \mathrm{Tr}_d \cap \mathbb{Z}^2}$ we conclude (by Koelman's Theorem 1.1) that

$$X_{\mathrm{Tr}_d} \cong \mathbb{V}(I_{\mathcal{A}_{\mathrm{Tr}_d}}) \subset \mathbb{P}^{\delta_{\mathrm{Tr}_d}}_{\mathbb{C}}, \text{ with } \delta_{\mathrm{Tr}_d} = \binom{d+2}{2} - 1, \text{ and}$$

$$I_{\mathcal{A}_{\mathrm{Tr}_d}} = \left\langle \left\{ z_{(i_1,j_1)}z_{(i_2,j_2)} - z_{(i'_1,j'_1)}z_{(i'_2,j'_2)} \,\left|\, \begin{array}{l} (i_1, j_1), (i_2, j_2), (i'_1, j'_1), (i'_2, j'_2) \in \mathrm{Tr}_d \cap \mathbb{Z}^2, \\ \text{with } (i_1, j_1) + (i_2, j_2) = (i'_1, j'_1) + (i'_2, j'_2) \end{array} \right. \right\} \right\rangle,$$

i.e., that $X_{\mathrm{Tr}_d}$ is isomorphic to the image of the so-called *d-uple Veronese embedding*

$$\nu_{2,d} : \mathbb{P}^2_{\mathbb{C}} \hookrightarrow \mathbb{P}^{\delta_{\mathrm{Tr}_d}}_{\mathbb{C}}, \ [\mathsf{X}_0 : \mathsf{X}_1 : \mathsf{X}_2] \longmapsto [\ldots : \mathsf{X}^{\boldsymbol{\alpha}} : \ldots]_{\boldsymbol{\alpha} \in \mathcal{E}_{2,d}}.$$

where the monomials $\{\mathsf{X}^{\boldsymbol{\alpha}} \,|\, \boldsymbol{\alpha} \in \mathcal{E}_{2,d}\}$ are arranged in a prescribed manner (e.g., lexicographically). In fact, in this case,

$$\beta_{\mathrm{Tr}_d} = \binom{\delta_{\mathrm{Tr}_d}+2}{2} - \sharp(2\mathrm{Tr}_d \cap \mathbb{Z}^2)$$

$$= \tfrac{1}{2}\binom{d+2}{2}\left(\binom{d+2}{2} + 1\right) - (2d^2 + 3d + 1)$$

$$= \tfrac{(d+1)(d+2)}{4}\left(\tfrac{(d+1)(d+2)}{2} + 1\right) - (2d^2 + 3d + 1) = \tfrac{1}{8}d\,(d+6)\left(d^2 - 1\right). \quad (1.2)$$

**Note 1.4.** For a `Magma` code for the computation of a minimal generating system of the ideal defining the projective toric surface associated to an *arbitrary* lattice polygon (and of much more, like Betti numbers etc.), see [3]. In the above mentioned particular case (in which we deal only with *quadrics*) it is enough (as we shall see in §2) to collect all vectorial relations $(i_1, j_1) + (i_2, j_2) = (i'_1, j'_1) + (i'_2, j'_2)$, and to determine a $\mathbb{C}$-linearly independent subset of the set of the corresponding quadratic binomials $z_{(i_1,j_1)} z_{(i_2,j_2)} - z_{(i'_1,j'_1)} z_{(i'_2,j'_2)}$ by simply performing Gaussian elimination.

## 2. THE ALGORITHM

An algorithm, implemented in `Python3` to compute a minimal generating set for the ideal $I_{\mathcal{A}_P}$, given the vertex set $\mathcal{V}(P)$ of the polygon $P$ is provided in the library `toricIdeal.py`. The algorithm is provided by the routine `minGenSet`, which receives a list of vertices and calls five subroutines to compute a minimal generating set of $I_{\mathcal{A}_P}$.

```python
import numpy
def minGenSet(p):
        intp=integerPoints(*vertToConst(p))
        (basis,genBin)=genBinom(intp)
        indepCol=findIndepCol(genBin)
        binom=findBinom(basis,genBin,indepCol)
        return(intp,binom)
```

