All Toric L.C.I.-Singularities Admit Projective Crepant Resolutions

Dimitrios I. Dais*, **Christian Haase**$^\diamond$ **and Günter M. Ziegler**$^\diamond$

*Mathematische Fakultät, Universität Tübingen, Auf der Morgenstelle 10, D-72076 Tübingen, Germany
e-mail: dais@wolga.mathematik.uni-tuebingen.de

$^\diamond$Fachbereich Mathematik, TU-Berlin, Sekr. MA 7.1, Straße des 17. Juni 136, D-10623 Berlin, Germany

e-mail: [haase, ziegler]@math.tu-berlin.de

Abstract

It is known that the underlying spaces of all abelian quotient singularities which are embeddable as complete intersections of hypersurfaces in an affine space can be overall resolved by means of projective torus-equivariant crepant birational morphisms in all dimensions. In the present paper we extend this result to the entire class of toric l.c.i.-singularities. Our proof makes use of Nakajima's classification theorem and of some special techniques from toric and discrete geometry.

1 Introduction

In the past two decades "crepant" birational morphisms were mainly used in algebraic geometry to reduce the canonical singularities of algebraic (not necessarily proper) d-folds, $d \geq 3$, to $\mathbb{Q}$-factorial terminal singularities, and to treat *minimal models* in high dimensions. From the late eighties onwards, crepant desingularizations $\widehat{Y} \longrightarrow Y$ of projective varieties Y with trivial dualizing sheaf and well-controllable singularities play also a crucial role in producing *Calabi-Yau manifolds*, which serve as internal target spaces for "non-linear supersymmetric sigma models" in the framework of the physical string-theory. This explains the recent mathematical interest in both local and global versions of the existence-problem of *smooth* birational models of such Y's.

• **The local problem.** This was at first focused on the high-dimensional "McKay correspondence" for the underlying spaces $\mathbb{C}^d/G$, $G \subset \mathrm{SL}(d,\mathbb{C})$, $d \geq 2$, of the Gorenstein quotient singularities, connecting the irreducible representations of G (or dually, the conjugacy classes of G), on the one hand, and the cohomology ring of the overlying spaces $\widehat{X}$ of (preferably projective), crepant, full desingularizations $\widehat{X} \longrightarrow X = \mathbb{C}^d/G$ of X, on the other (cf. [3, 37]). The problem setting was partially extended by proving that a one-to-one correspondence of McKay-type is true, too, for torus-equivariant, crepant, full desingularizations $\widehat{X} \longrightarrow X = U_\sigma$ of the underlying spaces of all Gorenstein toric singularities [5, §4]. As it turned out, the non-trivial (even) cohomology groups of $\widehat{X}$'s have the "expected" dimensions, depending on the "weights" (also called "ages" in [24]) of the conjugacy classes of the acting groups, and on the Ehrhart polynomial of the corresponding lattice polytopes, respectively, and being therefore *independent* of particular choices of representatives among these $\widehat{X}$'s. (See [3, Thm. 8.4] and [5, Thm. 4.4]). We should particularly note that, in both cases, $\widehat{X}$'s of this sort always exist in dimension $d \leq 3$. However, if $d \geq 4$, this is not always possible because even within their common class

$$\left\{ \begin{array}{c} \text{Gorenstein abelian} \\ \text{quotient singularities} \end{array} \right\} = \left\{ \begin{array}{c} \text{Gorenstein quotient} \\ \text{singularities} \end{array} \right\} \bigcap \left\{ \begin{array}{c} \text{Gorenstein toric} \\ \text{singularities} \end{array} \right\} \tag{1.1}$$

there are mostly terminal singularities (see Morrison-Stevens [31]). Hence, the high-dimensional bijections of McKay-type make sense only in combination with the following:

Question 1.1 (Existence Problem) *Under which conditions* (*or restrictions*) *on our starting-point data for these two classes of Gorenstein singularities do projective, crepant, full resolutions exist in dimensions* ≥ 4 ?

First answers via toric geometry for the case of Gorenstein abelian quotient singularities (1.1) were given in [10, 11, 12, 13]. To our surprise, the number of families of these singularities admitting resolutions of such special kind is *not* negligible as one would at first sight expect.

• **The global problem.** A wide class of CY-manifolds of particular interest is that one being constructible by resolving overall the (necessarily Gorenstein) singularities of the so-called *CY-varieties* via suitable projective crepant morphisms. In the case in which the singularities of such a CY-variety Y are of "mild nature" (like quotient or toroidal singularities), and as long as an appropriate stratification of the singular locus $\mathrm{Sing}(Y)$ of Y is available, the existence-problem of crepant full resolutions can be mostly reduced to the local one by performing standard glueing procedures. (In contrast to this, the conditions which would guarantee the preservation of the projectivity of the desingularizing morphisms globally are much more complicated and require additional information about the global geometry of Y). It is worth mentioning that also the Hodge numbers $h^{p,q}(\widehat{Y})$ of the overlying spaces of *all* crepant, full, global desingularizations $\widehat{Y} \longrightarrow Y$ of Y remain *invariant* (see Kontsevich [26] and Batyrev [2]). A method of working *formally* with Y's, even without assuming the existence of such special $\widehat{Y}$'s in dimensions ≥ 4, consists in introducing the so-called *string-theoretic* Hodge numbers $h^{p,q}_{\mathrm{str}}(Y)$ for Y's (cf. [5, 4]). Since as yet there is only a conjectural description of a candidate for the cohomology complex which probably leads to a mathematical definition of the "string-theoretic cohomology theory" globally (cf. [4, 4.4] and Borisov's new approach in [7, §4], [8, Conj. 9.23]), it would be important to know at least *when* the existence of *smooth* birational models for Y's is feasible or not.

• **The L.C.I.'s.** In the present paper we shall exclusively deal with one aspect of the local problem. We believe that a purely algebraic, *sufficient* condition for the existence of the desired resolutions in *all* dimensions is to require from our singularities to be, in addition, *l.c.i.'s*. In the toric category, where the Question 1.1 can be translated into a question concerning the existence of specific lattice triangulations of lattice polytopes, this conjecture was verified for abelian quotient singularities in [12] via Kei-ichi Watanabe's Theorem [43]. (For non-abelian groups acting on $\mathbb{C}^d$, it remains open). Furthermore, the authors of [12, cf. §8(iii)] asked for geometric analogues of the "joins" and "dilations" occuring in their Reduction Theorem also for toric non-quotient l.c.i.-singularities. As we shall see below, such a characterization (in a somewhat different context) **is** indeed possible by making use of another beautiful classification theorem due to Haruhisa Nakajima [32], which generalizes Watanabe's results to the entire class of toric l.c.i.'s. Based on this classification we prove the following:

Theorem 1.2 (Main Theorem) *The underlying spaces of all toric l.c.i.-singularities admit torus-equivariant, projective, crepant, full resolutions* (*i.e.,* "*smooth minimal models*") *in all dimensions.*

The proof of 1.2 relies on considerably simpler techniques than those of [12], basically because the vertices of the Nakajima's polytopes reside in the standard *rectangular* lattice within $\mathbb{R}^d$. Nevertheless, Watanabe's forests and *skew* lattices remain the right language if one wishes to read off the weights of *abelian group* actions by predeterminated eigencoordinates and diagonalizations in a direct manner. On the other hand, the common distinctive feature in both proofs is an *inductive* argument which makes things work in *all* dimensions.

• This paper is organized as follows: After recalling the algebraic hierarchy of singularities (see (1.2) below), and some basic notions and facts from toric geometry in §2, we explain in §3 why the existence of the desired desingularizations is equivalent to the existence of b.c.-triangulations of the lattice polytopes supporting the Gorenstein cones. Moreover, we give two first examples of lattice polytopes (namely the so-called Fano and $\mathbb{H}_d$-compatible polytopes) admitting such triangulations, and describe the corresponding exceptional prime divisors explicitly. In section 4 we provide convenient reformulations of Nakajima's classification. In §5 we give the proof of Main Theorem 1.2 by using certain maximal coherent triangulations, combined with the "Key-Lemma" 5.7 which guarantees their "basicness". An immediate algebraic application of 1.2 is contained in the second part of section 5, where it is shown that the monoidal "coordinate rings" $\mathbb{C}\left[\tau_P \cap \mathbb{Z}^d\right]$ of $U_{\tau_P^\vee}$'s for all Nakajima polytopes P have the Koszul-property. In §6 we present a simple method of computing the non-trivial cohomology group dimensions of the overlying spaces of all crepant, full resolutions of toric l.c.i.-singularities. Finally, in section 7 we apply our results for two "extreme" classes of toric g.c.i.-singularities which occur as direct generalizations of the classical A_{k-1}-singularities in arbitrary dimensions.

• **General terminology.** (a) First we recall some fundamental definitions from commutative algebra (cf. [27, 30]). Let R be a commutative ring with 1. The *height* $\mathrm{ht}(\mathfrak{p})$ of a prime ideal $\mathfrak{p}$ of R is the supremum of the lengths of all prime ideal chains which are contained in $\mathfrak{p}$, and the *dimension* of R is defined to be $\dim(R) := \sup\{\mathrm{ht}\,(\mathfrak{p})\,|\,\mathfrak{p}$ prime ideal of $R\}$. R is *Noetherian* if any ideal of it has a finite system of generators. R is a *local ring* if it is endowed with a *unique* maximal ideal $\mathfrak{m}$. A local ring R is *regular* (resp. *normal*) if $\dim(R) = \dim\left(\mathfrak{m}/\mathfrak{m}^2\right)$ (resp. if it is an integral domain and is integrally closed in its field of fractions). A finite sequence $a_1, \ldots, a_\nu$ of elements of a ring R is defined to be a *regular sequence* if a_1 is not a zero-divisor in R and for all i, $i = 2, \ldots, \nu$, a_i is not a zero-divisor of $R/\langle a_1, \ldots, a_{i-1}\rangle$. A Noetherian local ring R (with maximal ideal $\mathfrak{m}$) is *Cohen-Macaulay* if $\mathrm{depth}(R) = \dim(R)$, where the *depth* of R is defined to be the maximum of the lengths of all regular sequences whose members belong to $\mathfrak{m}$. A Cohen-Macaulay local ring R is *Gorenstein* if $\mathrm{Ext}_R^{\dim(R)}\,(R/\mathfrak{m}, R) \cong R/\mathfrak{m}$. A Noetherian local ring R is said to be a *complete intersection* if there exists a regular local ring R', such that $R \cong R'/\langle f_1, \ldots, f_q\rangle$ for a finite set of elements $\{f_1, \ldots, f_q\} \subset R'$ whose cardinality equals $q = \dim(R') - \dim(R)$. The hierarchy by inclusion of the above types of Noetherian local rings is known to be described by the following diagram:

$$\begin{array}{ccc}
\{\text{Noetherian local rings}\} & \supset & \{\text{normal local rings}\} \\
\cup & & \cup \\
\{\text{Cohen-Macaulay local rings}\} & & \{\text{regular local rings}\} \\
\cup & & \cap \\
\{\text{Gorenstein local rings}\} & \supset & \{\text{complete intersections ("c.i.'s")}\}
\end{array} \tag{1.2}$$

(b) An arbitrary Noetherian ring R and its associated affine scheme $\mathrm{Spec}(R)$ are called Cohen-Macaulay, Gorenstein, normal or regular, respectively, iff all the localizations $R_\mathfrak{m}$ with respect to all the members $\mathfrak{m} \in \text{Max-Spec}(R)$ of the maximal spectrum of R are of this type. In particular, if the $R_\mathfrak{m}$'s for all maximal ideals $\mathfrak{m}$ of R are c.i.'s, then one often says that R is a *local complete intersection* ("l.c.i.") to distinguish it from the "global" ones. (A *global complete intersection* ("g.c.i.") is defined to be a ring R of finite type over a field $\mathbf{k}$ (i.e., an affine $\mathbf{k}$-algebra), such that $R \cong \mathbf{k}\left[\mathsf{T}_1.., \mathsf{T}_d\right] \,/\, \langle \varphi_1\left(\mathsf{T}_1, .., \mathsf{T}_d\right), .., \varphi_q\left(\mathsf{T}_1, .., \mathsf{T}_d\right)\rangle$ for q polynomials $\varphi_1, \ldots, \varphi_q$ from $\mathbf{k}\left[\mathsf{T}_1, .., \mathsf{T}_d\right]$ with $q = d-\dim(R)$, cf. [23, 32]). Hence, the above inclusion hierarchy can be generalized for all Noetherian rings, just by omitting in (1.2) the word "local" and by substituting l.c.i.'s for c.i.'s.

(c) Throughout the paper we consider only *complex varieties* $(X, \mathcal{O}_X)$, i.e., integral separated schemes of finite type over $\mathbf{k} = \mathbb{C}$; thus, the punctual algebraic behaviour of X is determined by the stalks $\mathcal{O}_{X,x}$ of its structure sheaf $\mathcal{O}_X$, and X itself is said to have a given algebraic property (as in (b)) whenever all $\mathcal{O}_{X,x}$'s have the analogous property from (1.2) for all $x \in X$. Furthermore, via the GAGA-correspondence ([38], [20, §2]) which preserves the above quoted algebraic properties, we shall always work within the *analytic category* by using the so-called antiequivalence principle [19], i.e., the usual contravariant functor $(X, x) \rightsquigarrow \mathcal{O}_{X,x}^{\mathrm{hol}}$ between the category of isomorphy classes of germs of X and the corresponding category of isomorphy classes of analytic local rings at the marked points x).

(d) For a complex variety X, we denote by $\mathrm{Sing}(X) = \left\{x \in X \;\middle|\; \mathcal{O}_{X,x}^{\mathrm{hol}} \text{ is a non-regular local ring}\right\}$ its singular locus. By a *desingularization* (or *resolution of singularities*) $f : \widehat{X} \to X$ of a non-smooth X, we mean a "full" or "overall" desingularization (if not mentioned), i.e., $\mathrm{Sing}(\widehat{X}) = \varnothing$. When we deal with *partial* desingularizations, we mention it explicitly. A partial desingularization $f : X' \to X$ of a normal, Gorenstein complex variety X is called *non-discrepant* or simply *crepant*, if the (up to rational equivalence uniquely determined) difference $K_{X'} - f^*\left(K_X\right)$ vanishes. (K_X and $K_{X'}$ denote here canonical divisors of X and X', respectively). Furthermore, $f : X' \to X$ is *projective* if X' admits an f-ample Cartier divisor.

2 Some basic facts from toric geometry

In this section we introduce the brief toric glossary (a)-(k) and the notation which will be used in the subsequent sections. For further details the reader is referred to the textbooks of Oda [33], Fulton [17] and Ewald [16], and to the lecture notes [25].

(a) The *linear hull,* the *affine hull*, the *positive hull* and *the convex hull* of a set B of vectors of $\mathbb{R}^r$, $r \geq 1$, will be denoted by $\text{lin}(B)$, $\text{aff}(B)$, $\text{pos}(B)$ (or $\mathbb{R}_{\geq 0}\, B$) and $\text{conv}(B)$, respectively. The *dimension* $\dim(B)$ of a $B \subset \mathbb{R}^r$ is defined to be the dimension of its affine hull.

(b) Let N be a free $\mathbb{Z}$-module of rank $r \geq 1$. N can be regarded as a *lattice* in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r$. An $n \in N$ is called *primitive* if $\text{conv}(\{\mathbf{0}, n\}) \cap N$ contains no other points except $\mathbf{0}$ and n.

Let N be as above, $M := \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual lattice, $N_{\mathbb{R}}, M_{\mathbb{R}}$ their real scalar extensions, and $\langle .,. \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ the natural $\mathbb{R}$-bilinear pairing. A subset σ of $N_{\mathbb{R}}$ is called *convex polyhedral cone* (*c.p.c.*, for short) if there exist $n_1, \ldots, n_k \in N_{\mathbb{R}}$, such that $\sigma = \text{pos}(\{n_1, \ldots, n_k\})$. Its *relative interior* $\text{int}(\sigma)$ is the usual topological interior of it, considered as subset of $\text{lin}(\sigma) = \sigma + (-\sigma)$. The *dual cone* $\sigma^\vee$ of a c.p.c. σ is a c.p. cone defined by

$$\sigma^\vee := \{\mathbf{y} \in M_{\mathbb{R}} \mid \langle \mathbf{y}, \mathbf{x} \rangle \geq 0, \ \forall \mathbf{x}, \ \mathbf{x} \in \sigma\} \ .$$

Note that $(\sigma^\vee)^\vee = \sigma$ and $\dim(\sigma \cap (-\sigma)) + \dim(\sigma^\vee) = \dim(\sigma^\vee \cap (-\sigma^\vee)) + \dim(\sigma) = r$. A subset τ of a c.p.c. σ is called a *face* of σ (notation: $\tau \prec \sigma$), if $\tau = \{\mathbf{x} \in \sigma \mid \langle m_0, \mathbf{x} \rangle = 0\}$, for some $m_0 \in \sigma^\vee$. A c.p.c. $\sigma = \text{pos}(\{n_1, \ldots, n_k\})$ is called *simplicial* (resp. *rational*) if $n_1, \ldots, n_k$ are $\mathbb{R}$-linearly independent (resp. if $n_1, \ldots, n_k \in N_{\mathbb{Q}}$, where $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$). A *strongly convex polyhedral cone* (*s.c.p.c.*, for short) is a c.p.c. σ for which $\sigma \cap (-\sigma) = \{\mathbf{0}\}$, i.e., for which $\dim(\sigma^\vee) = r$. The s.c.p. cones are alternatively called *pointed cones* (having $\mathbf{0}$ as their apex).

(c) If $\sigma \subset N_{\mathbb{R}}$ is a rational c.p. cone, then the subsemigroup $\sigma \cap N$ of N is a monoid. The following proposition is due to Gordan, Hilbert and van der Corput and describes its fundamental properties.

Proposition 2.1 (Minimal generating system) *$\sigma \cap N$ is finitely generated as additive semigroup. Moreover, if σ is strongly convex, then among all the systems of generators of $\sigma \cap N$, there is a system $\mathbf{Hilb}_N(\sigma)$ of* minimal cardinality, *which is uniquely determined* (*up to the ordering of its elements*) *by the following characterization*:

$$\mathbf{Hilb}_N(\sigma) = \left\{ n \in \sigma \cap (N \smallsetminus \{\mathbf{0}\}) \;\middle|\; \begin{array}{l} n \text{ cannot be expressed as the sum of two} \\ \text{other vectors belonging to } \ \sigma \cap (N \smallsetminus \{\mathbf{0}\}) \end{array} \right\} \tag{2.1}$$

$\mathbf{Hilb}_N(\sigma)$ is called *the Hilbert basis of σ w.r.t. N.*

(d) For a lattice N of rank r having M as its dual, we define an r-dimensional *algebraic torus* $T_N \cong (\mathbb{C}^*)^r$ by setting $T_N := \text{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^*$. Every $m \in M$ assigns a character $\mathbf{e}(m) : T_N \to \mathbb{C}^*$. Moreover, each $n \in N$ determines an 1-parameter subgroup

$$\vartheta_n : \mathbb{C}^* \to T_N \quad \text{with} \quad \vartheta_n(\lambda)(m) := \lambda^{\langle m,n \rangle}, \quad \text{for} \quad \lambda \in \mathbb{C}^*, \ m \in M \ .$$

We can therefore identify M with the character group of T_N and N with the group of 1-parameter subgroups of T_N. On the other hand, for a rational s.c.p.c. σ with $M \cap \sigma^\vee = \mathbb{Z}_{\geq 0}\, m_1 + \cdots + \mathbb{Z}_{\geq 0}\, m_\nu$, we associate to the finitely generated monoidal subalgebra $\mathbb{C}[M \cap \sigma^\vee] = \oplus_{m \in M \cap \sigma^\vee} \mathbf{e}(m)$ of the $\mathbb{C}$-algebra $\mathbb{C}[M] = \oplus_{m \in M} \mathbf{e}(m)$ an affine complex variety

$$U_\sigma := \text{Max-Spec}(\mathbb{C}[M \cap \sigma^\vee]) ,$$

which can be identified with the set of semigroup homomorphisms :

$$U_\sigma = \left\{ u : M \cap \sigma^\vee \ \to \mathbb{C} \;\middle|\; \begin{array}{c} u(\mathbf{0}) = 1, \ u(m + m') = u(m) \cdot u(m') , \\ \text{for all} \quad m, m' \in M \cap \sigma^\vee \end{array} \right\} ,$$

where $\mathbf{e}(m)(u) := u(m) , \ \forall m, \ m \in M \cap \sigma^\vee$ and $\forall u, \ u \in U_\sigma$.

Proposition 2.2 (Embedding by binomials) *In the analytic category, U_σ, identified with its image under the injective map* $(\mathbf{e}(m_1), \ldots, \mathbf{e}(m_\nu)) : U_\sigma \hookrightarrow \mathbb{C}^\nu$*, can be regarded as an analytic set determined by a system of equations of the form*: (monomial) = (monomial). *This analytic structure induced on U_σ is independent of the semigroup generators $\{m_1, \ldots, m_\nu\}$ and each map $\mathbf{e}(m)$ on U_σ is holomorphic w.r.t. it. In particular, for $\tau \prec \sigma$, U_τ is an open subset of U_σ. Moreover, if σ is r-dimensional and $\#(\mathbf{Hilb}_M(\sigma^\vee)) = k$ $(\leq \nu)$, then k is nothing but the* embedding dimension *of U_σ, i.e. the minimal number of generators of the maximal ideal of the local $\mathbb{C}$-algebra $\mathcal{O}^{hol}_{U_\sigma,\ \mathbf{0}}$.*

Proof. See Oda [33, Prop. 1.2 and 1.3., pp. 4-7]. □

(e) A *fan* w.r.t. a free $\mathbb{Z}$-module N is a finite collection Δ of rational s.c.p. cones in $N_\mathbb{R}$, such that :

(i) any face τ of $\sigma \in \Delta$ belongs to Δ, and

(ii) for $\sigma_1, \sigma_2 \in \Delta$, the intersection $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

By $|\Delta| := \cup\{\sigma \mid \sigma \in \Delta\}$ one denotes the *support* and by $\Delta(i)$ the set of all i-dimensional cones of a fan Δ for $0 \leq i \leq r$. If $\varrho \in \Delta(1)$ is a ray, then there exists a unique primitive vector $n(\varrho) \in N \cap \varrho$ with $\varrho = \mathbb{R}_{\geq 0}\, n(\varrho)$ and each cone $\sigma \in \Delta$ can be therefore written as

$$\sigma = \sum_{\varrho \in \Delta(1),\ \varrho \prec \sigma} \mathbb{R}_{\geq 0}\, n(\varrho) \quad .$$

The set $\mathrm{Gen}(\sigma) := \{n(\varrho) \mid \varrho \in \Delta(1), \varrho \prec \sigma\}$ is called the *set of minimal generators* (within the pure first skeleton) of σ. For Δ itself one defines analogously $\mathrm{Gen}(\Delta) := \bigcup_{\sigma \in \Delta} \mathrm{Gen}(\sigma)$.