More specifically, the first subroutine, `vertToConst`, produces a complete system of facet-defining inequalities from the list $\mathcal{V}(P)$ and lower and upper bounds for the coordinates of the points of the polygon. First, the facets are distinguished among the line segments connecting any two vertices, using the fact that the polygon lies entirely in one of the two closed half-planes bounded by their supporting lines and then a constraint is created for each facet.

```python
def vertToConst(p):
        A=[]
        b=[]
        for i in range(len(p)-1):
                for j in range(i+1,len(p)):
                        big=True
                        small=True
                        coef=numpy.array([[(p[i,1]-p[j,1]),
                                (p[j,0]-p[i,0])]])
                        cst=numpy.inner(coef,p[j])
                        for point in p:
                                diff=numpy.inner(coef,point)-cst
                                if diff > 0:
                                        small=False
                                elif diff <0:
                                        big=False
                if big :
                        A.append(coef)
                        b.append(cst)
                if small :
                        A.append(-coef)
                        b.append(-cst)
        A=numpy.array(A)
```

```
        b=numpy.array(b)
        x=[min(p[:,0]),max(p[:,0])]
        y=[min(p[:,1]),max(p[:,1])]
        return(A,b,x,y)
```

The second one, `integerPoints` uses the constraints and by brute force finds the lattice points of the polygon.

```
def integerPoints(A,b,x,y):
        intPoints=[]
        for x0 in range(x[0],x[1]+1):
                for y0 in range(y[0],y[1]+1):
                        point=numpy.array([x0,y0])
                        diff=numpy.dot(A,point)-b
                        inInterior=True
                        for coord in diff:
                                if coord<0:
                                        inInterior=False
                                        break
                        if inInterior:
                                intPoints.append(point)
        intPoints=numpy.array(intPoints)
        return(intPoints)
```

The third one `genBinom` orders the basis elements

$$B = \{z_{(i,j)} z_{(i',j')} : (i,j), (i',j') \in P \cap \mathbb{Z}^2\}$$

of the $\mathbb{C}$-vector space $\mathrm{HP}_2(\mathbb{P}_{\mathbb{C}}^{\delta_P})$ and finds a generating set for the ideal $I_{\mathcal{A}_P}$ by collecting all vectorial relations of the form

$$(i_1, j_1) + (i_2, j_2) = (i'_1, j'_1) + (i'_2, j'_2)$$

where $(i_1, j_1), (i_2, j_2), (i'_1, j'_1), (i'_2, j'_2) \in P \cap \mathbb{Z}^2$. This is returned as a matrix containing the coefficients of the generating binomials w.r.t. the ordered basis $B$.

```
def genBinom(intPoints):
        basis=[]
        for i in range(len(intPoints)):
                for j in range(i,len(intPoints)):
                        bp=intPoints[i].tolist()+intPoints[j].tolist()
                        basis.append(bp)
        basis=numpy.array(basis)
        genBin=[]
        for i in range(len(basis)):
                for j in range(i+1,len(basis)):
                        if basis[i,0]+basis[i,2]==basis[j,0]+basis[j
                                ,2] and basis[i,1]+basis[i,3]==basis[j,1]+
                                basis[j,3]:
                                row=numpy.zeros(len(basis),
                                dtype=numpy.int)
                                row[i]+=1
                                row[j]-=1
                                genBin.append(row)
        genBin=numpy.transpose(numpy.array(genBin))
        return(basis, genBin)
```

 The fourth one `findIndepCol` performs a Gauss elimination on a matrix and finds
a basis of its column space by collecting the non-zero columns of the row echelon
form of it.

```python
from scipy.linalg import lu
def findIndepCol(A):
        U=lu(A,permute_l=True)[1]
        indepCol=[]
        for i in range(len(U)):
                for j in range(len(U[i])):
                        if U[i,j]!=0:
                                indepCol.append(j)
                                break
        return(indepCol)
```

 Finally, the fifth subroutine `findBinom` uses and matrix given by `genBinom` and
the list of $\mathbb{C}$-linearly independent columns found by `findIndepCol` to produce a
set of $\mathbb{C}$-linearly independent generating binomials of the ideal $I_{\mathcal{A}_P}$.

```python
def findBinom(basis,genBin,indepCol):
        binom=[]
        for i in indepCol:
                j1=-1
                j2=-1
                for j in range(len(genBin)):
                        if genBin[j,i]==1:
                                j1=j
                        elif genBin[j,i]==-1:
                                j2=j
                        if j1!=-1 and j2!=-1:
                                break
                        binomial="z_{{({},{})}}z_{{({},{})}}-z_
                                {{({},{})}}z_{{({},{})}}".format(basis[j1
                                ][0],basis[j1][1],basis[j1][2],basis[j1
                                ][3],basis[j2][0],basis[j2][1],basis[j2
                                ][2],basis[j2][3])
        binom.append(binomial)
        return(binom)
```

 The complexity of the `minGenSet` is polynomial of the class $\mathcal{O}(m^4 + n^2)$, where $n$
is the number of vertices and $m$ an integer bounding absolutely the coordinates of
the vertices.