(f) The *toric variety* $X(N, \Delta)$ associated to a fan Δ w.r.t. the lattice N is by definition the identification space

$$X(N, \Delta) := \Big(\Big(\coprod_{\sigma \in \Delta} U_\sigma\Big) \,/ \sim\Big) \tag{2.2}$$

with $U_{\sigma_1} \ni u_1 \sim u_2 \in U_{\sigma_2}$ if and only if there is a $\tau \in \Delta$, such that $\tau \prec \sigma_1 \cap \sigma_2$ and $u_1 = u_2$ within U_τ. $X(N, \Delta)$ is called *simplicial* if all the cones of Δ are simplicial. $X(N, \Delta)$ is compact iff $|\Delta| = N_\mathbb{R}$ [33, Thm. 1.11, p. 16]. Moreover, $X(N, \Delta)$ admits a canonical T_N-action which extends the group multiplication of $T_N = U_{\{\mathbf{0}\}}$:

$$T_N \times X(N, \Delta) \ni (t, u) \longmapsto t \cdot u \in X(N, \Delta) \tag{2.3}$$

where, for $u \in U_\sigma \subset X(N, \Delta)$, $(t \cdot u)(m) := t(m) \cdot u(m)$, $\forall m,\ m \in M \cap \sigma^\vee$. The orbits w.r.t. the action (2.3) are parametrized by the set of all the cones belonging to Δ. For a $\tau \in \Delta$, we denote by orb(τ) (resp. by $V(\tau)$) the orbit (resp. the closure of the orbit) which is associated to τ. If $\tau \in \Delta$, then $V(\tau) := V(\tau; \Delta) := X(N(\tau), \mathrm{Star}(\tau; \Delta))$ is itself a toric variety w.r.t.

$$N(\tau) := N \,/\, N_\tau \ , \qquad \mathrm{Star}(\tau; \Delta) := \{\overline{\sigma} \mid \sigma \in \Delta,\ \tau \prec \sigma\} \ ,$$

where N_τ is the sublattice $N\cap$ lin(τ) of N and $\overline{\sigma} = (\sigma + (N_\tau)_\mathbb{R}) \,/\, (N_\tau)_\mathbb{R}$ denotes the image of σ in $N(\tau)_\mathbb{R} = N_\mathbb{R} / (N_\tau)_\mathbb{R}$.

(g) The behaviour of toric varieties with regard to the algebraic properties (1.2) has as follows.

Theorem 2.3 (Normality and CM-property) *All toric varieties are normal and Cohen-Macaulay.*

Proof. For a proof of the normality property see [33, Thm. 1.4, p. 7]. The CM-property for toric varieties was first shown by Hochster in [22]. See also Kempf [25, Thm. 14, p. 52], and Oda [33, 3.9, p. 125]. □

In fact, by the definition (2.2) of $X(N, \Delta)$, all the algebraic properties of this kind are *local* with respect to its affine covering, i.e., it is enough to be checked for the affine toric varieties U_σ for all (maximal) cones σ of the fan Δ.

Definition 2.4 (Multiplicities and basic cones) Let N be a free Z-module of rank r and $\sigma \subset N_{\mathbb{R}}$ a simplicial, rational s.c.p.c. of dimension $d \leq r$. σ can be obviously written as $\sigma = \varrho_1 + \cdots + \varrho_d$, for distinct rays $\varrho_1, \ldots, \varrho_d$. *The multiplicity* $\mathrm{mult}(\sigma; N)$ of σ with respect to N is defined as the index

$$\mathrm{mult}\,(\sigma; N) := |\, N_\sigma \;:\; \mathbb{Z}n(\varrho_1) + \cdots + \mathbb{Z}n(\varrho_d)\, |$$

If $\mathrm{mult}(\sigma; N) = 1$, then σ is called a *basic cone* w.r.t. N.

Theorem 2.5 (Smoothness criterion) *The affine toric variety U_σ is smooth iff σ is basic w.r.t. N.* (*Correspondingly, an arbitrary toric variety $X\,(N, \Delta)$ is smooth if and only if it is simplicial and each s.c.p. cone $\sigma \in \Delta$ is basic w.r.t. N.*)

Proof. See [25, ch. I, Thm. 4, p. 14], and [33, Thm. 1.10, p. 15]. □

Next Theorem is due to Stanley [39, §6], who worked directly with the monoidal $\mathbb{C}$-algebra $\mathbb{C}\,[M \cap \sigma^\vee]$, as well as to Ishida [23, §7], Danilov and Reid [35, p. 294], who provided a purely algebraic-geometric characterization of the Gorensteinness property.

Theorem 2.6 (Gorenstein property) *The following conditions are equivalent*:

(i) *U_σ is Gorenstein.*

(ii) *There exists an element m_σ of M, such that $M \cap (\mathrm{int}\,(\sigma^\vee)) = m_\sigma + M \cap \sigma^\vee$.*

(iii) $\mathrm{Gen}(\sigma) \subset \mathbf{H}$*, where* $\mathbf{H}$ *denotes an affine hyperplane of* $(N_\sigma)_{\mathbb{R}}$ *that contains a lattice basis of* N_σ.

Moreover, if $\dim(\sigma) = r$*, then* m_σ *in* (ii) *is a uniquely determined primitive element of* $M \cap (\mathrm{int}\,(\sigma^\vee))$ *and* $\mathbf{H}$ *in* (iii) *equals* $\mathbf{H} = \{\mathbf{x} \in N_{\mathbb{R}} \mid \langle m_\sigma, \mathbf{x}\rangle = 1\}$.

A geometric interpretation of the remaining "finer" algebraic property, namely whether U_σ is a l.c.i. or not, in terms of the defining fan, is due to Nakajima and will be presented separately in §4, Thm. 4.7.

(h) A *map of fans* $\varpi : (N', \Delta') \to (N, \Delta)$ is a $\mathbb{Z}$-linear homomorphism $\varpi : N' \to N$ whose scalar extension $\varpi \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{R}} : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ satisfies the property:

$$\forall \sigma',\ \sigma' \in \Delta' \quad \exists\, \sigma,\ \sigma \in \Delta \quad \text{with} \quad \varpi \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{R}}\,(\sigma') \subset \sigma\,.$$

$\varpi \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{C}^*} : T_{N'} = N' \otimes_{\mathbb{Z}} \mathbb{C}^* \to T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ is a homomorphism from $T_{N'}$ to T_N and the scalar extension $\varpi^\vee \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{R}} : M_{\mathbb{R}} \to M'_{\mathbb{R}}$ of the dual $\mathbb{Z}$-linear map $\varpi^\vee : M \to M'$ induces canonically an *equivariant holomorphic map* $\varpi_* : X\,(N', \Delta') \to X\,(N, \Delta)$. This map is *proper* if and only if $\varpi^{-1}\,(|\Delta|) = |\Delta'|$. In particular, if $N = N'$ and Δ' is a refinement of Δ, then $\mathrm{id}_* : X\,(N, \Delta') \to X\,(N, \Delta)$ is *proper* and *birational* cf. [33, Thm. 1.15 and Cor. 1.18].

(i) By Carathéodory's Theorem concerning convex polyhedral cones (cf. [16, III 2.6 & V 4.2]) one can choose a refinement Δ' of any given fan Δ, so that Δ' becomes simplicial. Since further subdivisions of Δ' reduce the multiplicities of its cones, we may arrive (after finitely many subdivisions) at a fan $\widetilde{\Delta}$ having only basic cones. Hence, for every toric variety $X\,(N, \Delta)$ there exists a refinement $\widetilde{\Delta}$ of Δ consisting of exclusively basic cones w.r.t. N, i.e., such that $f = \mathrm{id}_* : X(N, \widetilde{\Delta}) \longrightarrow X\,(N, \Delta)$ is a T_N-equivariant (full) desingularization.

(j) The group of T_N-invariant Weil divisors of a toric variety $X\,(N, \Delta)$ has the set $\{V(\varrho) \mid \varrho \in \Delta(1)\}$ as $\mathbb{Z}$-basis. In fact, such a divisor D is of the form $D = D_\psi$, where $D_\psi := -\sum_{\varrho \in \Delta(1)} \psi(n(\varrho))V(\varrho)$ and $\psi : |\Delta| \to \mathbb{R}$ a *PL-Δ-support function*, i.e., an $\mathbb{R}$-valued, positively homogeneous function on $|\Delta|$ with $\psi(N \cap |\Delta|) \subset \mathbb{Z}$ which is piecewise linear and upper convex on each $\sigma \in \Delta$. (*Upper convex* on a $\sigma \in \Delta$ means that $\psi\,|_\sigma(\mathbf{x} + \mathbf{x}') \geq \psi\,|_\sigma(\mathbf{x}) + \psi\,|_\sigma(\mathbf{x}')$, for all $\mathbf{x}, \mathbf{x}' \in \sigma$). For example, the canonical divisor $K_{X(N,\Delta)}$ of $X\,(N, \Delta)$ equals D_ψ for ψ a PL-Δ-support function with $\psi(n(\varrho)) = 1$, for all rays $\varrho \in \Delta\,(1)$. A divisor $D = D_\psi$ is Cartier iff ψ is a *linear* Δ-support function (i.e., $\psi\,|_\sigma$ is overall linear on each $\sigma \in \Delta$). Obviously, D_ψ is $\mathbb{Q}$-Cartier iff $k \cdot \psi$ is a linear Δ-support function for some $k \in \mathbb{N}$.

Theorem 2.7 (Ampleness criterion) *A T_N-invariant ($\mathbb{Q}$-) Cartier divisor $D = D_\psi$ of a toric variety $X(N,\Delta)$ of dimension r is ample if and only if there exists a $\kappa \in \mathbb{N}$, such that $\kappa \cdot \psi$ is a* strictly upper convex *linear Δ-support function, i.e., iff for every $\sigma \in \Delta(r)$ there is a unique $m_\sigma \in M = \mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$, such that $\kappa \cdot \psi(\mathbf{x}) \leq \langle m_\sigma, \mathbf{x}\rangle$, for all $\mathbf{x} \in |\Delta|$, with equality being valid iff $\mathbf{x} \in \sigma$.*

Proof. It follows from [25, Thm. 13, p. 48]. □

(k) Throughout the paper, by a *polytope* in an euclidean space, is meant the convex hull of finitely many points or, equivalently, a bounded polyhedron. A *lattice polytope* P embedded in a given euclidean space is a polytope whose set vert(P) of vertices belongs to a reference lattice within this space. If M is a free $\mathbb{Z}$-module of rank r, $N = \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$ its dual, and $P \subset M_{\mathbb{R}} \cong \mathbb{R}^r$ an r-dimensional lattice polytope w.r.t. M, then there is a unique fan $\Delta^{(P)}$ in $N_{\mathbb{R}}$, the so-called *normal fan* of P, so that the corresponding r-dimensional toric variety $X(N,\Delta^{(P)})$ is projective and endowed with a distinguished T_N-invariant ample Cartier divisor $D_P := D_\psi$ which is induced by the strictly upper convex support function $\psi : N_{\mathbb{R}} \to \mathbb{R}$, with $\psi(\mathbf{x}) := \min\{\langle \mathbf{y}, \mathbf{x}\rangle \mid \mathbf{y} \in P\}$; and conversely, regarding a projective toric variety $X(N,\Delta)$ and a T_N-invariant ample Cartier divisor $D = D_\psi$ on it as our starting-point data, we win a characteristic r-dimensional lattice polytope $P = P_D$ assigned to D, with $P_D = \{\mathbf{y} \in M_{\mathbb{R}} \mid \langle \mathbf{y}, \mathbf{x}\rangle \geq \psi(\mathbf{x}), \forall \mathbf{x}, \mathbf{x} \in N_{\mathbb{R}}\}$ (cf. Oda [33, §2.4]).

3 Torus-equivariant crepant projective resolutions of Gorenstein toric singularities via b.c.-triangulations

We shall henceforth focus our attention to Gorenstein toric singularities and to their desired resolutions.

(a) Let N be a free $\mathbb{Z}$-module of rank $r \geq 2$ and $\sigma \subset N_{\mathbb{R}}$ a rational s.c.p.c. of dimension $d \leq r$. We identify U_σ with $X(N,\Delta)$, where Δ denotes the fan consisting of σ together with all of its faces. Since $N(\sigma) = N/N_\sigma$ is torsion free, there exists a lattice decomposition $N = N_\sigma \oplus \breve{N}$, inducing a decomposition of its dual $M = M_\sigma \oplus \breve{M}$, where $M_\sigma = \mathrm{Hom}_{\mathbb{Z}}(N_\sigma, \mathbb{Z})$ and $\breve{M} = \mathrm{Hom}_{\mathbb{Z}}(\breve{N},\mathbb{Z})$. Writing σ as $\sigma = \sigma' \oplus \{\mathbf{0}\}$ with σ' a d-dimensional cone in $(N_\sigma)_{\mathbb{R}}$, we obtain decompositions

$$T_N \cong T_{N_\sigma} \times T_{\breve{N}} \quad \text{and} \quad M \cap \sigma^\vee = \left(M \cap (\sigma')^\vee\right) \oplus \breve{M} \ ,$$

which give rise to the analytic isomorphisms:

$$\begin{array}{ccccccc} U_\sigma & \cong & U_{\sigma'} \times T_{\breve{N}} & \cong & U_{\sigma'} \times T_{N(\sigma)} & \cong & U_{\sigma'} \times (\mathbb{C}^*)^{r-d} \\ \| & & & & & & \| \\ X(N,\Delta) & & & & & & X(N_\sigma, \Delta') \times (\mathbb{C}^*)^{r-d} \end{array}$$

with Δ' the fan consisting of σ' together with all of its faces (cf. [17, p. 29], and [16, Thm.VI.2.12, p. 223]). U_σ can be therefore viewed as as a fiber bundle over $U_{\sigma'}$ having an $(r-d)$-dimensional algebraic torus as its typical fibre. Obviously, the study of the algebraic properties (mentioned in §1) for U_σ can be reduced to that of the corresponding properties of $U_{\sigma'}$. (For instance, the singular locus of U_σ equals $\mathrm{Sing}(U_\sigma) = \mathrm{Sing}(U_{\sigma'}) \times (\mathbb{C}^*)^{r-d}$). In fact, the main reason for preferring to work with $U_{\sigma'}$ (or with the germ $(U_{\sigma'}, \mathrm{orb}(\sigma'))$) instead of U_σ, is that since $\mathrm{lin}(\sigma') = (N_\sigma)_{\mathbb{R}}$, the orbit $\mathrm{orb}(\sigma') \in U_{\sigma'}$ is the unique fixed closed point under the action of T_{N_σ} on $U_{\sigma'}$.

Definition 3.1 (Singular representatives) If σ is non-basic w.r.t. N, then $U_{\sigma'}$ will be called *the singular representative* of U_σ and $\mathrm{orb}(\sigma') \in U_{\sigma'}$ the associated *distinguished* singular point within the singular locus $\mathrm{Sing}(U_{\sigma'})$ of $U_{\sigma'} = X(N_\sigma, \Delta')$.

Definition 3.2 (Splitting codimension) If σ is non-basic w.r.t. N, then it is also useful to introduce the notion of the "*splitting codimension*" of $\mathrm{orb}(\sigma') \in U_{\sigma'}$ as the number

$$\min \left\{ \varkappa \in \{2,\dots,d\} \;\middle|\; \begin{array}{c} U_{\sigma'} \cong U_{\sigma''} \times \mathbb{C}^{d-\varkappa}, \ \text{ for some } \ \sigma'' \prec \sigma' \\ \text{with } \ \dim(\sigma'') = \varkappa \ \text{ and } \ \mathrm{Sing}(U_{\sigma''}) \neq \varnothing \end{array} \right\} .$$

(In [10, p. 231] and [12, p. 202] there is a misprint in this definition: one must replace therein max by min.) If this number equals d, then $(U_{\sigma'}, \mathrm{orb}\,(\sigma'))$ will be called an *msc-singularity*, i.e., a singularity having the maximum splitting codimension.

(b) Gorenstein toric affine varieties are completely determined by suitable lattice polytopes.

Definition 3.3 (Lattice equivalence) If N_1 and N_2 are two free $\mathbb{Z}$-modules (not necessarily of the same rank) and $P_1 \subset (N_1)_{\mathbb{R}}$, $P_2 \subset (N_2)_{\mathbb{R}}$ two lattice polytopes w.r.t. them, we shall say that P_1 and P_2 are *lattice equivalent* to each other, and denote this by $P_1 \sim P_2$, if P_1 is affinely equivalent to P_2 via an affine map $\varpi : (N_1)_{\mathbb{R}} \to (N_2)_{\mathbb{R}}$, such that the restiction $\varpi\,|_{\mathrm{aff}(P)} : \mathrm{aff}(P) \to \mathrm{aff}(P')$ is a bijection mapping P_1 onto the (necessarily equidimensional) polytope P_2 , and, in addition, N_{P_1} is mapped bijectively onto the lattice N_{P_2}, where N_{P_j} is the affine sublattice $\mathrm{aff}(P_j) \cap N_j$ of N_j, $j = 1, 2$. If $N_1 = N_2 =: N$ and $\mathrm{rk}(N) = \dim(P_1) = \dim(P_2)$, then these ϖ's are exactly the *affine integral transformations* which are composed of unimodular N-transformations and N-translations.

Let now $U_\sigma = X\,(N, \Delta)$ be a d-dimensional affine toric variety as in (a) and $U_{\sigma'} = X\,(N_\sigma, \Delta')$. Assuming that U_σ is Gorenstein, we may pass to another analytically isomorphic "standard" representative as follows: Denote by $\mathbb{Z}^d$ the standard rectangular lattice in $\mathbb{R}^d$ and by $(\mathbb{Z}^d)^\vee$ its dual lattice within $(\mathbb{R}^d)^\vee = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^d, \mathbb{R})$. Since $\dim(\sigma') = \mathrm{rk}(N_\sigma) = d$, or equivalently, since $(\sigma')^\vee$ is strongly convex in $(M_\sigma)_{\mathbb{R}}$, Thm. 2.6 (iii) implies

$$\mathrm{Gen}\,(\sigma') \subset \mathbf{H}^{(d)} \quad \text{with} \quad \mathbf{H}^{(d)} := \{\mathbf{x} \in (N_\sigma)_{\mathbb{R}} \mid \langle m_{\sigma'}, \mathbf{x}\rangle = 1\},$$

for a unique primitive $m_{\sigma'} \in M_\sigma$. Clearly, $\sigma' \cap \mathbf{H}^{(d)}$ is a $(d-1)$-dimensional lattice polytope (w.r.t. N_σ). We choose a specific $\mathbb{Z}$-module isomorphism $\Upsilon : N_\sigma \xrightarrow{\cong} \mathbb{Z}^d$ inducing an $\mathbb{R}$-vector space isomorphism $\Phi = \Upsilon \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{R}} : (N_\sigma)_{\mathbb{R}} \xrightarrow{\cong} \mathbb{R}^d$, such that

$$\Phi\,(m_{\sigma'}) = (1, \underbrace{0, 0, ..., 0, 0}_{(d-1)\text{-times}}) \Longrightarrow \Phi\left(\mathbf{H}^{(d)}\right) = \left\{\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1 = 1\right\} =: \bar{\mathbf{H}}^{(d)}.$$

Obviously, $P := \Phi\left(\sigma' \cap \mathbf{H}^{(d)}\right) \subset \bar{\mathbf{H}}^{(d)}$ is a lattice $(d-1)$-polytope (w.r.t. $\mathbb{Z}^d$). Defining

$$\tau_P := \mathrm{pos}(P) = \left\{\kappa\, \mathbf{x} \in \mathbb{R}^d \mid \kappa \in \mathbb{R}_{\geq 0},\ \mathbf{x} \in P\right\}, \quad \Delta_P := \{\ \tau_P \text{ together with all of its faces}\},$$

(cf. Figure **1**) we obtain easily the following Lemma:

Lemma 3.4 (i) *There exists a torus-equivariant analytic isomorphism*

$$U_{\sigma'} = X\,(N_\sigma, \Delta') \cong U_{\tau_P} = X(\mathbb{Z}^d, \Delta_P) \quad (= \text{Max-Spec}(\mathbb{C}\left[(\mathbb{Z}^d)^\vee \cap \tau_P^\vee\right]))$$

mapping $\mathrm{orb}(\sigma')$ *onto* $\mathrm{orb}(\tau_P)$.

(ii) *If* $Q \subset \bar{\mathbf{H}}^{(d)}$ *is a lattice* $(d-1)$*-polytope* (*w.r.t.* $\mathbb{Z}^d$), *then* $P \sim Q$ *iff there exists a torus-equivariant analytic isomorphism* $U_{\tau_P} \cong U_{\tau_Q}$ *mapping* $\mathrm{orb}\,(\tau_P)$ *onto* $\mathrm{orb}\,(\tau_Q)$.

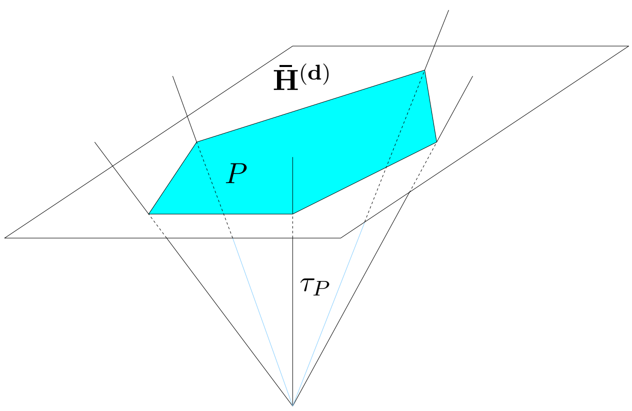


Figure 1

Definition 3.5 (Standard representatives) Any member of the isomorphy class of the underlying space $U_{\tau_P} = X(\mathbb{Z}^d, \Delta_P)$ of the distinguished Gorenstein point $\mathrm{orb}(\tau_P)$ (as in 3.4(ii)) is said to be *a standard representative* of U_σ associated to the lattice polytope P, and, in particular, *a singular standard representative* of U_σ, whenever σ is non-basic w.r.t. N. (In this case, the splitting codimension of $\mathrm{orb}(\tau_P)$ is defined to be the splitting codimension of $\mathrm{orb}(\sigma')$.)

(c) Suppose that σ is a non-basic c.p. cone w.r.t. N. From the above discussion it is now clear that for desingularizing U_σ, it suffices to resolve a singular representative $U_{\sigma'}$, and for U_σ Gorenstein, a standard singular representative U_{τ_P} of it. In the latter case, for any torus-equivariant partial desingularization $f = \mathrm{id}_* : X(\mathbb{Z}^d, \widehat{\Delta}_P) \longrightarrow X(\mathbb{Z}^d, \Delta_P) = U_{\tau_P}$ coming from a refinement $\widehat{\Delta}_P$ of Δ_P (cf. §2, (h)-(i)) there are one-to-one correspondences:

$$\begin{array}{ccc} \varrho & \in & \widehat{\Delta}_P\,(1) \smallsetminus \Delta_P\,(1) \\ \updownarrow & & \updownarrow \\ n\,(\varrho) & \in & \mathrm{Gen}(\widehat{\Delta}_P) \smallsetminus \mathrm{Gen}\,(\tau_P) \\ \updownarrow & & \updownarrow \\ D_{n(\varrho)} := V\,(\varrho) = V(\varrho;\widehat{\Delta}_P) & \in & \left\{ \begin{array}{c} \text{exceptional prime divisors} \\ \text{with respect to } f \end{array} \right\} \end{array} \tag{3.1}$$

Moreover, as we shall see below in proposition 3.14, it is possible to describe certain intrinsic algebraic-geometric properties of those f's which are crepant or / and projective exclusively in terms of lattice triangulations of the polytope P defining U_{τ_P}. For this reason, before proceeding to this description, we recall some central notions from the theory of polytopal subdivisions which will be crucially utilized in the rest of the paper.

Definition 3.6 (Polytopal subdivisions and refinements) (i) A *polytopal complex* is a finite family $\mathcal{S}$ of polytopes in an euclidean space $\mathbb{R}^\ell$, so that the intersection of any two of its polytopes constitutes always a common face of each of them. The *dimension* $\dim(\mathcal{S})$ of such an $\mathcal{S}$ is defined to be the largest possible dimension of a polytope belonging to it. $\mathcal{S}$ is called a *pure* polytopal complex if every polytope in $\mathcal{S}$ is contained in one of dimension $\dim(\mathcal{S})$.

(ii) Let $\mathcal{V}$ denote a finite set of points in an euclidean space, such that $P = \mathrm{conv}(\mathcal{V})$ is a k-dimensional polytope. A *polytopal subdivision* $\mathcal{S}$ of P is a finite family $\mathcal{S} = \{P_1, P_2, \ldots, P_\nu\}$ of k-dimensional polytopes, such that:

A. $\mathcal{S}$ is a pure k-dimensional polytopal complex.

B. The space supporting P is the union of spaces supporting $P_1, P_2, \ldots, P_\nu$.

C. $\mathrm{vert}(P_i) \subseteq \mathcal{V}$, for all $i \in \{1, 2, \ldots, \nu\}$.

(iii) A polytopal subdivision $\mathcal{S}$ of P as in (ii) is called a *triangulation* of P if each P_i, $1 \le i \le \nu$, is a k-dimensional simplex.

(iv) Suppose that $\mathcal{S} = \{P_1, P_2, \ldots, P_\nu\}$, $\mathcal{S}' = \{P'_1, P'_2, \ldots, P'_\mu\}$ are two polytopal subdivisions of P. Then $\mathcal{S}'$ is a *refinement* of $\mathcal{S}$ if for each j, $1 \le j \le \mu$, there exists an i, $1 \le i \le \nu$, such that $P'_j \subseteq P_i$.

Definition 3.7 (Coherent subdivisions) A polytopal subdivision $\mathcal{S}$ of a polytope $P \subset \mathbb{R}^k$ is called *coherent* (or, alternatively, *regular*, cf. [44, 5.3]) if P is the image $\pi\,(Q) = P$ of a polytope $Q \subset \mathbb{R}^{k+1}$ under the projection map

$$\mathbb{R}^{k+1} \ni (x_1, \ldots, x_k, x_{k+1}) \overset{\pi}{\longmapsto} (x_1, \ldots, x_k) \in \mathbb{R}^k \tag{3.2}$$

so that $\mathcal{S} = \{\pi\,(F) : F \text{ is a lower face of } Q\}$, where *the lower faces of* Q are the faces for which some outward normal vector has negative $(k+1)$-st coordinate. (The set of all lower faces of Q is sometimes called the *lower envelope* of Q).

The next two Lemmas describe further useful conditions which are equivalent to the coherency of $\mathcal{S}$.

Lemma 3.8 (Coherency and strictly upper convex functions) *A polytopal subdivision $\mathcal{S}$ of P is coherent iff there exists* a strictly upper convex $\mathcal{S}$-support function $\psi : |\mathcal{S}| \to \mathbb{R}$, *i.e. a piecewise-linear real function defined on the underlying space $|\mathcal{S}|$, for which*

$$\psi(t\ \mathbf{x} + (1-t)\ \ \mathbf{y}) \geq t\ \psi(\mathbf{x}) + (1-t)\ \ \psi(\mathbf{y}), \text{ for all } \ \mathbf{x}, \mathbf{y} \in |\mathcal{S}|, \text{ and } \ t \in [0,1] \ ,$$

so that its domains of linearity are exactly the polytopes of $\mathcal{S}$ having maximal dimension.

Proof. If $\mathcal{S}$ is coherent, then $\mathcal{S} = \{\pi(F) : F$ is a lower face of $Q\}$, with $\pi : \mathbb{R}^{k+1} \to \mathbb{R}^k$ the projection (3.2) and Q a polytope in $\mathbb{R}^{k+1}$. The function $\psi : |\mathcal{S}| \to \mathbb{R}$ defined by setting

$$\psi(\mathbf{x}) := \ \max \ \{t \in \mathbb{R} \ | \ (\mathbf{x}, -t) \in Q\}, \quad \text{for all } \mathbf{x} = (x_1, \ldots, x_k) \in |\mathcal{S}| = P,$$

is strictly upper convex. (Some authors prefer to work with convex support functions instead of upper convex ones and use min and $(\mathbf{x}, t)$ instead of max and $(\mathbf{x}, -t)$. But this is just a sign convention).

Conversely, if $\psi : |\mathcal{S}| \to \mathbb{R}$ is assumed to be a strictly upper convex support function, then $\mathcal{S}$ is coherent in the sense of 3.7 by defining Q to be the polytope $\mathrm{conv}\{(\mathbf{x}, -\psi(\mathbf{x})) \in \mathbb{R}^{k+1} \ | \ \mathbf{x} \in P\}$. $\square$

Lemma 3.9 (Coherency and "heights") *Let $\mathcal{V}$ be a finite set of points in $\mathbb{R}^k$ and $P = \mathrm{conv}(\mathcal{V})$. A* height function *on $\mathcal{V}$ is defined to be a function $\omega : \mathcal{V} \to \mathbb{R}$. (The values $\omega(\mathbf{v})$, $\mathbf{v} \in \mathcal{V}$, are called* "heights"*). Every height function ω on $\mathcal{V}$ induces a coherent polytopal subdivision $\mathcal{S}_\omega$ of the polytope $P = \mathrm{conv}(\mathcal{V})$ with $\mathrm{vert}(\mathcal{S}_\omega) \subseteq \mathcal{V}$; and conversely, each coherent polytopal subdivision $\mathcal{S}$ of $P = \mathrm{conv}(\mathcal{V})$ with $\mathrm{vert}(\mathcal{S}) \subseteq \mathcal{V}$ is of the form $\mathcal{S} = \mathcal{S}_\omega$, for some height function ω.*

Proof. Let ω be a height function on $\mathcal{V}$. The heights can be used to "lift" the point configuration $\mathcal{V}$ into the next dimension and to define $Q_\omega := \mathrm{conv}(\{(\mathbf{v}, \omega(\mathbf{v})) \in \mathbb{R}^{k+1} \ | \ \mathbf{v} \in \mathcal{V}\})$. The lower envelope of the polytope Q_ω is a pure polytopal complex having dimension equal to $\dim(P)$. Its image under the projection (3.2) determines a (necessarily coherent) polytopal subdivision $\mathcal{S}_\omega$ of P with $\mathrm{vert}(\mathcal{S}_\omega) \subseteq \mathcal{V}$. In fact, if $\{\mathbf{v}_{i_1}, .., \mathbf{v}_{i_\mu}\}$ are the vertices a polytope belonging to $\mathcal{S}_\omega$, then $\{(\mathbf{v}_{i_1}, \omega(\mathbf{v}_{i_1})), .., (\mathbf{v}_{i_\mu}, \omega(\mathbf{v}_{i_\mu}))\}$ is the vertex set of a face of the lower envelope of Q_ω.

Let now $\mathcal{S}$ denote an arbitrary coherent polytopal subdivision $\mathcal{S}$ of P with $\mathrm{vert}(\mathcal{S}) \subseteq \mathcal{V}$. By Lemma 3.8 there exists a strictly upper convex support function $\psi : |\mathcal{S}| \to \mathbb{R}$. Using the height function $\omega := (-\psi)\,|_{\mathcal{V}}$ we obtain $\mathcal{S} = \mathcal{S}_\omega$. $\square$

Remark 3.10 For "generic" choices of ω's the coherent polytopal subdivisions $\mathcal{S}_\omega$ are *triangulations* of P (cf. [18, p. 215 and p. 228], and [42, p. 64]).

Definition 3.11 (Lattice subdivisions) A *lattice subdivision* $\mathcal{S}$ of a lattice polytope P is a polytopal subdivision of P, such that the set $\mathrm{vert}(\mathcal{S})$ of the vertices of $\mathcal{S}$ belongs to the reference lattice (and $\mathrm{vert}(P) \subseteq \mathrm{vert}(\mathcal{S})$). A *lattice triangulation* of a lattice polytope P is a lattice subdivision of P which, in addition, is a triangulation (in the sense of 3.6).

Definition 3.12 (Maximal and basic triangulations) (i) A lattice polytope P is called *elementary* if the lattice points belonging to it are exactly its vertices. A lattice simplex is said to be *basic* or *unimodular* if its vertices constitute a part of an affine $\mathbb{Z}$-basis of the reference lattice (or equivalently, if its relative, normalized volume equals 1).

(ii) A lattice triangulation $\mathcal{T}$ of a lattice polytope P is defined to be *maximal* (resp. *basic*), if it consists only of elementary (resp. basic) simplices.

Definition 3.13 ("b.c."-triangulations) A *b.c.-triangulation* will be used as abbreviation for a basic, coherent triangulation of a lattice polytope.

Reverting to Gorenstein affine toric varieties, we explain how torus-equivariant crepant or / and projective desingularizations can be constructed by means of lattice triangulations.

Proposition 3.14 (Crepant desingularizations and triangulations) *Every torus-equivariant partial crepant desingularization of a standard representative U_{τ_P} of a Gorenstein affine toric variety U_σ (as in 3.5, with $P \subset \bar{\mathbf{H}}^{(d)}$ a lattice polytope w.r.t. $\mathbb{Z}^d$), induced by a subdivision of Δ_P into simplicial s.c.p. cones, is of the form*

$$f = f_{\mathcal{T}} : X(\mathbb{Z}^d, \widehat{\Delta}_P(\mathcal{T})) \longrightarrow X(\mathbb{Z}^d, \Delta_P) = U_{\tau_P} \tag{3.3}$$

where $\widehat{\Delta}_P = \widehat{\Delta}_P(\mathcal{T}) := \{\sigma_{\mathbf{s}}, \mathbf{s} \in \mathcal{T}\}$ is determined by a lattice triangulation $\mathcal{T}$ of P with

$$\sigma_{\mathbf{s}} := \left\{ \kappa\, \mathbf{x} \in \mathbb{R}^d \;\middle|\; \kappa \in \mathbb{R}_{\geq 0},\; \mathbf{x} \in \mathbf{s} \right\}.$$

By (3.1) the set of exceptional prime divisors equals $\left\{ D_n = V(\mathbb{R}_{\geq 0}\, n) \;\middle|\; n \in (P \smallsetminus \mathrm{vert}(P)) \cap \mathbb{Z}^d \right\}$. Moreover, such an $f_{\mathcal{T}}$ has the following properties:

(i) $f_{\mathcal{T}}$ is maximal w.r.t. discrepancy $\Longleftrightarrow \mathcal{T}$ *is maximal.*

(ii) $f_{\mathcal{T}}$ is full (*i.e.,* $X(\mathbb{Z}^d, \widehat{\Delta}_P(\mathcal{T}))$ *is overall smooth*) $\Longleftrightarrow \mathcal{T}$ *is basic.*

(iii) $f_{\mathcal{T}}$ is projective (*i.e.,* $X(\mathbb{Z}^d, \widehat{\Delta}_P(\mathcal{T}))$ *is quasiprojective*) $\Longleftrightarrow \mathcal{T}$ *is coherent.*

Proof. Let $f : X(\mathbb{Z}^d, \widehat{\Delta}_P) \longrightarrow X(\mathbb{Z}^d, \Delta_P) = U_{\tau_P}$ denote an arbitrary torus-equivariant partial desingularization of U_{τ_P} induced by a subdivision $\widehat{\Delta}_P$ of Δ_P into simplicial s.c.p. cones. The discrepancy of f equals

$$K_{X\left(\mathbb{Z}^d,\, \widehat{\Delta}_P\right)} - f^*\left(K_{U_{\tau_P}}\right) = \left[- \sum_{\varrho' \in \left(\widehat{\Delta}_P(1) \smallsetminus \Delta_P(1)\right)} \widehat{D}_{n(\varrho')} - \sum_{\varrho \in \Delta_P(1)} \widehat{D}_{n(\varrho)} \right] - f^*\left(- \sum_{\varrho \in \Delta_P(1)} D_{n(\varrho)} \right)$$

where $D_{n(\varrho)} := V(\varrho, \Delta_P)$, $\widehat{D}_{n(\varrho)} := V(\varrho, \widehat{\Delta}_P)$, for all rays ϱ of Δ_P and $\widehat{\Delta}_P$, respectively, and

$$f^*\left(- \sum_{\varrho \in \Delta_P(1)} D_{n(\varrho)} \right) = - \sum_{\varrho \in \Delta_P(1)} \widehat{D}_{n(\varrho)} - \sum_{\varrho' \in \left(\widehat{\Delta}_P(1) \smallsetminus \Delta_P(1)\right)} \mu_{\varrho'}\, \widehat{D}_{n(\varrho')}$$

with $\mu_{\varrho'}$'s $\in \mathbb{Q}_{\geq 0}$. If $\phi_{T_{\mathbb{Z}^d}}$ is the rational differential form generating the dualizing sheaf of the torus $T_{\mathbb{Z}^d}$, then the dualizing sheaf of U_{τ_P} is isomorphic to $\mathbb{C}\left[(\mathbb{Z}^d)^\vee \cap \mathrm{int}(\tau^\vee)\right] \cdot \phi_{T_{\mathbb{Z}^d}}$. Since U_{τ_P} is Gorenstein, $\mathbb{C}\left[(\mathbb{Z}^d)^\vee \cap \mathrm{int}(\tau^\vee)\right] \cdot \phi_{T_{\mathbb{Z}^d}}$ is generated by $\mathbf{e}((1,0,...,0,0)) \cdot \phi_{T_{\mathbb{Z}^d}}$ (cf. Thm. 2.6 and subsection (b)), and $K_{U_{\tau_P}}$ is trivial. The preservation of Gorensteinness for $X(\mathbb{Z}^d, \widehat{\Delta}_P)$ is equivalent to say that, for each member of its affine cover $\{U_{\widehat{\sigma}} \mid \widehat{\sigma} \in \widehat{\Delta}_P(d)\}$, the sheaf of sections of the canonical divisor $K_{X\left(\mathbb{Z}^d,\, \widehat{\Delta}_P\right)}$ over $U_{\widehat{\sigma}}$ is isomorphic to $\mathbb{C}\left[(\mathbb{Z}^d)^\vee \cap \mathrm{int}((\widehat{\sigma})^\vee)\right] \cdot \phi_{T_{\mathbb{Z}^d}}$ and is therefore generated by $\mathbf{e}((1,0,...,0,0)) \cdot \phi_{T_{\mathbb{Z}^d}}$. The order of vanishing for the divisor $\mathrm{div}(\mathbf{e}((1,0,...,0,0)) \cdot \phi_{T_{\mathbb{Z}^d}})$ which is associated to this common single generator along the $\widehat{D}_{n(\varrho')}$'s, $\varrho' \in (\widehat{\Delta}_P(1) \smallsetminus \Delta_P(1))$, equals

$$\mu_{\varrho'} = \mathrm{ord}_{\widehat{D}_{n(\varrho')}}\left(\mathrm{div}(\mathbf{e}((1,0,...,0,0)) \cdot \phi_{T_{\mathbb{Z}^d}})\right) = \langle (1, \underbrace{0,0,...,0,0}_{(d-1)\text{-times}}), n(\varrho') \rangle$$

(cf. Fulton [17, Lemma of p. 61]). From the above equations we deduce

$$K_{X\left(\mathbb{Z}^d,\, \widehat{\Delta}_P\right)} - f^*\left(K_{U_{\tau_P}}\right) = \sum_{\varrho' \in \left(\widehat{\Delta}_P(1) \smallsetminus \Delta_P(1)\right)} \left(\langle (1, \underbrace{0,0,...,0,0}_{(d-1)\text{-times}}), n(\varrho') \rangle - 1\right) \widehat{D}_{n(\varrho')}\,.$$

Thus, f is crepant iff

$$\mathrm{Gen}(\widehat{\Delta}_P) \subset \bar{\mathbf{H}}^{(d)} \tag{3.4}$$

i.e., iff f is of the form (3.3). Now property (i) is obvious. For (ii) observe that (3.4) implies for all $\mathbf{s} \in \mathcal{T}$: $\sigma_{\mathbf{s}}$ is a basic cone $\Longleftrightarrow$ $\mathbf{s}$ is a basic simplex. Concerning (iii), note that all torus-invariant Weil divisors of $X(\mathbb{Z}^d, \widehat{\Delta}_P(\mathcal{T}))$ are $\mathbb{Q}$-Cartier because this toric variety is $\mathbb{Q}$-factorial. Clearly, for every strictly upper convex linear $\widehat{\Delta}_P(\mathcal{T})$-support function ψ (in the sense of §2 (j)), the restriction $\psi\,|_{\mathcal{T}}$ is a strictly upper convex $\mathcal{T}$-support function (as in 3.8); and conversely, as it was explained in [12, §4], to any $\mathcal{T}$-support function ψ, one may canonically assign (eventually after suitable perturbation of the defining inequalities and / or multiplication by a scalar) a strictly upper convex linear $\widehat{\Delta}_P(\mathcal{T})$-support function ψ' (with $\psi'(\left|\widehat{\Delta}_P(\mathcal{T})\right| \cap \mathbb{Z}^d) \subset \mathbb{Q}$ or even in $\mathbb{Z}$). To finish the proof we apply Lemma 3.8 for ψ' and Theorem 2.7 for the divisor $D_{\psi'}$. $\square$

Remark 3.15 The birational morphisms $f_{\mathcal{T}}$, for $\mathcal{T}$'s maximal and coherent, can be decomposed into more elementary toric contractions (see Reid [36, (0.2)-(0.3)]). In several cases, these contractions are directly expressible as chains of blow-downs (cf. [11, 7.2 and §9]).

• Every lattice polytope P can be clearly embedded, up to an affine transformation, into $\bar{\mathbf{H}}^{(d)}$, with $d = \dim(P) + 1$, and its supporting cone $\tau_P \subset \mathbb{R}^d$ gives rise to the construction of an affine Gorenstein variety U_{τ_P}. Consequently, if we restrict the initial Question 1.1 of the introduction to the category of Gorenstein toric singularities (and their *torus-equivariant* resolutions), our previous discussion in subsection (b), together with the "bridge" which is built by proposition 3.14 and connects algebraic with discrete geometric statements, enable us to reformulate it as follows:

Question 3.16 *Under which conditions does a given lattice polytope P of dimension ≥ 3 admit of b.c.-triangulations?*

Remark 3.17 (i) All elementary triangles are basic, but already in dimension 3 there exist counterexamples of elementary simplices which are non-basic. Moreover, already in dimension 2 (i.e., for certain lattice polygons) there is a plethora of non-coherent (but necessarily basic) maximal triangulations. Hence, the problem of the existence of b.c.-triangulations turns out to be very subtle in general. The required extra "conditions" in the formulation of Question 3.16 depend essentially on the representatives of the coordinates of vertices of the given lattice polytope P within its lattice equivalence class. Unfortunately, regarding these integer coordinates as freely moving "parameters", we see that in high dimensions they are "too many" to handle (even for simplices and even if we reduce them by suitable unimodular transformations like Hermite normal form transformations). This is why a first realistic attempt to answer 3.16 partially (or at least to find *sufficient conditions* for the above existence problem) seems to be feasible only by the consideration of some *special families* of P's. In the present paper we deal with altogether three families of lattice polytopes and prove that they admit b.c.-triangulations (see below 3.18, 3.19, 3.20, 3.21, 4.2, and 5.1). The third one is exactly that corresponding to the toric l.c.i.-singularities and has some interesting members in common (and also not in common) with the first two (see §7).

(ii) For any finite set of points $\mathcal{V}$ in an $\mathbb{R}^d$, *all triangulations* $\mathcal{T}$ of the polytope $P = \mathrm{conv}(\mathcal{V})$ with $\mathrm{vert}(\mathcal{T}) \subseteq \mathcal{V}$ are parametrized by the vertices of a "gigantic" polytope $\mathbf{Un}\,(\mathcal{V})$, the so-called *universal polytope* of P (see Billera, Filliman & Sturmfels [6, §3], and de Loera, Hoşten, Santos & Sturmfels [15, §1-§4]). $\mathbf{Un}\,(\mathcal{V})$ contains a subpolytope $\mathbf{Sec}\,(\mathcal{V})$ whose vertices parametrize only the coherent $\mathcal{T}$'s. $\mathbf{Sec}\,(\mathcal{V})$ is in most of the cases considerably "big" too, and is called the *secondary polytope* of P. (For the main concepts of the theory of secondary polytopes the reader is referred to [6], Oda & Park [34], Ziegler [44, Lecture 9], as well as to the treatment of Gelfand, Kapranov & Zelevinsky [18, Ch. 7]. In practice, working with examples for which the cardinality of the given $\mathcal{V}$'s is relatively small, an enumeration of the vertices of $\mathbf{Sec}\,(\mathcal{V})$ can be easily achieved by making use of the MAPLE-package PUNTOS [14] of de Loera).

(iii) In the particular case in which $P \subset \bar{\mathbf{H}}^{(d)} \subset \mathbb{R}^d$ is a lattice polytope (w.r.t. $\mathbb{Z}^d$) and $\mathcal{V} = P \cap \mathbb{Z}^d$, the b.c.-triangulations of P correspond to a *very special* (not necessarily non-empty) "mysterious" subset $\mathbf{BC}(\mathcal{V})$ of $\mathrm{vert}(\mathbf{Sec}\,(\mathcal{V}))$. Thus, since 3.16 asks for conditions under which $\mathbf{BC}(\mathcal{V}) \neq \varnothing$, the expected theoretical answer(s) would surely require a much more extensive study for $\mathbf{Sec}\,(\mathcal{V})$ itself. At this point, we should also stress that in high dimensions "exotic pathological counterexamples" exist! For instance, Hibi and Ohsugi [21] discovered recently a 9-dimensional 0/1-polytope (with 15 vertices) having basic triangulations, but none of whose coherent triangulations is basic.

(iv) Passing by a connected vertex path from one vertex of $\mathbf{Sec}\,(\mathcal{V})$ to another, we perform a finite series of "bistellar operations" which are nothing but "flops" in the algebraic-geometric terminology [34, §3].

(d) We next present two characteristic families of lattice polytopes which admit specific b.c.-triangulations $\mathcal{T}$ leading to projective, crepant, full desingularization morphisms $f_{\mathcal{T}}$ with explicitly describable exceptional prime divisors.

Definition 3.18 (Fano polytopes) A lattice polytope Q is called a *Fano polytope* if $Q \sim P$, where $P \subset \mathbb{R}^d$ denotes a lattice polytope (w.r.t. $\mathbb{Z}^d$) containing exactly one lattice point in its relative interior, which, together with the vertices of each facet, forms an affine lattice basis of $(\mathbb{Z}^d)_P$.

Proposition 3.19 *Let $P \subset \bar{\mathbf{H}}^{(d)} \subset \mathbb{R}^d$ be a Fano polytope (w.r.t. $\mathbb{Z}^d$) with $\mathrm{int}(P) \cap \mathbb{Z}^d = \{n_0\}$.*

(i) *The canonical lattice triangulation $\mathcal{T}^{\mathrm{can}} := \{\{n_0\} \star F \mid F$ face of $P\}$ constructed by "joins" (i.e., by considering the pyramids over the faces of P with n_0 as apex) is a b.c.-triangulation of P.*

(ii) *The induced torus-equivariant projective, crepant, full desingularization*

$$f_{\mathcal{T}^{\mathrm{can}}} : X(\mathbb{Z}^d, \widehat{\Delta}_P\,(\mathcal{T}^{\mathrm{can}})) \longrightarrow X(\mathbb{Z}^d, \Delta_P) = U_{\tau_P}$$

possesses exactly one exceptional prime divisor

$$D_{n_0} = V\,(\mathbb{R}_{\geq 0} n_0) = X((\mathbb{Z}^d)(\mathbb{R}_{\geq 0} n_0), \mathrm{Star}(\mathbb{R}_{\geq 0} n_0; \widehat{\Delta}_P\,(\mathcal{T}^{\mathrm{can}})))$$

which is a projective, toric Fano manifold.