## 3. Applications

▶ **Veronese surfaces**. If $P = \mathrm{Tr}_2 := \mathrm{conv}(\{(0,0),(2,0),(0,2)\})$ (with $d = 2$ as in
1.3 (ii)), then the algorithm produces the following minimal generating set of $I_{\mathcal{A}_{\mathrm{Tr}_2}}$:

$$z_{(0,0)}z_{(2,0)} - z_{(1,0)}^2, \qquad z_{(0,0)}z_{(0,2)} - z_{(0,1)}^2,$$
$$z_{(0,0)}z_{(1,1)} - z_{(0,1)}z_{(1,0)}, \quad z_{(0,1)}z_{(1,1)} - z_{(0,2)}z_{(1,0)},$$
$$z_{(0,2)}z_{(2,0)} - z_{(1,1)}^2, \qquad z_{(0,1)}z_{(2,0)} - z_{(1,0)}z_{(1,1)}.$$

Analogously, if $P = \mathrm{Tr}_3 := \mathrm{conv}(\{(0,0),(3,0),(0,3)\})$ (with $d = 3$), then the 27
quadrics

$$
\begin{array}{lll}
z_{(1,1)}z_{(3,0)} - z_{(2,0)}z_{(2,1)}, & z_{(0,3)}z_{(2,1)} - z_{(1,2)}^2, & z_{(0,0)}z_{(1,2)} - z_{(0,2)}z_{(1,0)}, \\
z_{(0,3)}z_{(3,0)} - z_{(1,2)}z_{(2,1)}, & z_{(0,1)}z_{(3,0)} - z_{(1,0)}z_{(2,1)}, & z_{(0,1)}z_{(2,1)} - z_{(1,1)}^2, \\
z_{(0,0)}z_{(2,0)} - z_{(1,0)}^2, & z_{(0,2)}z_{(3,0)} - z_{(1,1)}z_{(2,1)}, & z_{(0,2)}z_{(2,1)} - z_{(0,3)}z_{(2,0)}, \\
z_{(0,2)}z_{(2,1)} - z_{(1,1)}z_{(1,2)}, & z_{(0,2)}z_{(3,0)} - z_{(1,2)}z_{(2,0)}, & z_{(0,1)}z_{(1,2)} - z_{(0,2)}z_{(1,1)}, \\
z_{(0,0)}z_{(2,1)} - z_{(1,0)}z_{(1,1)}, & z_{(0,0)}z_{(2,1)} - z_{(0,1)}z_{(2,0)}, & z_{(0,1)}z_{(2,1)} - z_{(0,2)}z_{(2,0)}, \\
z_{(0,0)}z_{(3,0)} - z_{(1,0)}z_{(2,0)}, & z_{(0,0)}z_{(0,2)} - z_{(0,1)}^2, & z_{(0,0)}z_{(1,1)} - z_{(0,1)}z_{(1,0)}, \\
z_{(0,1)}z_{(0,3)} - z_{(0,2)}^2, & z_{(1,0)}z_{(3,0)} - z_{(2,0)}^2, & z_{(0,1)}z_{(1,2)} - z_{(0,3)}z_{(1,0)}, \\
z_{(0,2)}z_{(1,2)} - z_{(0,3)}z_{(1,1)}, & z_{(1,2)}z_{(3,0)} - z_{(2,1)}^2, & z_{(0,1)}z_{(2,1)} - z_{(1,0)}z_{(1,2)}, \\
z_{(0,0)}z_{(0,3)} - z_{(0,1)}z_{(0,2)}, & z_{(0,1)}z_{(3,0)} - z_{(1,1)}z_{(2,0)}, & z_{(0,0)}z_{(1,2)} - z_{(0,1)}z_{(1,1)}
\end{array}
$$

generate minimally $I_{\mathcal{A}_{\mathrm{Tr}_3}}$ (cf. (1.2)).

▶ **Toric log del Pezzo surfaces**. These are of the form $X_P$, where $P = \ell\mathring{Q}$ is the polar of an LDP-polygon $Q \subset \mathbb{R}^2$ dilated by its index $\ell$. (An *LDP-polygon* $Q \subset \mathbb{R}^2$ is a convex polygon which contains the origin in its interior, and its vertices belong to $\mathbb{Z}^2$ and are primitive. The *index* of a polygon of this kind is defined to be

$$
\ell := \min\{\,\kappa \in \mathbb{Z}_{>0}\,|\,\mathcal{V}(\kappa\mathring{Q}) \subset \mathbb{Z}^2\,\}.
$$

Kasprzyk, Kreuzer & Nill [6, §6] developed an algorithm by means of which one creates an LDP-polygon, for given $\ell \geq 2$, by fixing a "special" edge and following a prescribed successive addition of vertices, and produced in this way the long lists of *all* LDP-polygons for $\ell \leq 17$. An explicit study for each of these 15346 LDP-polygons is available on the webpage [1].)