Proof. (i) follows directly from [12, Thm. 3.5] and Lemma 5.3 below. For (ii) note that P is, in particular, a *reflexive* polytope (cf. [1, 4.1.5]). This is equivalent to say that its polar polytope $P^* \subset (\mathbb{R}^d)^\vee$ with respect to $\mathrm{aff}(P)$ (having n_0 as its "origin") is again a lattice polytope (w.r.t. $(\mathbb{Z}^d)^\vee$). Since the rays of the fan $\mathrm{Star}(\mathbb{R}_{\geq 0} n_0; \widehat{\Delta}_P\,(\mathcal{T}^{\mathrm{can}}))$ are exactly the 1-dimensional cones determined by joining n_0 with the vertices of P, D_{n_0} is the $(d-1)$-dimensional projective toric variety associated to the normal fan of P^* (see §2 (k)). Thus, the fan $\mathrm{Star}(\mathbb{R}_{\geq 0} n_0; \widehat{\Delta}_P\,(\mathcal{T}^{\mathrm{can}}))$ is *strongly polytopal* (see [16, V.4.3 and V.4.4, p. 159]), and is composed of exclusively basic cones. Consequently, the exceptional prime divisor D_{n_0} has to carry the analytic structure of a smooth toric variety which is *Fano*, i.e., whose antidualizing sheaf is ample (cf. [1, 2.1.6 & 2.2.23]). □

Definition 3.20 ($\mathbb{H}_d$-compatible polytopes) Let $\mathbb{H}_d$ denote the affine hyperplane arrangement (of type $\widetilde{\mathcal{A}}_d$) in $\mathbb{R}^d$ consisting of the union of hyperplanes

$$\left\{ \left\{ \mathbf{x} \in \mathbb{R}^d \mid x_i = \kappa \right\}, 1 \leq i \leq d,\ \kappa \in \mathbb{Z} \right\} \cup \left\{ \left\{ \mathbf{x} \in \mathbb{R}^d \mid x_i - x_j = \lambda \right\}, 1 \leq i < j \leq d,\ \lambda \in \mathbb{Z} \right\} .$$

A lattice polytope Q will be called a $\mathbb{H}_d$*-compatible polytope* if $Q \sim P$, where $P \subset \mathbb{R}^d$ denotes a lattice polytope (w.r.t. $\mathbb{Z}^d$), such that the affine hulls $\mathrm{aff}(F)$ for all facets F of P belong to $\mathbb{H}_d$. The affine hyperplane arrangement $\mathbb{H}_d$ induces a basic triangulation $\mathcal{T}_{\mathbb{H}_d}$ of the *whole* space $\mathbb{R}^d$. In fact, $\mathcal{T}_{\mathbb{H}_d}$ is also coherent because there exists an overall well-defined strictly upper convex function on $|\mathcal{T}_{\mathbb{H}_d}|$ being constructible by means of appropriate sums of Heaviside functions (see [25, Ch. 3], and [12, Prop. 6.1]).

Theorem 3.21 *For $d \geq 2$, let $P \subset \bar{\mathbf{H}}^{(d)} \subset \mathbb{R}^d$ be a $(d-1)$-dimensional $\mathbb{H}_d$-compatible polytope w.r.t. $\mathbb{Z}^d$, and let $\mathcal{T}_{\mathbb{H}_d}\,|_P$ denote the triangulation of P determined by the restriction of $\mathcal{T}_{\mathbb{H}_d}$ to $|P|$. Then $\mathcal{T}_{\mathbb{H}_d}\,|_P$ is a b.c-triangulation, too, and the corresponding torus-equivariant projective, crepant, full desingularization*

$$f_{\mathcal{T}_{\mathbb{H}_d}|_P} : X(\mathbb{Z}^d, \widehat{\Delta}_P\,(\mathcal{T}_{\mathbb{H}_d}\,|_P)) \longrightarrow X(\mathbb{Z}^d, \Delta_P) = U_{\tau_P}$$

possesses exceptional prime divisors $D_n = V(\mathbb{R}_{\geq 0} n) = X((\mathbb{Z}^d)(\mathbb{R}_{\geq 0} n), \ \mathrm{Star}(\mathbb{R}_{\geq 0} n; \widehat{\Delta}_P(\mathcal{T}_{\mathbb{H}_d} |_P)))$ *for which*

$$D_n \cong \widehat{W}^{(d)}, \quad \text{if} \ \ n \in (\mathrm{int}\,(P)) \cap \mathbb{Z}^d,$$

and

$$D_n \cong \left\{ \text{a quasiprojective } \ (d-1) \text{-dimensional subvariety of } \ \widehat{W}^{(d)} \right\} \ ,$$

if $n \in (\partial P \backslash \mathrm{vert}(P)) \cap \mathbb{Z}^d$, *where* $\widehat{W}^{(d)}$ *denotes the projective toric Fano manifold obtained by a torus-equivariant, crepant, projective, full resolution* $\widehat{W}^{(d)} \longrightarrow W^{(d)}$ *of a* $(d-1)$*-dimensional projective, toric Fano variety* $W^{(d)}$ *(with at most Gorenstein singularities), induced by the triangulation* $\mathcal{T}_{\mathbb{H}_d}$. *In particular, as projective variety,* $W^{(d)}$ *admits an embedding* $W^{(d)} \hookrightarrow \mathbb{P}_{\mathbb{C}}^{d(d-1)}$ *and has the degree* $\binom{2\,(d-1)}{d-1}$ *w.r.t. this embedding.*

Proof. The first assertion is obvious. Let now $n \in (\mathrm{int}(P)) \cap \mathbb{Z}^d$. The *star* of n with respect to $\mathcal{T}_{\mathbb{H}_d}$ (in the sense of the theory of simplicial complexes) is lattice equivalent to a pure simplicial complex consisting of the triangulation $\mathcal{T}_{\mathbb{H}_d} |_{\mathcal{Z}^{(d)}}$ of a $(d-1)$-dimensional *lattice zonotope* $\mathcal{Z}^{(d)} \subset \mathbb{R}^{d-1}$ with d "zones" into basic simplices. (Our reference lattice here is that one being generated by $\mathrm{aff}(P) \cap \mathbb{Z}^d$). The zonotope $\mathcal{Z}^{(d)}$ can be also viewed as the convex hull of the union of the $[-1,0]$-cube with the $[0,1]$-cube, or, alternatively, as the Minkowski sum of d segments:

$$\mathcal{Z}^{(d)} = \left\{ \mathbf{x} = (x_1,..,x_{d-1}) \in \mathbb{R}^{d-1} \ \middle| \ \begin{array}{l} |x_i| \leq 1, \ \forall i, \ \ 1 \leq i \leq d-1, \ \ \text{and} \\ |x_i - x_j| \leq 1, \ \text{ for all } \ i,j, \text{ s.t. } \ 1 \leq i,j \leq d-1 \end{array} \right\} =$$
$$= \ \mathrm{conv}\left(([-1,0]^{d-1}) \cup ([0,1]^{d-1}) \right) =$$
$$= \frac{1}{2} \left([-e_1, e_1] + \cdots + [-e_{d-1}, e_{d-1}] + [-(e_1 + e_2 + \cdots + e_{d-1}), e_1 + e_2 + \cdots + e_{d-1}] \right)$$

with $\{e_1, e_2, \ldots, e_{d-1}\}$ denoting the standard basis of unit vectors of $\mathbb{R}^{d-1}$. Obviously,

$$\mathrm{vert}\left(\mathcal{Z}^{(d)}\right) = \left\{ \pm (e_{i_1} + e_{i_2} + \cdots + e_{i_k}) \ \middle| \ \begin{array}{l} \text{for all subsets of indices} \\ 1 \leq i_1 < i_2 < \cdots < i_k \leq d-1 \\ \text{and all } \ k, \ \ 1 \leq k \leq d-1 \end{array} \right\}$$

and

$$\#(\mathrm{vert}(\mathcal{Z}^{(d)})) = \#(\{\text{facets of } \mathcal{Z}^{(d)\,*}\}) = 2\left[\binom{d-1}{1} + \binom{d-1}{2} + \cdots + \binom{d-1}{d-1} \right] = 2\left(2^{d-1} - 1\right),$$

where $\mathcal{Z}^{(d)\,*}$ denotes the polar of $\mathcal{Z}^{(d)}$,

$$\mathcal{Z}^{(d)\,*} = \left\{ \mathbf{y} = (y_1, \ldots, y_{d-1}) \in (\mathbb{R}^{d-1})^{\vee} \ \middle| \ \begin{array}{l} |y_{i_1} + y_{i_2} + \cdots + y_{i_k}| \leq 1, \ \text{for all} \\ 1 \leq i_1 < i_2 < \cdots < i_k \leq d-1 \\ \text{and all } \ k, \ \ 1 \leq k \leq d-1 \end{array} \right\} ,$$

having the following $2\,(d-1) + 2\binom{d-1}{2} = d\,(d-1)$ vertices:

$$\mathrm{vert}\left(\mathcal{Z}^{(d)\,*}\right) = \left\{ \pm e_1^{\vee}, \pm e_2^{\vee}, \ldots, \pm e_{d-1}^{\vee} \right\} \cup \left\{ \pm \left(e_i^{\vee} - e_j^{\vee}\right) \ \middle| \ 1 \leq i < j \leq d-1 \right\}.$$

($\left\{e_1^{\vee}, e_2^{\vee}, \ldots, e_{d-1}^{\vee}\right\}$ denotes here the $\mathbb{R}$-basis of $(\mathbb{R}^{d-1})^{\vee}$ which is dual to $\{e_1, e_2, \ldots, e_{d-1}\}$).

Note that $\mathcal{Z}^{(d)}$, $\mathcal{Z}^{(d)\,*}$ are reflexive polytopes and can be inscribed in the cube $[-1,1]^{d-1}$. (Figures **2** and **3** illustrate them for $d = 3$ and $d = 4$). Let now $W^{(d)}$ be the $(d-1)$-dimensional projective toric variety being associated to the normal fan of $\mathcal{Z}^{(d)\,*}$ (as in §2 (k)). This fan is strongly polytopal because its rays are exactly the 1-dimensional cones in $\mathbb{R}^{d-1}$ determined by joining the origin with the vertices of $\mathcal{Z}^{(d)}$. On the other hand, by Theorem 2.6 and [1, 2.2.23], we see that $W^{(d)}$ is a

Gorenstein toric Fano variety (which is *singular* for $d \geq 4$). Moreover, the exceptional prime divisor $D_n = V(\mathbb{R}_{\geq 0}n) = X((\mathbb{Z}^d)(\mathbb{R}_{\geq 0}n), \mathrm{Star}(\mathbb{R}_{\geq 0}n; \widehat{\Delta}_P(\mathcal{T}_{\mathbb{H}_d}|_P)))$ is analytically isomorphic to $\widehat{W}^{(d)}$, where $\widehat{W}^{(d)}$ is that projective, toric Fano manifold which occurs as the overlying space of the torus-equivariant, projective, crepant desingularization of $W^{(d)}$ induced by restricting the b.c.-triangulation $\mathcal{T}_{\mathbb{H}_d}$ onto $\mathcal{Z}^{(d)}$. Of course, $W^{(d)}$ can be embedded into $\mathbb{P}_{\mathbb{C}}^{d(d-1)}$ via the map

$$W^{(d)} \ni w \longmapsto \left[w : \mathbf{e}\left(e_1^{\vee}\right)(w) : \mathbf{e}\left(-e_1^{\vee}\right)(w) : \cdots : \mathbf{e}\left(-(e_{d-2}^{\vee} - e_{d-1}^{\vee})\right)(w)\right] \in \mathbb{P}_{\mathbb{C}}^{d(d-1)},$$

defined by evaluating the torus characters at the points of $\mathcal{Z}^{(d)*} \cap \mathbb{Z}^d = \{\mathbf{0}\} \cup \mathrm{vert}(\mathcal{Z}^{(d)*})$ (cf. Fulton [17, p. 69]), because $\mathcal{Z}^{(d)*}$ is the lattice polytope which is determined by the anticanonical divisor of $W^{(d)}$. By [42, Thm. 4.16, p. 36], the degree of $W^{(d)}$ with respect to this embedding is equal to the normalized volume $\mathrm{Vol}_{\mathrm{norm}}\left(\mathcal{Z}^{(d)*}\right)$ of $\mathcal{Z}^{(d)*}$. It is worth mentioning that the facets of $\mathcal{Z}^{(d)*}$ are exactly the subpolytopes of the form

$$\pm\, \mathrm{conv}(\{e_{i_1}^{\vee}, e_{i_2}^{\vee}, \ldots, e_{i_k}^{\vee}\} \cup \{e_{i_1}^{\vee} - e_{j_1}^{\vee}, e_{i_2}^{\vee} - e_{j_2}^{\vee}, \ldots, e_{i_k}^{\vee} - e_{j_k}^{\vee}\}) \tag{3.5}$$

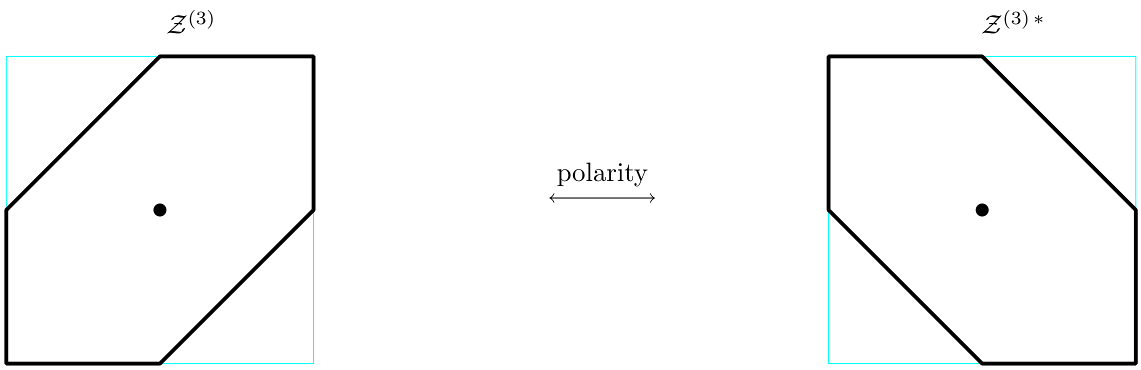
for all subsets of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq d-1$, with $1 \leq k \leq d-1$, and all possible indices

$$j_1 \in \{1, \ldots, d-1\} \smallsetminus \{i_1, i_2, \ldots, i_k\}, \ldots\ldots, j_k \in \{1, \ldots, d-1\} \smallsetminus \{i_1, i_2, \ldots, i_k\}\ .$$

Each facet (3.5) is nothing but the direct product of a $(d-(k+1))$-dimensional basic simplex with a $(k-1)$-dimensional basic simplex, for any k, $1 \leq k \leq d-1$, and consequently its relative, normalized volume equals $\binom{d-(k+1)+k-1}{k-1} = \binom{d-2}{k-1}$. Hence, since the normalized volume of a reflexive polytope is equal to the sum of the relative, normalized volumes of its facets, we get

$$\mathrm{Vol}_{\mathrm{norm}}\left(\mathcal{Z}^{(d)*}\right) = 2\left[\sum_{k=1}^{d-1} \binom{d-1}{k}\binom{d-2}{k-1}\right] = \tfrac{2}{d-1}\left[\sum_{k=1}^{d-1} \binom{d-1}{k}^2 \cdot k\right] = \binom{2\,(d-1)}{d-1}\ .$$

Finally, let us point out that if $n \in (\partial P \setminus \mathrm{vert}(P)) \cap \mathbb{Z}^d$, then by construction $\mathrm{Star}(\mathbb{R}_{\geq 0}n; \widehat{\Delta}_P(\mathcal{T}_{\mathbb{H}_d}|_P))$ is a subfan of the fan induced by the star of n with respect to the *entire* $\mathcal{T}_{\mathbb{H}_d}$. D_n can be therefore viewed as a torus-invariant non-compact subvariety of a projective toric variety which is analytically isomorphic to the above defined $W^{(d)}$. $\square$



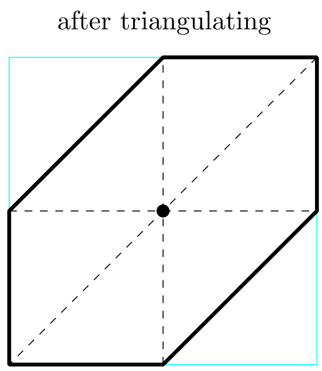
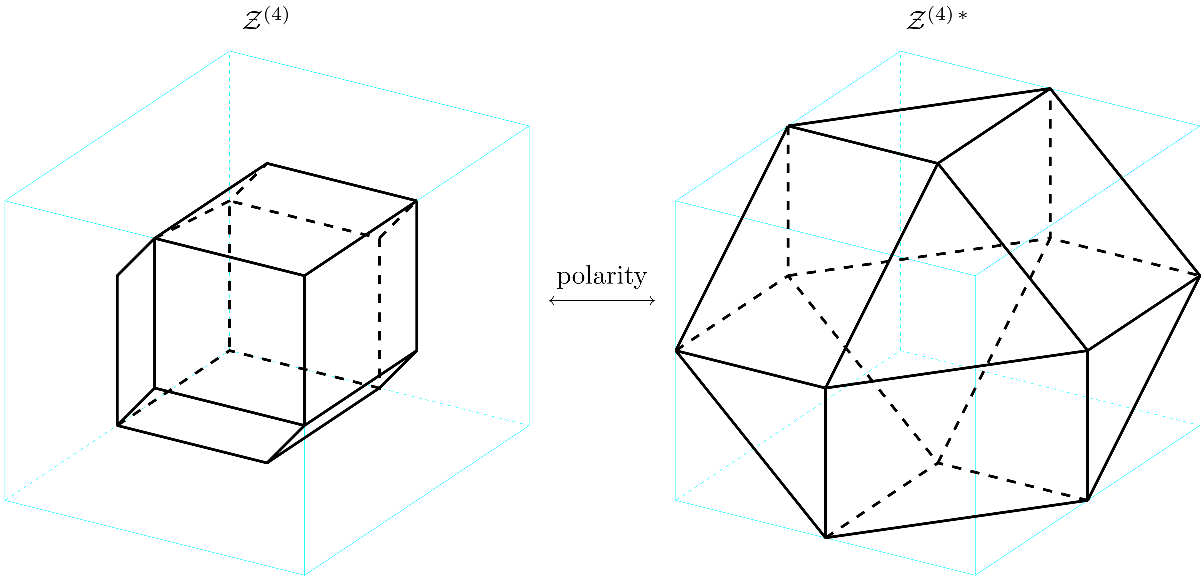


Figure 2



after triangulating

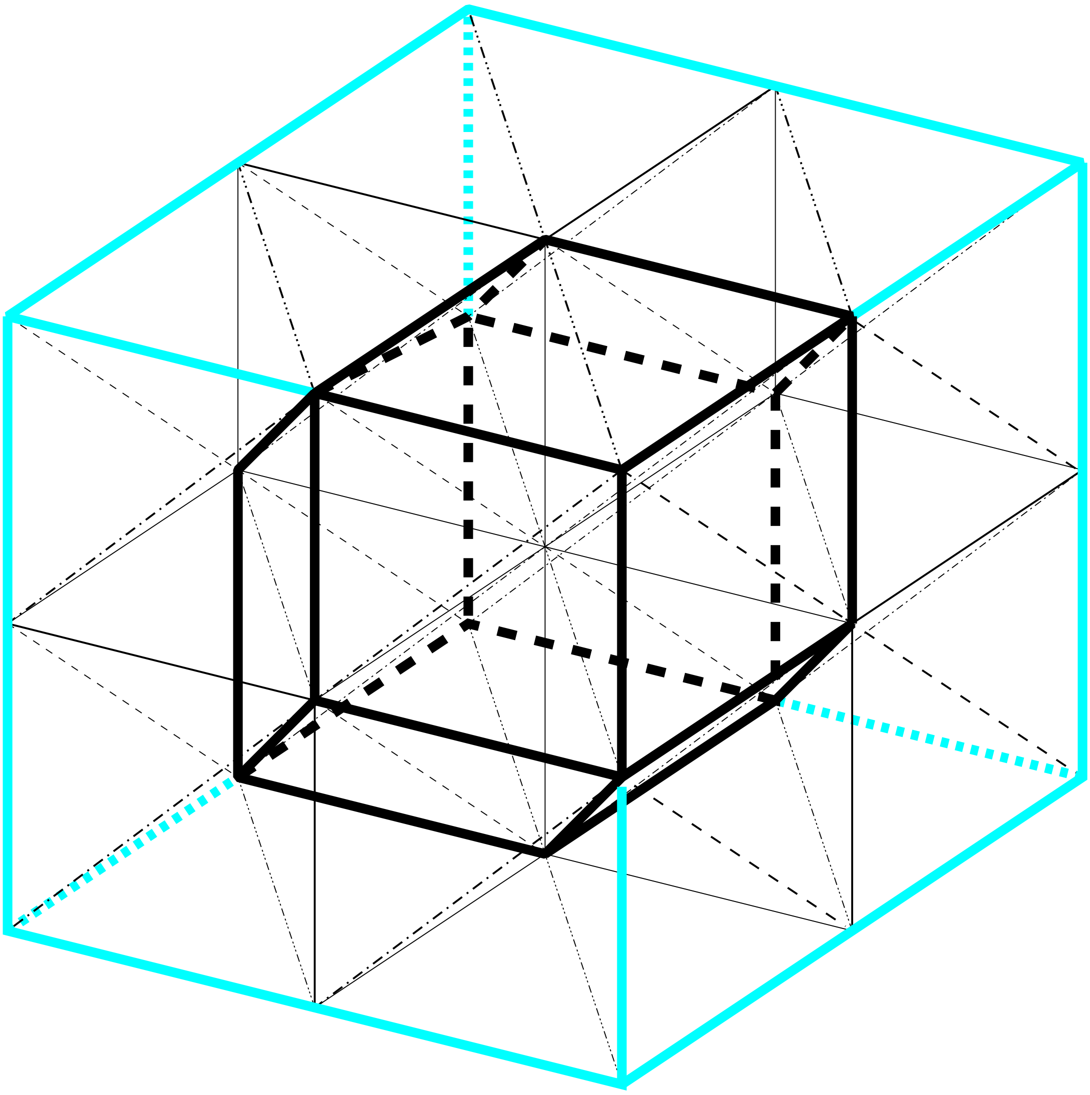


Figure 3

4 Nakajima's polytopes and Classification Theorem

Let $\mathbb{R}^d$ be the usual d-dimensional euclidean space, $\mathbb{Z}^d$ the usual rectangular lattice in $\mathbb{R}^d$ and $(\mathbb{Z}^d)^\vee$ its dual lattice in $(\mathbb{R}^d)^\vee = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^d, \mathbb{R})$. From now on, we shall represent the points of $\mathbb{R}^d$ by column vectors and the points of its dual $(\mathbb{R}^d)^\vee$ by row vectors.

Definition 4.1 A *sequence of free parameters of length* ℓ (w.r.t. $\mathbb{Z}^d$) is defined to be a finite sequence

$$\mathbf{m} := (m_1, m_2, \ldots, m_\ell), \quad 1 \leq \ell \leq d-1,$$

consisting of vectors $m_i := (m_{i,1}, m_{i,2}, \ldots, m_{i,d}), 1 \leq i \leq \ell$, of $\left(\mathbb{Z}^d\right)^\vee \smallsetminus \{(0, \ldots, 0)\}$ for which $m_{i,j} = 0$ for all i, $1 \leq i \leq \ell$, and all j, $1 \leq j \leq d$, with $i < j$. As $(\ell \times d)$-matrix such an $\mathbf{m}$ has the form:

$$\mathbf{m} = \left(\begin{array}{cccccc|c} m_{1,1} & 0 & 0 & \cdots & \cdots & 0 & 0 \ \cdots \ 0 \\ m_{2,1} & m_{2,2} & 0 & \cdots & \cdots & 0 & 0 \ \cdots \ 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & \cdots & \cdots & 0 & 0 \ \cdots \ 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \ \cdots \ \vdots \\ m_{\ell-1,1} & m_{\ell-1,2} & m_{\ell-1,3} & \cdots & \ddots & 0 & 0 \ \cdots \ 0 \\ m_{\ell,1} & m_{\ell,2} & m_{\ell,3} & \cdots & \cdots & m_{\ell,\ell} & \underbrace{0 \ \cdots \ 0}_{d-\ell \text{ zero-columns}} \end{array} \right) \tag{4.1}$$

Definition 4.2 (Nakajima's polytopes) Fixing the dimension d of our reference space, we define the polytopes $\left\{ P_{\mathbf{m}}^{(i)} \subset \bar{\mathbf{H}}^{(d)} \mid i \in \mathbb{N}, 1 \leq i \leq d \right\}$ lying on $\bar{\mathbf{H}}^{(d)} = \left\{ \mathbf{x} = (x_1, .., x_d)^\intercal \in \mathbb{R}^d \mid x_1 = 1 \right\}$ and being associated to an "admissible" free-parameter-sequence $\mathbf{m}$ as in (4.1) w.r.t. $\mathbb{Z}^d$ (with length $\ell = i-1$, for $2 \leq i \leq d$) by using induction on i; namely we define $P_{\mathbf{m}}^{(1)} := \{(1, \underbrace{0, 0, \ldots, 0, 0}_{(d-1)\text{-times}})\}$, and for $2 \leq i \leq d$,

$$P_{\mathbf{m}}^{(i)} := \mathrm{conv}\left(P_{\mathbf{m}}^{(i-1)} \cup \left\{ (\mathbf{x}', \langle m_{i-1}, \mathbf{x} \rangle, \underbrace{0, .., 0}_{(d-i)\text{-times}})^\intercal \;\middle|\; \mathbf{x} = (\underbrace{x_1, x_2, .., x_{i-1}}_{\substack{\| \\ \mathbf{x}'}}, \underbrace{0, .., 0}_{(d-i+1)\text{-times}})^\intercal \in P_{\mathbf{m}}^{(i-1)} \right\} \right).$$

($P_{\mathbf{m}}^{(i)}$ is obviously $(i-1)$-dimensional). For $\mathbf{m}$ to be "*admissible*" means that

$$\langle m_{i-1}, \mathbf{x} \rangle \geq 0, \quad \forall \mathbf{x}, \quad \mathbf{x} = (x_1, x_2, \ldots, x_{i-1}, \underbrace{0, \ldots, 0}_{(d-i+1)\text{-times}})^\intercal \in P_{\mathbf{m}}^{(i-1)}.$$

Any lattice $(i-1)$-polytope P which is lattice equivalent to a $P_{\mathbf{m}}^{(i)}$ (as defined above) will be called a *Nakajima polytope* (w.r.t. $\mathbb{R}^d$).

Example 4.3 (i) For $i = d = 1$, we have trivially $P_{\mathbf{m}}^{(1)} = \{1\}$.
(ii) For $d = 2$, $\mathbf{m} = (m_{1,1}, 0)$ we have $P_{\mathbf{m}}^{(1)} = \{(1,0)^\intercal\}$ and

$$P_{\mathbf{m}}^{(2)} = \mathrm{conv}\left(\{(1,0)^\intercal\} \cup \{(1, \langle m_1, (1,0) \rangle)^\intercal\}\right) = \mathrm{conv}\left(\{(1,0)^\intercal\} \cup \{(1, m_{1,1})^\intercal\}\right), \ m_{1,1} > 0.$$

(iii) For $i = d = 3$, and

$$\mathbf{m} = \begin{pmatrix} m_{1,1} & 0 & 0 \\ m_{2,1} & m_{2,2} & 0 \end{pmatrix}$$

we obtain

$$P_{\mathbf{m}}^{(3)} = \mathrm{conv}\left(\{(1,0,0)^\intercal, (1, m_{1,1}, 0)^\intercal, (1, 0, m_{2,1})^\intercal, (1, m_{1,1}, m_{2,1} + m_{1,1} m_{2,2})^\intercal\}\right)$$

with

$$m_{1,1} > 0, \quad m_{2,1} \geq 0, \quad m_{2,1} + m_{1,1}m_{2,2} \geq 0, \quad (m_{2,1}, m_{2,2}) \neq (0,0)\,.$$

(iv) Finally, for $i = d = 4$, and

$$\mathbf{m} = \begin{pmatrix} m_{1,1} & 0 & 0 & 0 \\ m_{2,1} & m_{2,2} & 0 & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & 0 \end{pmatrix}$$

we get

$$P_{\mathbf{m}}^{(4)} = \operatorname{conv}\left(\left\{ \begin{array}{c} (1,0,0,0)^{\intercal}, (1,m_{1,1},0,0)^{\intercal}, (1,0,m_{2,1},0)^{\intercal}, (1,m_{1,1},m_{2,1}+m_{1,1}m_{2,2},0)^{\intercal}, \\ (1,0,0,m_{3,1})^{\intercal}, (1,m_{1,1},0,m_{3,1}+m_{1,1}m_{3,2})^{\intercal}, (1,0,m_{2,1},m_{3,1}+m_{3,3}m_{2,1})^{\intercal}, \\ (1,m_{1,1},m_{2,1}+m_{1,1}m_{2,2},m_{3,1}+m_{3,2}m_{1,1}+m_{2,1}m_{3,3}+m_{1,1}m_{2,2}m_{3,3})^{\intercal} \end{array} \right\}\right)$$

with

$$\left\{ \begin{array}{l} m_{1,1} > 0,\ m_{2,1} \geq 0,\ m_{2,1} + m_{1,1}m_{2,2} \geq 0,\ m_{3,1} \geq 0,\ m_{3,1} + m_{1,1}m_{3,2} \geq 0, \\ m_{3,1} + m_{3,3}m_{2,1} \geq 0,\ m_{3,1} + m_{3,2}m_{1,1} + m_{2,1}m_{3,3} + m_{1,1}m_{2,2}m_{3,3} \geq 0, \\ (m_{2,1}, m_{2,2}) \neq (0,0), \quad (m_{3,1}, m_{3,2}, m_{3,3}) \neq (0,0,0) \end{array} \right.$$

In the Figures **4** and **5** we illustrate the lattice polytopes $P_{\mathbf{m}}^{(3)}, P_{\mathbf{m}}^{(4)}$, respectively, for

$$\mathbf{m} = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{m} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & -1 & -1 & 0 \end{pmatrix}.$$

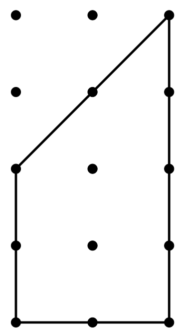


Figure 4

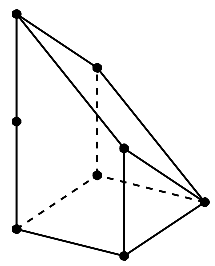


Figure 5

Lemma 4.4 (Description by inequalities) *The $(i-1)$-dimensional polytope $P_{\mathbf{m}}^{(i)}$ (w.r.t. $\mathbb{R}^d$) associated to an admissible free-parameter-sequence* $\mathbf{m}$ *can be written as a bounded solution set of a finite system of linear inequalities as follows*

$$P_{\mathbf{m}}^{(i)} = \left\{ \mathbf{x} = (x_1, .., x_d)^{\intercal} \in \mathbb{R}^d \;\middle|\; \begin{array}{l} x_1 = 1,\ 0 \leq x_{j+1} \leq \langle m_j, \mathbf{x} \rangle,\ \forall\ j,\ 1 \leq j \leq i-1 \\ \text{and} \quad x_\mu = 0,\ \forall\ \mu, \quad i+1 \leq \mu \leq d \end{array} \right\}.$$

Another useful geometric description of Nakajima polytopes can be provided by means of suitably cutted half-line prisms.

Definition 4.5 (Half-line prisms) Let $(i,d) \in \mathbb{N}^2$, $2 \leq i \leq d$, and $Q \subset \bar{\mathbf{H}}^{(d-1)}$ be a $(i-2)$-dimensional polytope. We define the *half-line prism* $\mathbf{Pr}_{\mathrm{hl}}^{(i)}(Q)$ over Q as the $(i-1)$-dimensional polyhedron

$$\mathbf{Pr}_{\mathrm{hl}}^{(i)}(Q) := Q \times \mathbb{R}_{\geq 0} = \left\{ (\mathbf{x}, t) \in \bar{\mathbf{H}}^{(d)} \mid \mathbf{x} \in Q, t \in \mathbb{R}_{\geq 0} \right\},$$

and identify $Q \subset \bar{\mathbf{H}}^{(d-1)}$ with $Q \times \{0\} \subset \bar{\mathbf{H}}^{(d-1)} \times \{0\} \hookrightarrow \bar{\mathbf{H}}^{(d)}$. (The only difference between $\mathbf{Pr}_{\mathrm{hl}}^{(i)}(Q)$ and a usual prism, is that the first one is "open from above").

Lemma 4.6 (Reduction Lemma) *A lattice polytope* $P \subset \bar{\mathbf{H}}^{(d)} \hookrightarrow \mathbb{R}^d$ *is a Nakajima polytope of dimension* $i-1$ (w.r.t. $\mathbb{R}^d$) ***iff*** $P \sim \{\text{a lattice point in } \bar{\mathbf{H}}^{(d)} \cap \mathbb{Z}^d\}$, *for* $i = 1$, *while for* $2 \leq i \leq d$,

$$P \sim (\mathbf{Pr}_{\mathrm{hl}}^{(i)}(Q)) \cap \left\{ \mathbf{x} = (1, x_2, .., x_d)^{\intercal} \in \bar{\mathbf{H}}^{(d)} \;\middle|\; \begin{array}{l} x_i \leq \sum_{j=1}^{i-1} \lambda_j x_j \quad \text{and} \\ x_\mu = 0, \ \forall \ \mu, \ \ i+1 \leq \mu \leq d \end{array} \right\},$$

where the facet Q *of the right-hand side is a Nakajima polytope of dimension* $i-2$ (*w.r.t.* $\mathbb{R}^{d-1}$, *identified with* $\mathbb{R}^{d-1} \times \{0\} \hookrightarrow \mathbb{R}^d$), *and* $(\lambda_1, \ldots, \lambda_{i-1}, -1, \underbrace{0, 0, \ldots, 0, 0}_{(d-i)\text{-times}}) \in (\mathbb{Z}^d)^\vee$ *expresses a* (*not identically zero*) *functional with non-negative values on* $Q \times \{0\} \hookrightarrow \bar{\mathbf{H}}^{(d)}$.

Proof. For $i = 1$ there is nothing to be shown. Let $i \in \{2, \ldots, d\}$. If $P \sim P_{\mathbf{m}}^{(i)}$ for $\mathbf{m}$ an admissible sequence of free parameters of length $i-1$ (w.r.t. $\mathbb{Z}^d$), then

$$P_{\mathbf{m}}^{(i)} = (\mathbf{Pr}_{\mathrm{hl}}^{(i)}(P_{\overline{\mathbf{m}}}^{(i-1)})) \cap \left\{ \mathbf{x} = (1, x_2, .., x_d)^{\intercal} \in \bar{\mathbf{H}}^{(d)} \;\middle|\; \begin{array}{l} x_i \leq \sum_{j=1}^{i-1} \lambda_j x_j \quad \text{and} \\ x_\mu = 0, \ \forall \ \mu, \ \ i+1 \leq \mu \leq d \end{array} \right\},$$

where $P_{\overline{\mathbf{m}}}^{(i-1)}$ is a Nakajima polytope of dimension $i-2$, with $P_{\overline{\mathbf{m}}}^{(i-1)} = \{(1, 0, ..., 0)\}$ for $i = 2$, and determined by the admissible sequence of free parameters $\overline{\mathbf{m}} = (m_1, m_2, \ldots, m_{i-2})$ of length $i-2$, for $i \geq 3$, so that $\mathbf{m} = \begin{pmatrix} \overline{\mathbf{m}} & 0 & 0 \\ m_{i-1} & & 0 \end{pmatrix}$ under the identification of $P_{\overline{\mathbf{m}}}^{(i-1)}$ with $P_{\overline{\mathbf{m}}}^{(i-1)} \times \{0\} \hookrightarrow P_{\mathbf{m}}^{(i)}$. Hence, it is enough to take $Q = P_{\overline{\mathbf{m}}}^{(i-1)}$ and $\lambda_1 = m_{i-1,1}$, $\lambda_2 = m_{i-1,2}$, $\ldots$, $\lambda_{i-1} = m_{i-1,i-1}$. And conversely, having the intersection of the half-line prism $\mathbf{Pr}_{\mathrm{hl}}^{(i)}(Q)$ with the half-space determined by the non-trivial non-negatively-valued functional $(\lambda_1, \ldots, \lambda_{i-1}, -1, 0, \ldots, 0) \in (\mathbb{Z}^d)^\vee$ on $Q \times \{0\}$ as our starting-point, we may construct (by the backtracking method, i.e., by passing from the last to the last but one row etc.) an admissible sequence $\mathbf{m}$ of free parameters of length $i-1$ (w.r.t. $\mathbb{Z}^d$), such that $P \sim P_{\mathbf{m}}^{(i)}$. $\square$

Now Nakajima's Classification Theorem [32, Thm. 1.5, p. 86] can be formulated as follows:

Theorem 4.7 (Nakajima's Classification of Toric L.C.I.'s) *Let* N *be a free* $\mathbb{Z}$*-module of rank* r, *and* $\sigma \subset N_{\mathbb{R}}$ *a s.c.p. cone of dimension* $d \leq r$. *Moreover, let* U_σ *denote the affine toric variety associated to* σ, *and* $U_{\sigma'}$ *as in* §3 (a). *Then* U_σ *is local complete intersection* ***if and only if*** *there exists an admissible sequence* $\mathbf{m}$ *of free parameters of length* $d-1$ (*w.r.t.* $\mathbb{Z}^d$), *such that for any standard representative* $U_{\tau_P} \cong U_{\sigma'}$ *of* U_σ *we have* $P \sim P_{\mathbf{m}}^{(d)}$, *i.e.,* P *is a Nakajima* $(d-1)$*-dimensional polytope* (*w.r.t.* $\mathbb{R}^d$).

Remark 4.8 (i) Theorem 4.7 was first proved in dimension 3 by Ishida [23, Thm. 8.1, p. 136]. Previous classification results, due to Watanabe [43], cover essentially only the class of the $\mathbb{Q}$-factorial toric l.c.i.'s in all dimensions. (The term "Watanabe simplex" introduced in [12, 5.13], can be used, up to lattice equivalence, as a synonym for a Nakajima polytope which is simultaneously a simplex.)

(ii) Obviously, U_σ is a l.c.i. $\Longleftrightarrow U_{\sigma'} \cong U_{\tau_P}$ is a g.c.i. (Since in the setting of [12], it was always assumed that $d = r$, the abelian quotient "g.c.i."-spaces were abbreviated therein simply as "c.i.'s").

(iii) For P a non-basic Nakajima polytope, $(U_{\tau_P}, \mathrm{orb}(\tau_P))$ is a toric g.c.i.-singularity.

(iv) For P a Nakajima $(d-1)$-polytope and τ_P non-basic w.r.t. $\mathbb{Z}^d$, $\mathrm{orb}(\tau_P) \in U_{\tau_P}$ has splitting codimension $\varkappa$, with $2 \leq \varkappa \leq d-1$, iff P is lattice-equivalent to the join $\check{P} \star \mathbf{s}$ of a $(\varkappa-1)$-dimensional (non-basic) Nakajima polytope $\check{P}$ with a basic $(d-\varkappa-1)$-simplex $\mathbf{s}$, which lie in adjacent lattice hyperplanes, and $\varkappa$ is minimal w.r.t. this property.

(v) It is easy for every $P \subset \bar{\mathbf{H}}^{(d)}$, with $P \sim P_{\mathbf{m}}^{(d)}$, to verify that

$$d \leq \#(\mathrm{vert}(P)) \leq 2^{d-1} \quad \text{and} \quad d \leq \#(\{\text{facets of } P\}) \leq 2\,(d-1) \tag{4.2}$$

5 Proof of Main Theorem and of Koszul-property

(a) By Prop. 3.14 (ii), (iii), and Thm. 4.7, our Main Theorem 1.2 is equivalent to the following:

Theorem 5.1 *All Nakajima polytopes admit b.c.-triangulations in all dimensions.*

Our proof of 5.1 relies on the construction of the desired lattice triangulations via the classical "pulling operation" of vertices of a point configuration and the "Key-Lemma" 5.7.

Definition 5.2 (Pulling vertices) Consider a finite set of points $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset \mathbb{R}^d$ and let $\mathcal{S} = \{P_1, P_2, ..., P_\nu\}$ denote a polytopal subdivision of $\text{conv}(\mathcal{V})$ with $\text{vert}(\mathcal{S}) \subseteq \mathcal{V}$. For any $i \in \{1, \ldots, k\}$, we define a refinement $\mathsf{p}_{\mathbf{v}_i}(\mathcal{S})$ of $\mathcal{S}$ (called the *pulling* of $\mathbf{v}_i$) as follows:

(i) $\mathsf{p}_{\mathbf{v}_i}(\mathcal{S})$ contains all P_j's for which $\mathbf{v}_i \notin P_j$, and

(ii) if $\mathbf{v}_i \in P_j$, then $\mathsf{p}_{\mathbf{v}_i}(\mathcal{S})$ contains all the polytopes having the form $\text{conv}(F \cup \mathbf{v}_i)$, with F a facet of P_j such that $\mathbf{v}_i \notin F$.

Lemma 5.3 *The refinements of $\mathcal{S}$ obtained by pulling* **all** *the points of $\mathcal{V}$ (in arbitrary order) are triangulations of* $\text{conv}(\mathcal{V})$ *with vertex set $\mathcal{V}$.*

Proof. This is an easy exercise (cf. [28, §2]). □

Example 5.4 (Realization of pullings by "full flags") Suppose that $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset \mathbb{R}^d$ is the set of vertices of a d-dimensional polytope P. In this special case, the triangulation $\mathcal{T}$ obtained after performing the pulling operation for all points of $\mathcal{V}$ has a nice geometric realization due to Stanley (see [40, §1]). For every face F of P define $\mathbf{v}(F) := \mathbf{v}_j$, where $j := \min\{i \mid \mathbf{v}_i \in F\}$. A *full flag* of P is a chain $\mathcal{F}$ of faces $F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_d = P$, such that $\dim(F_i) = i$, for all i, $0 \leq i \leq d$, and $\mathbf{v}(F_{i-1}) \neq \mathbf{v}(F_i)$, for all i, $1 \leq i \leq d$. For any full flag $\mathcal{F}$ define $\mathbf{v}(\mathcal{F}) := \{\mathbf{v}(F_0), .., \mathbf{v}(F_d)\}$. Then the simplices of the triangulation

$$\mathcal{T} = \mathsf{p}_{\mathbf{v}_k}(\mathsf{p}_{\mathbf{v}_{k-1}} \cdots \cdots (\mathsf{p}_{\mathbf{v}_2}(\mathsf{p}_{\mathbf{v}_1}(\{P\})))) \tag{5.1}$$

constructed by pulling the points of $\mathcal{V}$ in the order $\mathbf{v}_1, \ldots, \mathbf{v}_k$ (by starting from the "trivial" subdivision $\{P\}$) are exactly the elements of the set $\{\text{conv}(\mathbf{v}(\mathcal{F})) \mid \mathcal{F} \text{ a full flag of } P\}$.

The pulling of a vertex point of a polytope (and therefore triangulations of the form (5.1) too) are known to be coherent (cf. Lee [28, p. 448], and [29, p. 275]). In fact, a slightly stronger statement is also true:

Lemma 5.5 (Coherency preservation by pulling operation) *Let $\mathcal{V}$ be a set of finite points in $\mathbb{R}^d$ and $\mathcal{S} = \{P_1, P_2, ..., P_\nu\}$ an arbitrary coherent polytopal subdivision of* $\text{conv}(\mathcal{V})$ *with* $\text{vert}(\mathcal{S}) \subseteq \mathcal{V}$. *Then the refinement $\mathsf{p}_{\mathbf{v}_0}(\mathcal{S})$ of $\mathcal{S}$, for a $\mathbf{v}_0 \in \mathcal{V}$, forms a coherent polytopal subdivision of* $\text{conv}(\mathcal{V})$.