(i) Up to unimodular transformation the only *reflexive hexagon* (i.e., the only LDP-hexagon of index 1) is

$$
Q := \text{conv}\,(\{(0,1),(1,1),(1,0),(0,-1),(-1,-1),(-1,0)\})
$$

having

$$
\mathring{Q} := \text{conv}\,(\{(1,0),(1,-1),(0,-1),(-1,0),(-1,1),(0,1)\})
$$

as its polar, and $X_{\mathring{Q}} \cong \mathbb{V}(I_{\mathcal{A}_{\mathring{Q}}}) \subset \mathbb{P}^6_{\mathbb{C}}$ with $I_{\mathcal{A}_{\mathring{Q}}}$ minimally generated by the 9 quadrics:

$$
\begin{array}{lll}
z_{(-1,0)}z_{(1,-1)} - z_{(0,-1)}z_{(0,0)}, & z_{(-1,0)}z_{(1,0)} - z_{(0,0)}^2, & z_{(-1,0)}z_{(1,0)} - z_{(-1,1)}z_{(1,-1)}, \\
z_{(-1,0)}z_{(0,0)} - z_{(-1,1)}z_{(0,-1)}, & z_{(-1,0)}z_{(0,1)} - z_{(-1,1)}z_{(0,0)}, & z_{(0,-1)}z_{(1,0)} - z_{(0,0)}z_{(1,-1)}, \\
z_{(-1,0)}z_{(1,0)} - z_{(0,-1)}z_{(0,1)}, & z_{(-1,1)}z_{(1,0)} - z_{(0,0)}z_{(0,1)}, & z_{(0,0)}z_{(1,0)} - z_{(0,1)}z_{(1,-1)}.
\end{array}
$$

(ii) For the LDP-triangle $Q$ *of index* 2 with vertex set

$$
\mathcal{V}(Q) := \{(0,1),(8,1),(-4,-1)\}
$$

we obtain

$$
\mathcal{V}(2\mathring{Q}) = \{(1,-2),(0,-2),(-1,6)\}(\text{cf. Fig. 2})
$$

and $X_{2\mathring{Q}} \cong \mathbb{V}(I_{\mathcal{A}_{2\mathring{Q}}}) \subset \mathbb{P}^6_{\mathbb{C}}$ with $I_{\mathcal{A}_{2\mathring{Q}}}$ minimally generated by the 7 quadrics:

$$
\begin{array}{ll}
z_{(-1,6)}z_{(1,-2)} - z_{(0,2)}^2, & z_{(0,0)}z_{(0,2)} - z_{(0,1)}z_{(0,1)}, \\
z_{(0,-1)}z_{(0,2)} - z_{(0,0)}z_{(0,1)}, & z_{(0,-2)}z_{(0,2)} - z_{(0,0)}^2, \\
z_{(0,-2)}z_{(0,2)} - z_{(0,-1)}z_{(0,1)}, & z_{(0,-2)}z_{(0,0)} - z_{(0,-1)}^2, \\
z_{(0,-2)}z_{(0,1)} - z_{(0,-1)}z_{(0,0)}.
\end{array}
$$

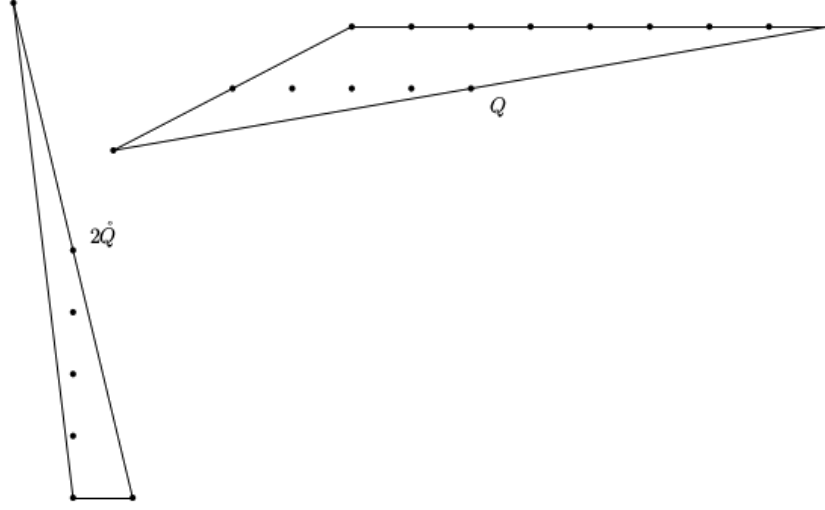FIGURE 2.