Proof. Let $\omega : \mathcal{V} \to \mathbb{R}$ be a height function on $\mathcal{V}$, for which $\mathcal{S} = \mathcal{S}_\omega$ (in the notation of Lemma 3.9). The (maximal dimensional) polytopes $P_1, ..., P_\nu$ of $\mathcal{S}$ are images of the lower facets of the polytope $Q_\omega := \text{conv}(\{(\mathbf{v}, \omega(\mathbf{v})) \in \mathbb{R}^{d+1} \mid \mathbf{v} \in \text{vert}(\mathcal{S})\})$ under the projection $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ w.r.t. the last coordinate. Let $\mathfrak{l}_1, \ldots, \mathfrak{l}_\nu \in (\mathbb{R}^d)^\vee$ denote functionals for which

$$\mathfrak{l}_i(\mathbf{v}) - \omega(\mathbf{v}) \leq c_i, \quad \text{for all } \mathbf{v} \in \text{vert}(\mathcal{S}), \quad 1 \leq i \leq \nu,$$

and appropriate c_i's $\in \mathbb{R}$, so that the equality is valid only for $\mathbf{v} \in P_i$, i.e., so that $\mathfrak{l}_i$ "determines" P_i. We define

$$t_0 := \max\{\mathfrak{l}_i(\mathbf{v}_0) - c_i \mid 1 \leq i \leq \nu\} = \min\{t \in \mathbb{R} : (\mathbf{v}_0, t) \in Q_\omega\} .$$

Obviously, $(\mathbf{v}_0, t_0)$ belongs to the boundary of Q_ω (see Fig. **6**). Without loss of generality, we may assume that $\nu \geq 2$ and that the maximum of the differences $\mathfrak{l}_i(\mathbf{v}_0) - c_i$ is achieved for $1 \leq i \leq j$, but not for $j+1 \le i \le \nu$, for some index $j \in \{1,2,..,\nu-1\}$. (This means that $\mathbf{v}_0 \in P_i$ for $1 \le i \le j$, but $\mathbf{v}_0 \notin P_i$ for $j+1 \le i \le \nu$). We define

$$\omega' : \mathrm{vert}\left(\mathsf{p}_{\mathbf{v}_0}\left(\mathcal{S}\right)\right) \longrightarrow \mathbb{R}, \quad \text{with} \quad \omega'\left(\mathbf{v}\right) := \begin{cases} \omega\left(\mathbf{v}\right), & \text{if } \ \mathbf{v} \neq \mathbf{v}_0 \\ t_0 - \varepsilon, & \text{if } \ \mathbf{v} = \mathbf{v}_0 \end{cases}$$

where $\varepsilon > 0$ is chosen to be small enough for ensuring that $\mathfrak{l}_i\left(\mathbf{v}_0\right) - \omega'\left(\mathbf{v}_0\right) < c_i$, for all i, $j+1 \le i \le \nu$, i.e., for setting $\left(\mathbf{v}_0, \omega'\left(\mathbf{v}_0\right)\right)$ into a "general position" w.r.t. the lower envelope of Q_ω. If we define $Q_{\omega'} := \mathrm{conv}(\{(\mathbf{v}, \omega'(\mathbf{v})) \in \mathbb{R}^{d+1} \mid \mathbf{v} \in \mathrm{vert}(\mathsf{p}_{\mathbf{v}_0}(\mathcal{S}))\})$, then the faces of $Q_{\omega'}$ are the faces of Q_ω which do not contain $\left(\mathbf{v}_0, \omega'\left(\mathbf{v}_0\right)\right)$, together with the faces of type $\mathrm{conv}(\{(\mathbf{v}_0, \omega'\left(\mathbf{v}_0\right))\} \cup F)$, where F is a face of some facet of Q_ω containing $\left(\mathbf{v}_0, t_0\right)$. Thus, the projection $\pi(\mathrm{conv}(\{(\mathbf{v}_0, \omega'\left(\mathbf{v}_0\right))\} \cup F))$ onto $\mathbb{R}^d$ is a subset of the projection of this facet, and $\mathsf{p}_{\mathbf{v}_0}\left(\mathcal{S}\right)$ is exactly the polytopal subdivision of $\mathrm{conv}(\mathcal{V})$ induced by the above defined height function ω' (again in the sense of Lemma 3.9). $\square$

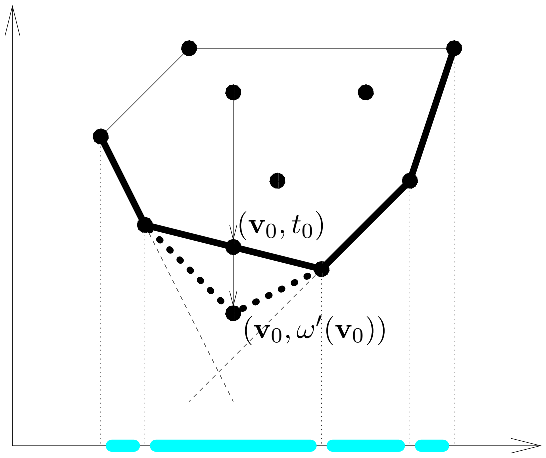


Figure 6

Theorem 5.6 *Let $\mathcal{V}$ be a set of finite points in $\mathbb{R}^d$ and $\mathcal{S}$ a coherent polytopal subdivision of* $\mathrm{conv}(\mathcal{V})$ *with* $\mathrm{vert}(\mathcal{S}) \subseteq \mathcal{V}$. *Then $\mathcal{S}$ can be always refined to a coherent triangulation $\mathcal{T}$ of* $\mathrm{conv}(\mathcal{V})$, *such that* $\mathrm{vert}\left(\mathcal{T}\right) = \mathcal{V}$.

Proof. By sequentially pulling *all* the points of $\mathcal{V}$ (in arbitrary order), and by using Lemmas 5.3 and 5.5, we may always construct such a coherent triangulation $\mathcal{T}$ of $\mathrm{conv}(\mathcal{V})$. $\square$

Lemma 5.7 (Key-Lemma) *Let* $\mathbf{s} \subset \mathbb{R}^d$ *be a $(d-2)$-dimensional simplex with*

$$\mathrm{vert}\left(\mathbf{s}\right) = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{d-1}\} \subset \bar{\mathbf{H}}^{(d-1)} \cap \mathbb{Z}^d \hookrightarrow \bar{\mathbf{H}}^{(d)} \cap \mathbb{Z}^d, \quad d \ge 2,$$

and let $\mathbf{s}' \subset \mathbb{R}^d$ *denote a $(d-1)$-dimensional simplex with*

$$\mathrm{vert}\left(\mathbf{s}'\right) = \left\{\mathbf{v}_1', \mathbf{v}_2', \ldots, \mathbf{v}_{d-1}', \mathbf{v}_d'\right\} \subset \left(\mathbf{Pr}_{\mathrm{hl}}^{(d)}\left(\mathbf{s}\right)\right) \cap \mathbb{Z}^d \hookrightarrow \bar{\mathbf{H}}^{(d)} \cap \mathbb{Z}^d.$$

If $\mathbf{s}$ *is a basic simplex and* $\mathbf{s}'$ *an elementary simplex (w.r.t.* $\mathbb{Z}^d$*), then* $\mathbf{s}'$ *has to be basic too.*

Proof. The property for a lattice simplex to be elementary or basic remains invariant among all the members of its lattice equivalence class. Since $\mathbf{s}$ is embedded into $\bar{\mathbf{H}}^{(d)} \hookrightarrow \mathbb{R}^d$ and is assumed to be basic, there exists an affine integral transformation $\Phi : \mathbb{R}^d \to \mathbb{R}^d$, such that $\Phi\left(\mathbf{v}_1\right) = e_1$, $\Phi\left(\mathbf{v}_i\right) = e_1 + e_i$, for all i, $2 \le i \le d-1$, and $\Phi(\bar{\mathbf{H}}^{(d)}) = \bar{\mathbf{H}}^{(d)}$, where $\{e_1, \ldots, e_{d-1}, e_d\}$ is the standard basis of unit vectors of $\mathbb{R}^d$ (!). This induces the lattice equivalence:

$$\mathbf{s} \sim \widetilde{\mathbf{s}}, \quad \text{with} \quad \widetilde{\mathbf{s}} := \ \mathrm{conv}\left(\{e_1, e_1+e_2, e_1+e_3, \ldots, e_1+e_{d-1}\}\right) \ .$$

We define $\widetilde{\mathbf{s}}' := \Phi(\mathbf{s}')$. Since $\widetilde{\mathbf{s}}'$ is $(d-1)$-dimensional and

$$\mathrm{vert}(\widetilde{\mathbf{s}}') = \left\{\Phi(\mathbf{v}_1'), \Phi(\mathbf{v}_2'), \ldots, \Phi\left(\mathbf{v}_{d-1}'\right), \Phi(\mathbf{v}_d')\right\} \subset \left(\mathbf{Pr}_{\mathrm{hl}}^{(d)}(\widetilde{\mathbf{s}})\right) \cap \mathbb{Z}^d,$$

$d-1$ among the d vertices of $\widetilde{\mathbf{s}}'$, e.g., up to enumeration of indices, say the first $d-1$ ones, must be of the form

$$\Phi(\mathbf{v}_1') = e_1 + \gamma_1 \cdot e_d, \quad \Phi(\mathbf{v}_i') = e_1 + e_i + \gamma_i \cdot e_d, \quad \forall i, \quad 2 \leq i \leq d-1,$$

for a $(d-1)$-tuple $(\gamma_1, \gamma_2, \ldots, \gamma_{d-1}) \in (\mathbb{Z}_{\geq 0})^{d-1}$. Moreover,

$$\Phi(\mathbf{v}_d') \in \{\mathbb{R}_{\geq 0}(e_1 + \gamma_1 \cdot e_d) \cap \mathbb{Z}^d\} \cup \left(\bigcup_{i=2}^{d-1} \{\mathbb{R}_{\geq 0}(e_1 + e_i + \gamma_i \cdot e_d) \cap \mathbb{Z}^d\}\right).$$

On the other hand, $\widetilde{\mathbf{s}}' \cap \mathbb{Z}^d = \mathrm{vert}(\widetilde{\mathbf{s}}')$ means that

$$\Phi(\mathbf{v}_d') \in \{e_1 + (\gamma_1 \pm 1) \cdot e_d\} \cup \left(\bigcup_{i=2}^{d-1} \{e_1 + e_i + (\gamma_i \pm 1) \cdot e_d\}\right),$$

because otherwise $\widetilde{\mathbf{s}}'$ could not be elementary (cf. Figure **7**). Now since the matrices

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \cdots & \gamma_{d-1} & \gamma_1 \pm 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & & \vdots & 1 \text{ (i-th row)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \cdots & \gamma_{d-1} & \gamma_i \pm 1 \end{pmatrix}$$

for all i, $2 \leq i \leq d-1$, have always determinants equal to ± 1, we obtain $\mathrm{mult}(\widetilde{\mathbf{s}}'; \mathbb{Z}^d) = 1$, and consequently both $\widetilde{\mathbf{s}}'$ and $\mathbf{s}'$ have to be basic simplices. $\square$

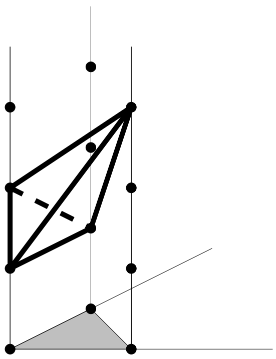


Figure 7

Proof of Theorem 5.1: By Thm. 4.7 it suffices to show that all $(d-1)$-dimensional Nakajima polytopes $P \subset \bar{\mathbf{H}}^{(d)} \subset \mathbb{R}^d$ (with $P \sim P_{\mathbf{m}}^{(d)}$, where $\mathbf{m}$ is an admissible sequence of length $d-1$) admit b.c.-triangulations. We shall use induction on the dimension d of the ambient space. For $d \leq 3$ this is obviously trivial. The proof will take place for any fixed $d \geq 4$ by assuming that the assertion is true for $d-1$. By Reduction Lemma 4.6, we may write P as the intersection

$$P = \left(\mathbf{Pr}_{\mathrm{hl}}^{(d)}(Q)\right) \cap \left\{\mathbf{x} = (1, x_2, .., x_d)^{\intercal} \in \bar{\mathbf{H}}^{(d)} \;\middle|\; x_d \leq \sum_{j=1}^{d-1} \lambda_j x_j\right\}$$

of a half-line prism over Nakajima $(d-2)$-polytope $Q \subset \bar{\mathbf{H}}^{(d-1)} \hookrightarrow \bar{\mathbf{H}}^{(d)} \subset \mathbb{R}^d$ with a half-space in $\bar{\mathbf{H}}^{(d)}$ determined by a non-trivial, non-negatively-valued functional $(\lambda_1, \ldots, \lambda_{i-1}, -1, 0, \ldots, 0) \in (\mathbb{Z}^d)^\vee$ on $Q \times \{0\}$. By induction hypothesis, Q possesses a coherent triangulation, say $\mathfrak{T}$, with $\mathrm{vert}(\mathfrak{T}) = Q \cap \mathbb{Z}^d$, into *basic* simplices**.** This means that

$$\mathcal{S}_{\mathfrak{T}} := \bigcup_{\text{all simplices } \mathbf{s} \in \mathfrak{T}} P_{\mathbf{s}}, \text{ where } P_{\mathbf{s}} := (\mathbf{Pr}_{\mathrm{hl}}^{(d)}(\mathbf{s})) \cap \left\{ \mathbf{x} = (1, x_2, .., x_d)^\intercal \in \bar{\mathbf{H}}^{(d)} \mid x_d \leq \sum_{j=1}^{d-1} \lambda_j x_j \right\},$$

forms a polytopal lattice subdivision of P into polytopes constructed by the half-line prisms over all the simplices of $\mathfrak{T}$.

• The polytopal subdivision $\mathcal{S}_{\mathfrak{T}}$ itself is coherent. Indeed, if the coherent triangulation $\mathfrak{T}$ of Q is induced by a height function $\omega : \mathrm{vert}(\mathfrak{T}) \to \mathbb{R}$ (as in Lemma 3.9), then $\mathcal{S}_{\mathfrak{T}}$ will be induced by the height function $\omega' : \mathrm{vert}(\mathcal{S}_{\mathfrak{T}}) \to \mathbb{R}$ defined by

$$\omega'(\mathbf{v}') := \omega(\mathbf{v}), \quad \text{for all } \mathbf{v}' \in \mathrm{vert}(\mathcal{S}_{\mathfrak{T}}), \ \mathbf{v}' = (\mathbf{v}, t), \ \mathbf{v} \in \mathrm{vert}(\mathfrak{T}), \ t \in \mathbb{Z}_{\geq 0}.$$

• As we mentioned in Theorem 5.6, pulling sequentially all the points of $\mathcal{V} := |\mathcal{S}_{\mathfrak{T}}| \cap \mathbb{Z}^d$ (in arbitrary order), we arrive at a coherent triangulation $\mathcal{T}$ of P, which is simultaneously a maximal lattice triangulation.

• To show that $\mathcal{T}$ is a b.c.-triangulation w.r.t. $\mathbb{Z}^d$, it is therefore enough to verify its "basicness". Since $\mathcal{T}$ is by construction a refinement of $\mathcal{S}_{\mathfrak{T}}$ (see 3.6 (iv)), all subtriangulations $\{\mathcal{T}|_{P_{\mathbf{s}}} : \mathbf{s} \text{ simplices of } \mathfrak{T}\}$ obtained by the restrictions of $\mathcal{T}$ onto $P_{\mathbf{s}}$'s have to be maximal lattice triangulations too. As all the simplices of them are elementary with vertices belonging to the set of lattice points of half-line prisms over basic simplices, we prove that all these subtriangulations have to be basic by applying Lemma 5.7. Since these subtriangulations fit together to give $\mathcal{T}$, $\mathcal{T}$ has to be basic as well. This completes the proof of Theorems 5.1 and 1.2. $\square$

Example 5.8 Fixing b.c.-triangulations $\mathfrak{T}$ of the "bases" of the Nakajima polytopes which were shown in Figures **4** and **5**, we construct in Figures **8** and **9**, respectively, the subdivisions $\mathcal{S}_{\mathfrak{T}}$ and afterwards two b.c.-triangulations $\mathcal{T}$ by pulling vertices. More precisely, in Figure **8** we pull the available points in the order $(1,0,2)^\intercal$, $(1,1,1)^\intercal$, $(1,1,2)^\intercal$, $(1,1,0)^\intercal$, $(1,2,1)^\intercal$, $(1,2,2)^\intercal$, $(1,2,3)^\intercal$ (and the remaining ones in arbitrary order).

In Figure **9** we pull the points in the order $(1,0,0,1)^\intercal$ and $(1,1,0,1)^\intercal$ (and the remaining ones in arbitrary order). The obtained subdivision is again a b.c.-triangulation.

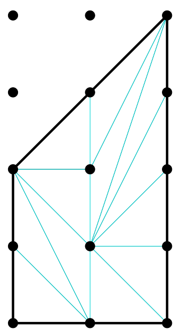


Figure 8

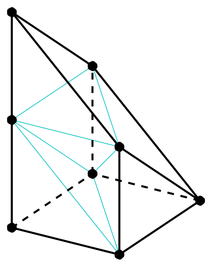


Figure 9

Remark 5.9 Theorem 1.2 has various applications to global geometrical constructions. For instance, the Calabi-Yau varieties which arise from (compactified, non-degenerate) hypersurfaces or ideal-theoretic complete intersections of hypersurfaces embedded into compact toric Fano varieties, and have at most l.c.i.-singularities, admit crepant, full, global desingularizations in all dimensions (cf. [1, 4, 5]).

(b) Another application of the proof of our Main Theorem 1.2 is of purely algebraic nature and is related to the so-called *Koszul property* of graded algebras. (We restrict ourselves to graded algebras defined over the field $\mathbb{C}$ of complex numbers).

Definition 5.10 (Koszul $\mathbb{C}$-algebras) A graded $\mathbb{C}$-algebra R is called a *Koszul algebra* if $\mathbb{C}$ (regarded as the R-module $R/\mathfrak{m}$ for $\mathfrak{m}$ a maximal homogeneous ideal) has a linear free resolution (in the sense of homological algebra), i.e., if there exists an exact sequence

$$\cdots \longrightarrow \mathfrak{R}_{i+1} \xrightarrow{\varphi_{i+1}} \mathfrak{R}_i \xrightarrow{\varphi_i} \cdots \xrightarrow{\varphi_2} \mathfrak{R}_1 \xrightarrow{\varphi_1} \mathfrak{R}_0 \longrightarrow R/\mathfrak{m} \longrightarrow 0$$

of graded free R-modules all of whose matrices (determined by the φ_i's) have entries which are linear forms (i.e., forms of degree 1). Every Koszul algebra is generated by its component of degree 1 and is defined by relations of degree 2.

Definition 5.11 ("Non-faces") Let $\mathcal{V}$ be a finite set of points in $\mathbb{R}^d$ and $\mathcal{T}$ a triangulation of $\mathrm{conv}(\mathcal{V})$. A simplex whose vertices belong to $\mathcal{V}$ but itself does not belong to $\mathcal{T}$ is defined to be a *non-face* of $\mathcal{T}$. A *minimal non-face* of $\mathcal{T}$ is a non-face of $\mathcal{T}$ which is minimal with respect to the inclusion.

Proposition 5.12 (Koszulness and b.c.-triangulations) *If a $(d-1)$-dimensional lattice polytope $P \subset \bar{\mathbf{H}}^{(d)} \hookrightarrow \mathbb{R}^d$ (w.r.t. $\mathbb{Z}^d$) admits a b.c.-triangulation whose minimal non-faces are 1-dimensional, then $R_P = \mathbb{C}\left[\tau_P \cap \mathbb{Z}^d\right]$ is a Koszul algebra.*

Proof. See Bruns, Gubeladze & Trung [9, 2.1.3., p. 142]. □

Proposition 5.13 (From Nakajima polytopes to Koszulness) *The coordinate rings ($\mathbb{C}$-algebras) $R_P = \mathbb{C}\left[\tau_P \cap \mathbb{Z}^d\right]$ of the affine toric varieties $U_{\tau_P^\vee}$ being associated to the duals of the cones τ_P are Koszul for all $(d-1)$-dimensional Nakajima polytopes P.*

Proof. By proposition 5.12 it suffices to prove that the b.c.-triangulations of such a P which were constructed in the proof of Theorem 5.1 have exclusively 1-dimensional minimal non-faces. We shall again use induction on d. Assume that the assertion is true for $d-1$. Let $\pi : \bar{\mathbf{H}}^{(d)} \to \bar{\mathbf{H}}^{(d-1)}$ denote the projection w.r.t. the last coordinate, and $\mathcal{T}$ a b.c.-triangulation of a $(d-1)$-dimensional Nakajima polytope $P \subset \bar{\mathbf{H}}^{(d)}$ induced by extending a b.c.-triangulation $\mathfrak{T}$ of a $(d-2)$-dimensional Nakajima polytope $Q \subset \bar{\mathbf{H}}^{(d-1)}$ as in the proof of 5.1.

• *Fact.* By construction, each face of $\mathcal{T}$ is mapped by π onto a face of $\mathfrak{T}$.

Now choose an arbitrary *non-face* $\mathbf{s}$ of $\mathcal{T}$ of dimension ≥ 2. It is enough to show that $\mathbf{s}$ contains an 1-dimensional non-face of $\mathcal{T}$. We examine the two possible cases separately:

(i) If the projection $\pi(\mathbf{s})$ of $\mathbf{s}$ is a *face* of the b.c.-triangulation $\mathfrak{T}$ of Q, we consider the simplex $\mathbf{s}'$ of $\mathcal{T}$ which contains the barycenter $\mathbf{bar}(\mathbf{s})$ of $\mathbf{s}$ in its relative interior. (Such a simplex $\mathbf{s}'$ always exists, though it might be of dimension strictly smaller than that of $\mathbf{s}$). Since both $\pi(\mathbf{s}')$ and $\pi(\mathbf{s})$ are faces of $\mathfrak{T}$, and $\pi(\mathbf{bar}(\mathbf{s}))$ belongs to the intersection of their relative interiors, we have $\pi(\mathbf{s}') = \pi(\mathbf{s})$. (Any point of $|\mathfrak{T}|$ belongs to the relative interior of exactly one simplex of $\mathfrak{T}$). For each vertex $\mathbf{u} \in \pi(\mathbf{s}') = \pi(\mathbf{s})$, we define:

$$t_{\mathbf{s}}^{\max}(\mathbf{u}) := \max\{t \in \mathbb{Z}_{\geq 0} \mid (\mathbf{u}, t) \in \mathrm{vert}(\mathbf{s})\}, \quad t_{\mathbf{s}'}^{\max}(\mathbf{u}) := \max\{t \in \mathbb{Z}_{\geq 0} \mid (\mathbf{u}, t) \in \mathrm{vert}(\mathbf{s}')\}$$

and

$$t_{\mathbf{s}}^{\min}(\mathbf{u}) := \min\{t \in \mathbb{Z}_{\geq 0} \mid (\mathbf{u}, t) \in \mathrm{vert}(\mathbf{s})\}, \quad t_{\mathbf{s}'}^{\min}(\mathbf{u}) := \min\{t \in \mathbb{Z}_{\geq 0} \mid (\mathbf{u}, t) \in \mathrm{vert}(\mathbf{s}')\},$$

respectively. Since $\mathbf{s}$ is a non-face and $\mathbf{s}'$ a face of $\mathcal{T}$, $\mathbf{s}$ cannot be contained in $\mathbf{s}'$; so there must be at least one vertex $\mathbf{v}_0 = (\mathbf{u}_0, t_0) \in \mathbf{s} \smallsetminus \mathbf{s}'$ of $\mathcal{T}$, $\mathbf{u}_0 \in \mathrm{vert}(\pi(\mathbf{s}))$, for which

$$\textit{either} \quad t_0 > t_{\mathbf{s}'}^{\max}(\mathbf{u}_0) \quad (*) \quad \textit{or} \quad t_0 < t_{\mathbf{s}'}^{\min}(\mathbf{u}_0) \quad (**)$$

• *Claim* A. In case $(*)$ (resp. in case $(**)$) there is at least one vertex $\mathbf{u}_{\blacktriangledown}$ of $\pi(\mathbf{s}') = \pi(\mathbf{s})$, such that $t_{\mathbf{s}}^{\min}(\mathbf{u}_{\blacktriangledown}) < t_{\mathbf{s}'}^{\max}(\mathbf{u}_{\blacktriangledown})$ (resp. $t_{\mathbf{s}}^{\max}(\mathbf{u}_{\blacktriangledown}) > t_{\mathbf{s}'}^{\min}(\mathbf{u}_{\blacktriangledown})$).

• *Proof of Claim* A. The proof will be done only for the case $(*)$ because case $(**)$ can be treated similarly. Suppose that $t_{\mathbf{s}}^{\min}(\mathbf{u}) \geq t_{\mathbf{s}'}^{\max}(\mathbf{u})$, for all vertices $\mathbf{u} \in \pi(\mathbf{s}') = \pi(\mathbf{s})$. Let

$$\operatorname{vert}(\mathbf{s}) = \left\{ \mathbf{v}_i = (1, v_{i,1}, v_{i,2}, \ldots, v_{i,d-1}) \in \bar{\mathbf{H}}^{(d)} \cap \mathbb{Z}^d \ \mid 1 \leq i \leq \ \dim(\mathbf{s}) + 1 \right\}$$

be an enumeration of the vertex set of $\mathbf{s}$, and $\mathbf{bar}(\mathbf{s}) = (1, b_1, b_2, \ldots, b_{d-1})$ the coordinates of the barycenter of $\mathbf{s}$ with

$$\mathbf{bar}(\mathbf{s}) = \frac{1}{\dim(\mathbf{s})+1} \sum_{i=1}^{\dim(\mathbf{s})+1} \mathbf{v}_i, \ \text{ i.e.,} \quad b_j = \frac{1}{\dim(\mathbf{s})+1} \sum_{i=1}^{\dim(\mathbf{s})+1} v_{i,j}, \ \forall j, \ 1 \leq j \leq d-1.$$

The projection of $\mathbf{bar}(\mathbf{s})$ equals

$$\pi(\mathbf{bar}(\mathbf{s})) = (1, b_1, b_2, \ldots, b_{d-2}) = \frac{1}{\dim(\mathbf{s})+1} \sum_{i=1}^{\dim(\mathbf{s})+1} \pi(\mathbf{v}_i) = \sum_{\mathbf{u} \in \operatorname{vert}(\pi(\mathbf{s}))} \mathfrak{r}(\mathbf{u}) \ \mathbf{u},$$

where

$$\mathfrak{r}(\mathbf{u}) := \tfrac{1}{\dim(\mathbf{s})+1} \ \# \left\{ \pi^{-1}(\mathbf{u}) \cap \ \operatorname{vert}(\mathbf{s}) \right\} = \tfrac{1}{\dim(\mathbf{s})+1} \ \# \left\{ \text{all } \mathbf{v}_i\text{'s mapped onto } \mathbf{u} \text{ under } \pi \right\} .$$

Now let $\mathbf{v}_{\bullet} = (\mathbf{u}_{\bullet}, t_{\bullet})$ denote an arbitrary point of $\mathbf{s}'$ with $\mathbf{u}_{\bullet} = \pi(\mathbf{bar}(\mathbf{s}))$. Obviously,

$$t_{\bullet} \leq \sum_{\mathbf{u} \in \operatorname{vert}(\pi(\mathbf{s}))} \mathfrak{r}(\mathbf{u}) \ t_{\mathbf{s}'}^{\max}(\mathbf{u}) \leq \sum_{\mathbf{u} \in \operatorname{vert}(\pi(\mathbf{s}))} \mathfrak{r}(\mathbf{u}) \ t_{\mathbf{s}}^{\min}(\mathbf{u}) \leq \tfrac{1}{\dim(\mathbf{s})+1} \sum_{i=1}^{\dim(\mathbf{s})+1} v_{i,d-1} = b_{d-1} \ .$$

If the last inequality were not strict, we would conclude that

$$v_{i,d-1} = t_{\mathbf{s}}^{\min}(\mathbf{u}), \ \forall i, \ 1 \leq i \leq \ \dim(\mathbf{s}) + 1, \quad \text{and} \quad \forall \mathbf{u}, \quad \mathbf{u} \in \operatorname{vert}(\pi(\mathbf{s})) \, .$$

Since $\mathbf{u}_0 \in \operatorname{vert}(\pi(\mathbf{s}))$ and $t_0 = t_{\mathbf{s}}^{\min}(\mathbf{u}_0) > t_{\mathbf{s}'}^{\max}(\mathbf{u}_0)$, this would mean that the second inequality is necessarily strict. Hence, in each case, either the second or the third inequality has to be *strict.* This implies that the last coordinate of *all* points of $\mathbf{s}'$ having the point $\pi(\mathbf{bar}(\mathbf{s}))$ as their projection under π is $< b_{d-1}$, and therefore $\mathbf{bar}(\mathbf{s}) \notin \mathbf{s}'$, which contradicts our initial assumption.

• *Claim* B. $\operatorname{conv}(\{\mathbf{v}_0, (\mathbf{u}_{\blacktriangledown}, t_{\mathbf{s}}^{\min}(\mathbf{u}_{\blacktriangledown}))\})$ in case $(*)$ (resp. $\operatorname{conv}(\{\mathbf{v}_0, (\mathbf{u}_{\blacktriangledown}, t_{\mathbf{s}}^{\max}(\mathbf{u}_{\blacktriangledown}))\})$ in case $(**)$) is indeed an 1-dimensional non-face of $\mathcal{T}$.

• *Proof of Claim* B. If it were a face of $\mathcal{T}$, then it would obviously possess non-empty intersection with the face $\operatorname{conv}(\{(\mathbf{u}_0, t_{\mathbf{s}'}^{\max}(\mathbf{u}_0)), (\mathbf{u}_{\blacktriangledown}, t_{\mathbf{s}'}^{\max}(\mathbf{u}_{\blacktriangledown}))\})$ (resp. with the face $\operatorname{conv}(\{(\mathbf{u}_0, t_{\mathbf{s}'}^{\min}(\mathbf{u}_0)), (\mathbf{u}_{\blacktriangledown}, t_{\mathbf{s}'}^{\min}(\mathbf{u}_{\blacktriangledown}))\})$). But this would mean that $\mathcal{T}$ cannot be a triangulation.

(ii) Suppose now that $\pi(\mathbf{s})$ is a *non-face* of the b.c.-triangulation $\mathfrak{T}$ of Q. In this case, by induction hypothesis, $\pi(\mathbf{s})$ contains an 1-dimensional minimal non-face of $\mathfrak{T}$, say $\operatorname{conv}(\{\mathbf{u}, \mathbf{u}'\})$. Then both $\pi^{-1}(\mathbf{u})$ and $\pi^{-1}(\mathbf{u}')$ have to be faces of $\mathcal{T}$ (cf. [44, 7.10]), and for any two vertices $\mathbf{v}, \mathbf{v}'$ of $\mathbf{s}$, with $\mathbf{v} \in \pi^{-1}(\mathbf{u}) \subset \mathbf{s}$ and $\mathbf{v}' \in \pi^{-1}(\mathbf{u}') \subset \mathbf{s}$, $\operatorname{conv}(\{\mathbf{v}, \mathbf{v}'\})$ constitutes necessarily an 1-dimensional non-face of $\mathcal{T}$ (by the above mentioned fact). This completes the proof. $\square$

Remark 5.14 In fact we have shown the stronger statement that $R_P \cong \mathbb{C}\left[\mathsf{T}_1, \mathsf{T}_2, \ldots, \mathsf{T}_{\#(P \cap \mathbb{Z}^d)}\right] / I_P$, where the binomial ideal I_P has a Gröbner basis of degree 2 (cf. [9]).

6 On the computation of cohomology group dimensions

To compute the non-trivial (even) cohomology group dimensions of the overlying spaces of crepant, full resolutions of toric l.c.i.-singularities we need some basic concepts from enumerative combinatorics (cf. [41, §4.6]).

Let N be a free $\mathbb{Z}$-module, $P \subset N_{\mathbb{R}}$ a lattice polytope of dimension k w.r.t. N, and ν a positive integer. Let $\mathbf{Ehr}(P, \nu) := \#(\nu P \cap N_P) = \sum_{j=0}^{k} \mathbf{a}_j(P)\, \nu^j \in \mathbb{Q}[\nu]$ denote the *Ehrhart polynomial* of P with N_P the affine sublattice $\mathrm{aff}(P) \cap N$ of N, and

$$\mathfrak{Ehr}(P; q) := 1 + \sum_{\nu=1}^{\infty} \mathbf{Ehr}(P, \nu)\ q^{\nu} \in \mathbb{Q}\,[\![q]\!]$$

the corresponding *Ehrhart series*. Writing $\mathfrak{Ehr}(P; q)$ as

$$\mathfrak{Ehr}(P; q) = \frac{\boldsymbol{\delta}_0(P) + \ \boldsymbol{\delta}_1(P)\ q + \cdots + \ \boldsymbol{\delta}_{k-1}(P)\ q^{k-1} + \ \boldsymbol{\delta}_k(P)\ q^k}{(1-q)^{k+1}}$$

we get the so-called $\boldsymbol{\delta}$-*vector* $\boldsymbol{\delta}(P) = (\boldsymbol{\delta}_0(P), \boldsymbol{\delta}_1(P), \ldots, \boldsymbol{\delta}_{k-1}(P), \boldsymbol{\delta}_k(P))$ of P. (We should mention that both $\mathbf{a}_j(P)$'s and $\boldsymbol{\delta}_j(P)$'s are invariant under lattice equivalence).

Lemma 6.1 *For all j, $0 \le j \le k$, the j-th coordinate of the $\boldsymbol{\delta}$-vector of P is given by the formula*:

$$\boldsymbol{\delta}_j(P) = \sum_{i=0}^{k} \left(\sum_{\xi=0}^{j} (-1)^{\xi} \binom{k+1}{\xi} (j-\xi)^i \right) \mathbf{a}_i(P) \tag{6.1}$$

Proof. Consider the sum

$$\sum_{\nu=0}^{\infty} \left(\sum_{i=0}^{k} \mathbf{a}_i(P)\, \nu^i \right) \sum_{\mu=0}^{k+1} (-1)^{\mu} \binom{k+1}{\mu} q^{\mu+\nu}$$

and compute the coefficient of q^j in its development. □

Theorem 6.2 (Cohomology Group Dimensions) *Let $X(\mathbb{Z}^d, \widehat{\Delta}_P) \longrightarrow U_{\tau_P}$ be any torus-equivariant crepant full resolution of a d-dimensional standard singular representative of a (singular) Gorenstein toric affine variety U_σ (as in* 3.1 *and in* §3 (c)*). Then the odd cohomology groups of its overlying space are trivial and the dimension of the even ones equals*:

$$\dim_{\mathbb{Q}} H^{2j}\left(X\left(\mathbb{Z}^d, \widehat{\Delta}_P\right); \mathbb{Q}\right) = \boldsymbol{\delta}_j(P), \quad \forall j, \quad 0 \le j \le d-1 \tag{6.2}$$

*and is therefore **independent** of the particular choice of a basic triangulation $\mathcal{T}$ of P by means of which one constructs the fan $\widehat{\Delta}_P$ $(= \widehat{\Delta}_P(\mathcal{T}))$.*

Proof. See Batyrev-Dais [5, Thm. 4.4., p. 909]. □

Proposition 6.3 *For torus-equivariant crepant full resolutions $X(\mathbb{Z}^d, \widehat{\Delta}_P) \to U_{\tau_P}$ of a d-dimensional standard singular representative of a toric affine variety U_σ which is a (singular) l.c.i. with $P \sim P_{\mathbf{m}}^{(d)}$ (as in Thm. 4.7), the non-trivial cohomology group dimensions of $X(\mathbb{Z}^d, \widehat{\Delta}_P)$ are computable by means of the formulae* (6.1), (6.2), *and the coefficients of the Ehrhart polynomial*

$$\mathbf{Ehr}(P, \nu) = \sum_{\mu_1=0}^{m_{1,1}\nu}\ \sum_{\mu_2=0}^{m_{2,1}\nu + m_{2,2}\mu_1} \cdots\cdots \sum_{\mu_{d-1}=0}^{m_{d-1,1}\nu + m_{d-1,2}\mu_1 + m_{d-1,3}\mu_2 + \cdots + m_{d-1,d-1}\mu_{d-2}} \mathbf{1} \tag{6.3}$$

which depend exlusively on the corresponding admissible free-parameter-sequence $\mathbf{m}$ defining $P_{\mathbf{m}}^{(d)}$. (Notice that $d \ge 2$ and by convention: $\mu_0 = \nu$ for $d = 2$.)

Proof. Since $P_{\mathbf{m}}^{(d)}$ contains always the "origin" of aff$(P_{\mathbf{m}}^{(d)})$ as one of its vertices, the ν times dilated $P_{\mathbf{m}}^{(d)}$ (with respect to aff$(P_{\mathbf{m}}^{(d)})$!) can be described by Lemma 4.4 as

$$\nu P_{\mathbf{m}}^{(d)} = \left\{ \mathbf{x} = (x_1, .., x_d)^{\intercal} \in \mathbb{R}^d \;\middle|\; x_1 = 1 \text{ and } \; 0 \leq x_{j+1} \leq m_{j,1}\,\nu + \sum_{2\leq\kappa\leq j} m_{j,\kappa}x_\kappa, \,\forall\, j,\, 1 \leq j \leq d-1 \right\}.$$

Thus, formula (6.3) expresses its canonical lattice point enumerator. □

Example 6.4 For $d = 2, 3, 4$, the Ehrhart polynomial of $P_{\mathbf{m}}^{(d)}$ equals $\mathbf{Ehr}(P_{\mathbf{m}}^{(2)}, \nu) = m_{1,1}\nu + 1$,

$$\mathbf{Ehr}(P_{\mathbf{m}}^{(3)}, \nu) = (\frac{1}{2}m_{2,2}m_{1,1}^2 + m_{2,1}m_{1,1})\,\nu^2 + (m_{1,1} + \frac{1}{2}m_{2,2}m_{1,1} + m_{2,1})\nu + 1$$

and $\mathbf{Ehr}(P_{\mathbf{m}}^{(4)}, \nu) = (m_{3,1}m_{2,1}m_{1,1} + \frac{1}{2}m_{3,2}m_{2,1}m_{1,1}^2 + \frac{1}{2}m_{3,3}m_{2,1}^2m_{1,1} +$
$+\frac{1}{6}m_{3,3}m_{2,2}^2m_{1,1}^3 + \frac{1}{2}m_{3,3}m_{2,2}m_{2,1}m_{1,1}^2 + \frac{1}{2}m_{3,1}m_{2,2}m_{1,1}^2 + \frac{1}{3}m_{3,2}m_{2,2}m_{1,1}^3)\;\nu^3 +$
$(m_{2,1}m_{1,1} + \frac{1}{2}m_{3,3}m_{2,1}^2 + \frac{1}{2}m_{3,1}m_{2,2}m_{1,1} + \frac{1}{4}m_{3,3}m_{2,2}^2m_{1,1}^2 + m_{3,1}m_{2,1} +$
$+\frac{1}{2}m_{2,2}m_{1,1}^2 + \frac{1}{2}m_{3,2}m_{1,1}^2 + \frac{1}{2}m_{3,2}m_{2,2}m_{1,1}^2 + \frac{1}{4}m_{3,3}m_{2,2}m_{1,1}^2 + \frac{1}{2}m_{3,3}m_{2,1}m_{1,1} +$
$m_{3,1}m_{1,1} + \frac{1}{2}m_{3,2}m_{2,1}m_{1,1} + \frac{1}{2}m_{3,3}m_{2,2}m_{2,1}m_{1,1})\;\nu^2 +$
$+(\frac{1}{2}m_{3,2}m_{1,1} + m_{2,1} + m_{1,1} + \frac{1}{2}m_{3,3}m_{2,1} + \frac{1}{2}m_{2,2}m_{1,1} +$
$+m_{3,1} + \frac{1}{12}m_{3,3}m_{2,2}^2m_{1,1} + \frac{1}{6}m_{3,2}m_{2,2}m_{1,1} + \frac{1}{4}m_{3,3}m_{2,2}m_{1,1})\;\nu + 1$, respectively.

7 Extreme classes: (d, k)-hypersurfaces and RP-singularities

Two-dimensional toric singularities are always msc-singularities. Moreover, the underlying spaces of the Gorenstein ones (more precisely, the standard singular representatives of them) are of the form

$$U_\tau = \text{Max-Spec}\left(\mathbb{C}\,[t, u, w] \;/\; \langle t^k - u\; w\rangle\right) \tag{7.1}$$

i.e., hypersurfaces depending on a free parameter $k \in \mathbb{Z}_{\geq 2}$. (These are nothing but the classically called A_{k-1}*-singularities*.) Obviously, $U_\tau = U_{\tau_{\mathrm{conv}(\{e_1, e_1+k\cdot e_2\})}}$ with conv$(\{e_1, e_1 + k \cdot e_2\})$ a lattice segment constructed by a dilation of a unit interval by the scalar k. In this section, we apply our results for two classes (7.2) and (7.3) of toric msc-g.c.i.-singularities which are direct generalizations of (7.1) and which are, in addition, "extreme", in the sense, that their corresponding Nakajima polytopes achieve exactly the lowest and the highest bound, respectively, for the number (4.2) of vertices/facets. Moreover, these Nakajima polytopes for both classes are simultaneously examples for $\mathbb{H}_d$-compatible polytopes.

(a) For $k \in \mathbb{N}$, $d \in \mathbb{Z}_{\geq 2}$, let $\mathbf{s}_k^{(d)} \subset \bar{\mathbf{H}}^{(d)} \hookrightarrow \mathbb{R}^d$ denote the $(d-1)$-simplex

$$\mathbf{s}_k^{(d)} := \text{conv}\left(\{e_1, e_1 + k\, e_2, e_1 + k(\,e_2 + e_3), \ldots, e_1 + k(\,e_2 + e_3 + \cdots + e_{d-1} + e_d)\}\right)$$

being constructed by the k-th dilation of a basic $(d-1)$-simplex.

Proposition 7.1 (On $(d;k)$-hypersurfaces) (i) $\mathbf{s}_k^{(d)}$ *is a Nakajima polytope* (*w.r.t.* $\mathbb{R}^d$).
(ii) *For the corresponding affine toric g.c.i.-variety we have*:

$$U_{\tau_{\mathbf{s}_k^{(d)}}} \cong \text{Max-Spec}\left(\mathbb{C}\,[t, u_1, u_2, u_3, \ldots, u_d] \;/\; \left\langle t^k - \textstyle\prod_{j=1}^d u_i \right\rangle\right) \tag{7.2}$$

(*This is called, in particular,* $(d;k)$-hypersurface).

(iii) $(U_{\tau_{\mathbf{s}_k^{(d)}}}, \text{orb}(\tau_{\mathbf{s}_k^{(d)}}))$ *is a singularity* (*in fact, an msc-singularity*) *if and only if* $k \geq 2$.

(iv) $\mathbf{s}_k^{(d)}$ *is a* $\mathbb{H}_d$*-compatible polytope.*

(v) *For all torus-equivariant crepant full desingularizations* $X(\mathbb{Z}^d, \widehat{\Delta}_{\mathbf{s}_k^{(d)}}) \longrightarrow U_{\tau_{\mathbf{s}_k^{(d)}}}$ *we obtain:*

$$\dim_{\mathbb{Q}} H^{2j}\left(X(\mathbb{Z}^d, \widehat{\Delta}_{\mathbf{s}_k^{(d)}}); \mathbb{Q}\right) = \sum_{i=0}^{j} (-1)^i \binom{d}{i} \binom{k\,(j-i)+d-1}{d-1}, \ \forall j, \ \ 0 \le j \le d-1.$$

Proof. (i) Obviously, $\mathbf{s}_k^{(d)} = P_{\mathbf{m}}^{(d)}$ with $\mathbf{m}$ denoting the $((d-1)\times d)$-matrix having entries $m_{1,1} = k$, $m_{i,i} = 1$ in its diagonal, $\forall i$, $2 \le i \le d-1$, and zero entries otherwise. For (ii), (iii), (iv) and (v) see Dais-Henk-Ziegler [12, Prop. 5.10, 6.1, and Cor. 7.4]. (Notice that actually $U_{\tau_{\mathbf{s}_k^{(d)}}} \cong \mathbb{C}^d / G\,(d;k)$ is an abelian quotient space with $G\,(d;k) \cong (\mathbb{Z}\,/\,k\,\mathbb{Z})^{d-1}$). $\square$

(b) Let $k_1, k_2, \ldots, k_{d-1}$ be a $(d-1)$-tuple of positive integers $(d \ge 2)$, and let

$$\begin{aligned}\mathbf{RP}\,(k_1, k_2, \ldots, k_{d-1}) &= \left\{(x_1, .., x_d)^{\intercal} \in \mathbb{R}^d \ \mid x_1 = 1, \ 0 \le x_{j+1} \le k_j, \ \forall j, \ 1 \le j \le d-1\right\} \\ &= \{1\} \times [0, k_1] \times [0, k_2] \times \cdots \times [0, k_{d-1}]\end{aligned}$$

denote the $(d-1)$-dimensional *rectangular parallelepiped* in $\bar{\mathbf{H}}^{(d)} \hookrightarrow \mathbb{R}^d$ having them as lengths of its edges.

Proposition 7.2 (On RP-singularities) (i) $\mathbf{RP}\,(k_1, k_2, \ldots, k_{d-1})$ *is a Nakajima polytope* (*w.r.t.* $\mathbb{R}^d$).

(ii) *For the corresponding affine toric g.c.i.-variety we have*:

$$U_{\tau_{\mathbf{RP}(k_1,..,k_{d-1})}} \cong \text{Max-Spec}(\mathbb{C}\,[t, u_1, u_2, .., u_{d-1}, w_1, w_2, .., w_{d-1}]\ /\ \left\langle \{t^{k_i} - u_i w_i \,|_{1 \le i \le d-1}\} \right\rangle) \tag{7.3}$$

(iii) $(U_{\tau_{\mathbf{RP}(k_1,..,k_{d-1})}}, \mathrm{orb}(\tau_{\mathbf{RP}(k_1,..,k_{d-1})}))$ *is an msc-singularity* (*unless* $d = 2$ *and* $k_1 = 1$).

(iv) $\mathbf{RP}\,(k_1, k_2, \ldots, k_{d-1})$ *is a* $\mathbb{H}_d$*-compatible polytope.*

(v) *For all torus-equivariant crepant full desingularizations*

$$X(\mathbb{Z}^d, \widehat{\Delta}_{\mathbf{RP}(k_1,k_2,\ldots,k_{d-1})}) \longrightarrow U_{\tau_{\mathbf{RP}(k_1,k_2,\ldots,k_{d-1})}}$$

we obtain for all j, $0 \le j \le d-1$:

$$\dim_{\mathbb{Q}} H^{2j}\left(X(\mathbb{Z}^d, \widehat{\Delta}_{\mathbf{RP}(k_1,..,k_{d-1})}); \mathbb{Q}\right) = \sum_{i=0}^{d-1} \left[\sum_{\xi=0}^{j} (-1)^{\xi} \binom{d}{\xi} (j-\xi)^i\right] \mathfrak{s}_i\,(k_1, k_2, \ldots, k_{d-1}),$$

where

$$\mathfrak{s}_0\,(k_1, .., k_{d-1}) = 1, \ \ \mathfrak{s}_i\,(k_1, .., k_{d-1}) = \sum_{1 \le \mu_1 < \mu_2 < \cdots < \mu_i \le d-1} k_{\mu_1} \cdot k_{\mu_2} \cdot \cdots \cdot k_{\mu_i}, \ \forall i, \ \ 1 \le i \le d-1,$$

are the elementary symmetric polynomials w.r.t. the variables $k_1, k_2, \ldots, k_{d-1}$.