(iii) For the LDP-pentagon $Q$ *of index* 3 with vertex set
$$\mathcal{V}(Q) := \{(0,1),(1,1),(1,0),(-2,-1),(-3,-1)\}$$
(which is unimodularly equivalent to the pentagon "$Q_2^{[3]}$" of [5]) we obtain
$$\mathcal{V}(3\mathring{Q}) = \{(2,-3),(0,-3),(-3,0),(-3,-9),(0,3)\},$$
and $X_{3\mathring{Q}} \cong \mathbb{V}(I_{\mathcal{A}_{3\mathring{Q}}}) \subset \mathbb{P}_{\mathbb{C}}^{38}$ with $I_{\mathcal{A}_{3\mathring{Q}}}$ minimally generated by a set of 646 quadrics!

(iv) For the LDP-quadrilateral $Q$ *of index* 4 with vertex set
$$\mathcal{V}(Q) := \{(-1,2),(3,2),(-1,-1),(-3,-2)\}$$
we obtain $\mathcal{V}(4\mathring{Q}) = \{(2,-1),(0,-2),(-12,16),(-4,8)\}$, and $X_{4\mathring{Q}} \cong \mathbb{V}(I_{\mathcal{A}_{4\mathring{Q}}}) \subset \mathbb{P}_{\mathbb{C}}^{45}$ with $I_{\mathcal{A}_{4\mathring{Q}}}$ minimally generated by a set of 918 quadrics!

(v) Finally, the LDP-triangle $Q$ *of index* 5 with vertex set
$$\mathcal{V}(Q) := \{(0,1),(15,1),(-15,-2)\}$$
we obtain $\mathcal{V}(5\mathring{Q}) = \{(1,-5),(0,-5),(-1,10)\}$, and $X_{5\mathring{Q}} \cong \mathbb{V}(I_{\mathcal{A}_{5\mathring{Q}}}) \subset \mathbb{P}_{\mathbb{C}}^{9}$ with $I_{\mathcal{A}_{5\mathring{Q}}}$ minimally generated by the following 21 quadrics:

$z_{(0,-5)}z_{(0,-1)} - z_{(0,-3)}^2,$  $\quad$ $z_{(0,-5)}z_{(0,1)} - z_{(0,-4)}z_{(0,0)},$  $\quad$ $z_{(0,-5)}z_{(0,0)} - z_{(0,-3)}z_{(0,-2)},$

$z_{(0,-5)}z_{(0,2)} - z_{(0,-4)}z_{(0,1)},$  $\quad$ $z_{(0,-5)}z_{(0,2)} - z_{(0,-3)}z_{(0,0)},$  $\quad$ $z_{(0,-3)}z_{(0,2)} - z_{(0,-2)}z_{(0,1)},$

$z_{(0,-4)}z_{(0,2)} - z_{(0,-2)}z_{(0,0)},$  $\quad$ $z_{(0,-2)}z_{(0,2)} - z_{(0,-1)}z_{(0,1)},$  $\quad$ $z_{(0,-5)}z_{(0,-1)} - z_{(0,-4)}z_{(0,-2)},$

$z_{(0,-5)}z_{(0,1)} - z_{(0,-3)}z_{(0,-1)},$  $\quad$ $z_{(0,-5)}z_{(0,-3)} - z_{(0,-4)}^2,$  $\quad$ $z_{(0,-4)}z_{(0,2)} - z_{(0,-1)}^2,$

$z_{(0,-5)}z_{(0,0)} - z_{(0,-4)}z_{(0,-1)},$  $\quad$ $z_{(0,-5)}z_{(0,1)} - z_{(0,-2)}^2,$  $\quad$ $z_{(0,-1)}z_{(0,2)} - z_{(0,0)}z_{(0,1)},$

$z_{(0,-2)}z_{(0,2)} - z_{(0,0)}^2,$  $\quad$ $z_{(0,0)}z_{(0,2)} - z_{(0,1)}^2,$  $\quad$ $z_{(0,-5)}z_{(0,2)} - z_{(0,-2)}z_{(0,-1)},$

$z_{(0,-5)}z_{(0,-2)} - z_{(0,-4)}z_{(0,-3)},$  $\quad$ $z_{(0,-3)}z_{(0,2)} - z_{(0,-1)}z_{(0,0)},$  $\quad$ $z_{(0,-4)}z_{(0,2)} - z_{(0,-3)}z_{(0,1)}.$

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