Proof. (i) $\mathbf{RP}\,(k_1, k_2, \ldots, k_{d-1})$ equals $P_{\mathbf{m}}^{(d)}$ with $\mathbf{m}$ denoting the $((d-1)\times d)$-matrix with entries $m_{i,1} = k_i$ in its first column, for all i, $1 \le i \le d-1$, and zero entries otherwise. Moreover, it has 2^{d-1} vertices; namely

$$\mathrm{vert}(\mathbf{RP}\,(k_1, .., k_{d-1})) = \{e_1 + \varepsilon_1 \cdot k_1 \cdot e_2 + \varepsilon_2 \cdot k_2 \cdot e_3 + \cdots + \varepsilon_{d-1} \cdot k_{d-1} \cdot e_d \,|\, \varepsilon_1, \ldots, \varepsilon_{d-1} \in \{0, 1\}\}.$$

(ii) As it was pointed out by Nakajima [32, p. 92], the set

$$\left\{e_1^{\vee}, e_2^{\vee}, \ldots, e_{d-1}^{\vee}, e_d^{\vee}, k_1 \cdot e_1^{\vee} - e_2^{\vee}, k_2 \cdot e_1^{\vee} - e_3^{\vee}, \ldots, k_{d-1} \cdot e_1^{\vee} - e_d^{\vee}\right\}$$

forms a system of generators for the monoid $\tau^{\vee}_{\mathbf{RP}(k_1,..,k_{d-1})} \cap (\mathbb{Z}^d)^{\vee}$. Using (2.1) it is easy to see that the above set is exactly the Hilbert basis of $\tau^{\vee}_{\mathbf{RP}(k_1,..,k_{d-1})}$ w.r.t. $(\mathbb{Z}^d)^{\vee}$. Hence,

$$\left\{\mathbf{e}\left(e_1^{\vee}\right), \mathbf{e}\left(e_2^{\vee}\right), \ldots, \mathbf{e}\left(e_{d-1}^{\vee}\right), \mathbf{e}\left(e_d^{\vee}\right), \mathbf{e}\left(k_1 \cdot e_1^{\vee} - e_2^{\vee}\right), \mathbf{e}\left(k_2 \cdot e_1^{\vee} - e_3^{\vee}\right), \ldots, \mathbf{e}\left(k_{d-1} \cdot e_1^{\vee} - e_d^{\vee}\right)\right\}$$

generates $\mathbb{C}[\tau^{\vee}_{\mathbf{RP}(k_1,..,k_{d-1})} \cap (\mathbb{Z}^d)^{\vee}]$, and the affine toric variety $U_{\tau_{\mathbf{RP}(k_1,..,k_{d-1})}}$ has embedding dimension $2d-1$ (and is, in particular, a g.c.i. of $d-1$ *binomials* by (i), Thm. 4.7, Rem. 4.8(ii), and Thm. 2.2). The map

$$\theta : \mathbb{C}\left[t, u_1, u_2, .., u_{d-1}, w_1, w_2, .., w_{d-1}\right] \longrightarrow \mathbb{C}[\tau^{\vee}_{\mathbf{RP}(k_1,..,k_{d-1})} \cap (\mathbb{Z}^d)^{\vee}]$$

defined by $\theta(t) := \mathbf{e}\left(e_1^{\vee}\right)$, $\theta(u_i) := \mathbf{e}\left(e_{i+1}^{\vee}\right)$, $\theta(w_i) := \mathbf{e}\left(k_i \cdot e_1^{\vee} - e_{i+1}^{\vee}\right)$, $\forall i$, $1 \leq i \leq d-1$, is a $\mathbb{C}$-algebra epimorphism. It suffices to show that $\mathrm{Ker}(\theta) = I$ with $I := \left\langle \left\{t^{k_i} - u_i w_i \mid 1 \leq i \leq d-1\right\}\right\rangle$. For $d = 2$ this is obvious. We shall hereafter assume that $d \geq 3$.

• *Claim* A. This ideal is contained in the kernel of θ, i.e., $I \subseteq \mathrm{Ker}(\theta)$.

• *Proof of Claim* A. Consider the lattice Λ of the defining binomial equations of $U_{\tau_{\mathbf{RP}(k_1,..,k_{d-1})}} \hookrightarrow \mathbb{C}^{2d-1}$,

$$\Lambda = \left\{ (a_1, a_2, \ldots, a_{2d-1}) \in \mathbb{Z}^{2d-1} \;\middle|\; \sum_{i=1}^{d} a_i\, e_i^{\vee} + \sum_{i=d+1}^{2d-1} a_i \left(k_{i-d} \cdot e_1^{\vee} - e_{i-d+1}^{\vee}\right) = 0 \right\}.$$

Since the extra relations can be written in the form

$$a_1 + \sum_{i=d+1}^{2d-1} a_i\, k_{i-d} = 0, \qquad a_j - a_{d+j-1} = 0, \quad \forall j, \quad 2 \leq j \leq d,$$

setting $\xi_{j-1} := -a_j = -a_{d+j-1}$, for all j, $2 \leq j \leq d$, as auxiliary parameters, we may express every point of Λ as follows

$$\begin{aligned}(a_1, a_2, \ldots, a_{2d-1}) &= (\textstyle\sum_{i=1}^{d-1} \xi_i\, k_i, -\xi_1, -\xi_2, \ldots, -\xi_{d-1}, -\xi_1, -\xi_2, \ldots, -\xi_{d-1}) = \\ &= \textstyle\sum_{i=1}^{d-1} \xi_i\ (k_i \cdot e_1^{\vee} - e_{i+1}^{\vee} - e_{d+i}^{\vee}).\end{aligned}$$

Now these $d-1$ vectors are also $\mathbb{Z}$-linearly independent. So they constitute a $\mathbb{Z}$-basis of Λ, and

$$k_i \cdot e_1^{\vee} - e_{i+1}^{\vee} - e_{d+i}^{\vee} = k_i \cdot e_1^{\vee} - (e_{i+1}^{\vee} + e_{d+i}^{\vee})$$

is the difference of two vectors with non-negative coordinates having disjoint support for all indices i, $1 \leq i \leq d-1$. Hence, $I \subseteq \mathrm{Ker}(\theta)$ by [42, 4.3-4.4, p. 32].

• *Claim* B. The opposite inclusion $I \supseteq \mathrm{Ker}(\theta)$ is true too.

• *Proof of Claim* B. For every $\kappa \in \mathbb{N}$, $1 \leq \kappa \leq d-1$, and every subset of indices $2 \leq i_1 < i_2 < \cdots < i_\kappa \leq d$ of length κ, we define the cone

$$\begin{aligned} C_{i_1, i_2, \ldots, i_\kappa} &:= \left\{ (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_d) \in \tau^{\vee}_{\mathbf{RP}(k_1,..,k_{d-1})} \;\middle|\; \begin{array}{l} \mathbf{y}_1 \geq 0,\ \mathbf{y}_{i_1} \geq 0,\ \mathbf{y}_{i_2} \geq 0,\ \ldots,\ \mathbf{y}_{i_\kappa} \geq 0 \\ \text{and}\ \ \mathbf{y}_j \leq 0,\ \forall j,\ j \in \{2, \ldots, d\} \smallsetminus \{i_1, \ldots, i_\kappa\} \end{array} \right\} \\ &= \mathrm{pos}\left(\{e_1^{\vee}\} \cup \left\{e_{i_1}^{\vee}, \ldots, e_{i_\kappa}^{\vee}\right\} \cup \left\{k_{j-1} \cdot e_1^{\vee} - e_j^{\vee} \mid j \in \{2, \ldots, d\} \smallsetminus \{i_1, \ldots, i_\kappa\}\right\}\right). \end{aligned}$$

Then

$$\tau^{\vee}_{\mathbf{RP}(k_1,..,k_{d-1})} = \bigcup_{\kappa=1}^{d-1} \ \bigcup_{2 \leq i_1 < i_2 < \cdots < i_\kappa \leq d} C_{i_1, i_2, \ldots, i_\kappa}$$

is a fan-subdivision of $\tau^{\vee}_{\mathbf{RP}(k_1,..,k_{d-1})}$ into 2^{d-1} s.c.p. (and actually basic) cones. Now since

$$\mathbf{Hilb}_{(\mathbb{Z}^d)^\vee}(\tau^{\vee}_{\mathbf{RP}(k_1,..,k_{d-1})}) = \{e_1^\vee, e_2^\vee, \ldots, e_d^\vee, k_1\cdot e_1^\vee - e_2^\vee, k_2\cdot e_1^\vee - e_3^\vee, \ldots, k_{d-1}\cdot e_1^\vee - e_d^\vee\},$$

each lattice point $m \in \tau^{\vee}_{\mathbf{RP}(k_1,..,k_{d-1})} \cap (\mathbb{Z}^d)^\vee$ can be written as a linear combination

$$m = \lambda \cdot e_1^\vee + \sum_{i=1}^{d-1} \mu_i \cdot e_{i+1}^\vee + \sum_{j=1}^{d-1} \nu_j \cdot (k_j \cdot e_1^\vee - e_{j+1}^\vee) \tag{7.4}$$

for *uniquely* determined coefficients $(\lambda, \mu_1, \ldots, \mu_{d-1}, \nu_1, \ldots, \nu_{d-1}) \in (\mathbb{Z}_{\geq 0})^{2d-1}$ which have to satisfy the extra conditions

$$\mu_1 \cdot \nu_1 = \mu_2 \cdot \nu_2 = \cdots = \mu_{d-1} \cdot \nu_{d-1} = 0 \tag{7.5}$$

because m necessarily belongs to a cone of the form $C_{i_1,i_2,\ldots,i_\kappa}$. For the rest of the proof it is enough to make use of an elegant trick due to Ishida (see [23, p. 143]). We define a homomorphism

$$g : \mathbb{C}[\tau^{\vee}_{\mathbf{RP}(k_1,..,k_{d-1})} \cap (\mathbb{Z}^d)^\vee] \longrightarrow \mathbb{C}\,[t, u_1, u_2, .., u_{d-1}, w_1, w_2, .., w_{d-1}]$$

of $\mathbb{C}$-vector spaces by mapping the character of any m as in (7.4) onto

$$\mathbb{C}[\tau^{\vee}_{\mathbf{RP}(k_1,..,k_{d-1})} \cap (\mathbb{Z}^d)^\vee] \ni \mathbf{e}(m) \longmapsto g(\mathbf{e}(m)) := t^\lambda\, u_1^{\mu_1} \cdots u_{d-1}^{\mu_{d-1}}\, w_1^{\nu_1} \cdots w_{d-1}^{\nu_{d-1}} .$$

g is obviously a section of θ (i.e., $\theta \circ g = \mathrm{Id}$). For proving $I \supseteq \mathrm{Ker}(\theta)$ it is therefore sufficient to show that

$$\mathbb{C}\,[t, u_1, \ldots, u_{d-1}, w_1, \ldots, w_{d-1}] = g(\mathbb{C}[\tau^{\vee}_{\mathbf{RP}(k_1,..,k_{d-1})} \cap (\mathbb{Z}^d)^\vee]) + I\ . \tag{7.6}$$

Let $\phi = t^p\, u_1^{q_1} \cdots u_{d-1}^{q_{d-1}}\, w_1^{r_1} \cdots w_{d-1}^{r_{d-1}}$ denote an arbitrary monomial in $\mathbb{C}\,[t, u_1, \ldots, u_{d-1}, w_1, \ldots, w_{d-1}]$. To conclude (7.6) we shall prove that

$$\phi \in g(\mathbb{C}[\tau^{\vee}_{\mathbf{RP}(k_1,..,k_{d-1})} \cap (\mathbb{Z}^d)^\vee]) + I\ . \tag{7.7}$$

Define $\mathfrak{A}_\phi := \{i \in \{1, \ldots, d-1\} \mid q_i \cdot r_i > 0\}$ and $\beta_\phi := \#(\mathfrak{A}_\phi)$. If $\beta_\phi = 0$, then $\phi = g(\mathbf{e}(m))$, for some $m \in \tau^{\vee}_{\mathbf{RP}(k_1,..,k_{d-1})} \cap (\mathbb{Z}^d)^\vee$ (by (7.4) and (7.5)). Let now $\beta_\phi \in \{1, .., d-1\}$ and consider an index $i_0 \in \mathfrak{A}_\phi$. Suppose that property (7.7) is false for ϕ. Without loss of generality, we may further assume that the product $q_{i_0} \cdot r_{i_0}$ of the degrees of ϕ in the variables u_{i_0} and w_{i_0} is chosen to be *minimal* with respect to the violation of (7.7).

Then

$$t^{p+k_{i_0}}\ u_1^{q_1} \cdots u_{i_0-1}^{q_{i_0-1}}\, u_{i_0}^{q_{i_0}-1}\, u_{i_0+1}^{q_{i_0+1}} \cdots u_{d-1}^{q_{d-1}}\, w_1^{r_1} \cdots w_{i_0-1}^{r_{i_0-1}}\, w_{i_0}^{r_{i_0}-1}\, w_{i_0+1}^{r_{i_0+1}} \cdots w_{d-1}^{r_{d-1}} - \phi =$$

$$= t^p\, u_1^{q_1} \cdots u_{i_0-1}^{q_{i_0-1}}\, u_{i_0}^{q_{i_0}-1}\, u_{i_0+1}^{q_{i_0+1}} \cdots u_{d-1}^{q_{d-1}}\, w_1^{r_1} \cdots w_{i_0-1}^{r_{i_0-1}}\, w_{i_0}^{r_{i_0}-1}\, w_{i_0+1}^{r_{i_0+1}} \cdots w_{d-1}^{r_{d-1}}\, \left(t^{k_{i_0}} - u_{i_0} w_{i_0}\right) \in I\ .$$

This implies that

$$t^{p+k_{i_0}}\ u_1^{q_1} \cdots u_{i_0-1}^{q_{i_0-1}}\, u_{i_0}^{q_{i_0}-1}\, u_{i_0+1}^{q_{i_0+1}} \cdots u_{d-1}^{q_{d-1}}\, w_1^{r_1} \cdots w_{i_0-1}^{r_{i_0-1}}\, w_{i_0}^{r_{i_0}-1}\, w_{i_0+1}^{r_{i_0+1}} \cdots w_{d-1}^{r_{d-1}}$$

does not belong to $g(\mathbb{C}[\tau^{\vee}_{\mathbf{RP}(k_1,..,k_{d-1})} \cap (\mathbb{Z}^d)^\vee]) + I$, contradicting the minimality assumption for $q_{i_0} \cdot r_{i_0}$. Hence, (7.7) is always true.

(iii) and (iv) are obvious. Finally, (v) follows from the determination of the coefficients of the Ehrhart polynomial

$$\mathbf{Ehr}\,(\mathbf{RP}\,(k_1, k_2, \ldots, k_{d-1})\,; \nu) = \prod_{i=1}^{d-1}\ (k_i\ \nu + 1)\ ,$$

combined with the formulae (6.1) and (6.2). □

Remark 7.3 As both toric g.c.i.-varieties (7.2) and (7.3) are constructible by means of $\mathbb{H}_d$-compatible Nakajima polytopes, the most natural choice of a crepant birational morphism to desingularize them is $f_{\mathcal{T}_{\mathbb{H}_d}|\text{restr.}}$. For this choice the precise nature of the occuring exceptional prime divisors is known by Thm. 3.21.

(c) At the end of the paper we devote a few words to *non-l.c.i.'s*: In complete analogy to the case of non-l.c.i. Gorenstein abelian quotient spaces (cf. [10, 11]), we expect that also the underlying spaces of toric non-l.c.i. Gorenstein singularities will be *only rarely* overall resolvable by crepant birational morphisms. Let us nevertheless give two examples of non-Nakajima polytopes admitting b.c.-triangulations.

Example 7.4 Let d be an odd integer ≥ 3. Define the $(d-1)$-dimensional lattice polytope

$$Q = \text{conv}\left(\left\{e_1 \pm e_2, e_1 \pm e_3, \ldots, e_1 \pm e_{d-1}, e_1 \pm e_d, e_1 \pm (\textstyle\sum_{j=2}^{d} e_j)\right\}\right)$$

$$= \left\{\mathbf{x} = (x_1, \ldots, x_d)^{\intercal} \in \mathbb{R}^d \mid x_1 = 1, -1 \leq x_1 + \sum_{i=2}^{d} \varepsilon_i x_i \leq 1, \forall \varepsilon_i \in \{\pm 1\} : \sum_{i=2}^{d} \varepsilon_i \in \{-1, 0, 1\}\right\}.$$

Since

$$\#(\{\text{facets of } Q\}) = d \cdot \binom{d-1}{\frac{1}{2}(d-1)} = \frac{d\,!}{\left[(\frac{1}{2}\,(d-1))!\right]^2} > 2\,(d-1)\,,$$

$Q \subset \bar{\mathbf{H}}^{(d)} \hookrightarrow \mathbb{R}^d$ cannot be lattice equivalent to a Nakajima polytope (by (4.2)). On the other hand, Q is a non-simplex, Fano polytope, and the Gorenstein non-l.c.i., non-quotient, msc-singularity $(U_{\tau_Q},\text{orb}(\tau_Q))$ can therefore be overall resolved by a crepant projective birational morphism (by Prop. 3.19).

Example 7.5 The zonotope $\mathcal{Z}^{(d)}$ defined earlier in the proof of Theorem 3.21 is $\mathbb{H}_d$-compatible, but for $d \geq 3$, it has $d(d-1) > 2(d-1)$ facets.

To create more examples of non-Nakajima polytopes having b.c.-triangulations, one may start with $\mathcal{Z}^{(d)}$ or with a Q as above in 7.4, and consider "joins" of it with further (finitely many) Nakajima polytopes, or, alternatively, combine or mix all those with suitably triangulated dilations of basic simplices. (For the "good" behaviour of joins and dilations under b.c.-triangulations, see [12, §3 and §6]).

References

[1] Batyrev V.V.: *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, Jour. of Algebraic Geometry **3**, (1994), 493-535.

[2] Batyrev V.V.: *Stringy Hodge numbers of varieties with Gorenstein canonical singularities*, alg-geom/9711008; to appear in the Proc. of Taniguchi Symposium 1997: "Integrable Systems and Algebraic Geometry", Kobe/Kyoto.

[3] Batyrev V.V.: *Non-Archimedian integrals and stringy Euler numbers of log-terminal pairs*, alg-geom/9803071; to appear in the Jour. of the European Math. Soc. **1**, (1999).

[4] Batyrev V.V., Borisov L.A.: *Mirror duality and string-theoretic Hodge numbers*, Inventiones Math. **126**, (1996), 183-203.

[5] Batyrev V.V., Dais D.I.: *Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry*, Topology **35**, (1996), 901-929.

[6] Billera L.J., Filliman P., Sturmfels B.: *Construction and complexity of secondary polytopes,* Advances in Math. **83**, (1990), 155-179.

[7] Borisov L.A.: *String cohomology of a toroidal singularity*, preprint, alg-geom/9802052.

[8] Borisov L.A.: *Vertex algebras and mirror symmetry*, preprint, alg-geom/9809094.

[9] Bruns W., Gubeladze J., Trung N. V.: *Normal polytopes, triangulations and Koszul algebras*, Jour. für die reine und ang. Math. **485**, (1997), 123-160.

[10] Dais D.I., Haus U.-U., Henk M.: *On crepant resolutions of* 2*-parameter series of Gorenstein cyclic quotient singularities*, Results in Math. **33**, (1998), 208-265.

[11] Dais D.I., Henk M.: *On a series of Gorenstein cyclic quotient singularities admitting a unique projective crepant resolution*, alg-geom/9803094; to appear in: "Combinatorial Convex Geometry and Toric Varieties", (ed. by G.Ewald & B.Teissier), Birkhäuser.

[12] Dais D.I., Henk M., Ziegler G.M.: *All abelian quotient c.i.-singularities admit projective crepant resolutions in all dimensions*, Advances in Math. **139**, (1998), 192-239.

[13] Dais D.I., Henk M., Ziegler G.M.: *On the existence of crepant resolutions of Gorenstein abelian quotient singularities in dimensions* ≥ 4, in preparation.

[14] de Loera J.A.: *Triangulations of Polytopes and Computational Algebra*, PhD Thesis, Cornell University, (1995). [The author's computer algebra program PUNTOS which determines coherent triangulations of the convex hull of point configurations is available via anonymous ftp at `ftp://geom.umn.edu`, directory `priv/deloera`.]

[15] de Loera J.A., Hoşten S., Santos F., Sturmfels B.: *The polytope of all triangulations of a point configuration*, Documenta Math. J. DMV **1**, (1996), 103-119.

[16] Ewald G.: *Combinatorial Convexity and Algebraic Geometry*, Graduate Texts in Mathematics, Vol. **168**, Springer-Verlag, (1996).

[17] Fulton W.: *Introduction to Toric Varieties*, Annals of Math. Studies, Vol. **131**, Princeton University Press, (1993).

[18] Gelfand I.M., Kapranov M.M., Zelevinsky A.V.: *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser, (1994).

[19] Grothendieck A.: *Techniques de construction en géométrie analytique,* Séminaire H. Cartan, É.N.S., 1960/61, Exp. 13.

[20] Grothendieck A.: *Géométrie algébrique et géométrie analytique*, (Notes by M.Raynaud). In : *Revêtements étales et Groupe Fondemental* (SGA 1), Exp. 12, Lecture Notes in Math., Vol. **224**, Springer-Verlag, (1971), pp. 311-343.

[21] Hibi T., Ohsugi H.: *A normal* $(0,1)$*-polytope none of whose regular triangulations is unimodular*, Discrete & Comp. Geometry **21** (1999), 201-204.

[22] Hochster M.: *Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes*, Annals of Math. **96**, (1972), 318-337.

[23] Ishida M.-N.: *Torus embeddings and dualizing complexes*, Tôhoku Math. Jour. **32**, (1980), 111-146.

[24] Ito Y., Reid M.: *The McKay correspondence for finite subgroups of* $\mathrm{SL}(3,\mathbb{C})$. In : "Higher Dimensional Complex Varieties", Proceedings of the International Conference held in Trento, Italy, June 15-24, 1994, (M.Andreatta & Th.Peternell, eds.); Walter de Gruyter, (1996), 221-240.

[25] Kempf G., Knudsen F., Mumford D., Saint-Donat D.: *Toroidal Embeddings I*, Lecture Notes in Mathematics, Vol. **339**, Springer-Verlag, (1973).

[26] Kontsevich M.: *p-adic integrals, loop spaces and generalized McKay's correspondence*, two talks given at E.N.S. (Paris) and M.P.I. (Bonn), (February-March 1996).

[27] Kunz E.: *Introduction to Commutative Algebra and Algebraic Geometry*, Birkhäuser, (1985).

[28] LEE C.W.: *Regular triangulations of convex polytopes.* In: "Applied Geometry and Discrete Mathematics-The Victor Klee Festschrift", (P.Gritzmann & B.Sturmfels, eds.), DIMACS Series in Discrete Math. and Theoretical Comp. Science, Vol. **4**, A.M.S., (1991), 443-456.

[29] LEE C.W.: *Subdivisions and triangulations of polytopes*, in: "Discrete and Computational Geometry", (J.E.Goodman & J.O'Rourke, eds.), CRC-Press, New York, (1997), pp. 271-290.

[30] MATSUMURA H.: *Commutative Ring Theory*, Cambridge Studies in Adv. Math., Vol. **8**, Cambridge University Press, (1986).

[31] MORRISON D.R., STEVENS G.: *Terminal quotient singularities in dimension three and four*, Proc. A.M.S. **90**, (1984), 15-20.

[32] NAKAJIMA H.: *Affine torus embeddings which are complete intersections*, Tôhoku Math. Jour. **38**, (1986), 85-98.

[33] ODA T.: *Convex Bodies and Algebraic Geometry. An Introduction to the theory of toric varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Bd. **15**, Springer-Verlag, (1988).

[34] ODA T., PARK H.S.: *Linear Gale transforms and GKZ-decompositions*, Tôhoku Math. Jour. **43**, (1991), 375-399.

[35] REID M.: *Canonical threefolds*, Journée de Géométrie Algébrique d'Angers, (A. Beauville, ed.), Sijthoff and Noordhoff, Alphen aan den Rijn, (1980), 273-310.

[36] REID M.: *Decompositions of toric morphisms.* In: "Arithmetic and Geometry II", (M.Artin & J.Tate, eds.), Progress in Math. **36**, Birkhäuser, (1983), 395-418.

[37] REID M.: *McKay correspondence*, preprint, alg-geom/9702016.

[38] SERRE J.-P.: *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier **6**, (1956), 1-42.

[39] STANLEY R.P.: *Hilbert functions of graded algebras*, Advances in Math. **28**, (1978), 57-81.

[40] STANLEY R.P.: *Decompositions of rational convex polytopes*, Annals of Discrete Math. **6**, (1980), 333-342.

[41] STANLEY R.P.: *Enumerative Combinatorics*, Vol. **I**, Wadsworth & Brooks/Cole Math. Series, (1986); second printing: Cambridge University Press, (1997).

[42] STURMFELS B.: *Gröbner Bases and Convex Polytopes*, University Lecture Series, Vol. **8**, A.M.S., (1996).

[43] WATANABE K.: *Invariant subrings which are complete intersections I, (Invariant subrings of finite Abelian groups)*, Nagoya Math. Jour. **77**, (1980), 89-98.

[44] ZIEGLER G.M.: *Lectures on Polytopes*, Graduate Texts in Mathematics, Vol. **152**, Springer-Verlag, (1995).