Geometric Combinatorics in the Study of Compact Toric Surfaces

Dimitrios I. Dais

Abstract. The compact toric surfaces can be classified up to isomorphism by means of suitable weighted directed graphs, and their invariants (both "conventional" and "stringy") are to be expressed by closed formulae depending on their combinatorial data. Moreover, the nonsingular ones possess uniquely determined anticanonical models which are toric log Del Pezzo surfaces.

Introduction

By the well-known *Enriques-Kodaira classification* one subdivides the class of the minimal models of nonsingular compact complex surfaces into several subclasses according to the values taken from their Kodaira dimension ($-\infty, 0, 1$, or 2) and from their other invariants (first Betti number, topological Euler characteristic, self-intersection of the canonical divisor etc.); see, e.g., [**2**, Ch. 8-9], [**3**, Ch. VI], [**9**, Ch. VII-X], [**21**, Ch. 4], [**36**, Ch. 1, §7], or [**37**, Ch. 1]. Nonsingular projective minimal surfaces having Kodaira dimension $-\infty$ (that is, either rational or ruled over a nonsingular compact curve of genus ≥ 1) occupy a distinguished position in the classification table because they correspond, in the MMP-terminology (by the *hard dichotomy*, cf. [**36**, Thm. 1.5.5]), to the so-called *Mori's fiber spaces* in dimension 2. In the mid 1980's Sakai [**51**], [**52**], provided a classification (analogous to the "Enriques' part" of the above mentioned) for *normal* projective (or, more generally, Moishezon) surfaces, and generalized the notion of minimal models for *normal pairs*. Even in this framework, projective normal surfaces with Kodaira dimension $-\infty$ play apparently a pivotal role in answering various questions which arise in two-dimensional birational geometry (cf. [**49**], [**52**], and [**33**]).

Compact *toric surfaces* are rational surfaces X_Δ equipped with an algebraic action of a two-dimensional (algebraic) torus $\mathbb{T}$, containing an open dense $\mathbb{T}$-orbit, and admitting at worst normal, log-terminal singularities. Since they are defined by means of *complete fans* Δ consisting of two-dimensional rational strongly convex polyhedral cones, lots of their main algebro-geometric properties are to be described by suitable combinatorial properties of these cones. The aim of the present paper is to stress the particular importance of a systematic use of *geometric combinatorics* in the study of X_Δ's as, for instance, in the computation of their invariants, in their classification (up to biregular isomorphism), in the examination of their minimal,

2000 *Mathematics Subject Classification.* 14Q10 (Primary); 14J26, 14M25 (Secondary).

antiminimal and anticanonical models (in the sense of Sakai) etc. More precisely, the paper is organized as follows:

▷ After having established in this section the general algebro-geometric terminology which will be used in the sequel, we survey briefly in §1 those parts of the theory of normal surfaces being involved in the treatment of compact toric surfaces.

▷ In §2 we recall some fundamental notions from toric geometry and fix our notation. A detailed study of the 2-dimensional s.c.p. cones (including their lattice-geometric properties and number-theoretic parametrization), as well as of the 2-dimensional toric singularities (including their quotient space structure, their defining equations, and their minimal resolutions) is presented in §3.

▷ In §4 we introduce the concept of the *combinatorial data* of a compact toric surface X_Δ. These data, together with the intersection numbers of pairs of $\mathbb{T}$-invariant Weil divisors and appropriate generalizations of Noether's formula are used to determine the classical invariants of X_Δ, with the self-intersection $K_{X_\Delta}^2$ of its canonical divisor considered as the prominent one (see formulae (4.15), (4.16), and (4.17)).

▷ Section 5 is devoted to the classification (up to biregular isomorphism) of compact toric surfaces by means of WVE^{2}C-*graphs* (see Theorems 5.8 and 5.10).

▷ In Section 6 we apply the general theory of §1 to obtain minimal models of normal pairs (X_Δ, D), and then we focus on the antiminimal and anticanonical models of *nonsingular* X_Δ's. The latter ones are necessarily *toric log Del Pezzo surfaces.* Though the complete classification of these surfaces (up to biregular isomorphism) remains an open combinatorial problem, some first partial results are accomplished by Theorems 6.10 and 6.12, in which it is shown that there are only 16 surfaces of this kind having index 1, and only 7 surfaces having index 2 and Picard number 1, respectively.

▷ In §7 we explain how one can compute the Euler-Poincaré characteristic of the coherent sheaf $\mathcal{O}_{X_\Delta}(D)$ associated to a Weil divisor D on X_Δ via a generalized *Riemann-Roch formula* whose correction terms depend on the combinatorial data of X_Δ.

▷ Finally, in §8 we compute the *stringy $\mathcal{E}$-function* of any compact toric surface.

• **General terminology**. (i) If X is a *complex variety*, i.e., an integral separated scheme of finite type over $\mathbb{C}$, then its punctual algebraic behaviour is determined by the stalks $\mathcal{O}_{X,x}$ of its structure sheaf $\mathcal{O}_X$, and X itself is said to have a given *algebraic property* (e.g., to be normal, Gorenstein, Cohen-Macaulay etc) whenever all $\mathcal{O}_{X,x}$'s have the analogous property for all $x \in X$. Furthermore, via the GAGA-correspondence (cf. [**23**, App. B]) we may work in the *analytic category* by means of the usual contravariant functor between the category of isomorphy classes of germs (X, x) and the corresponding category of isomorphy classes of analytic local rings at the marked points x.

(ii) Let X be a *normal* complex variety. We denote by $\mathrm{Div_P}(X)$, $\mathrm{Div_W}(X)$, and $\mathrm{Div_C}(X)$, the additive groups of *principal*, *Weil* and *Cartier divisors* on X, respectively (see [**23**, Ch. II, §6]), and by $\mathrm{Div_W}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp., $\mathrm{Div_C}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$) the group of $\mathbb{Q}$-*Weil* (resp., $\mathbb{Q}$-*Cartier*) *divisors.* Two Weil divisors D and D' are said to be *linearly equivalent*, written $D \sim D'$, if $D - D' \in \mathrm{Div_P}(X)$. (In many formulae involving Weil divisors we prefer to write "=" instead of "∼", but it will be always clear what is meant.) For the corresponding divisor class groups on X we introduce the notation

$$\mathrm{Cl}\,\mathrm{Div}_{\mathrm{W}}(X) := \mathrm{Div}_{\mathrm{W}}(X)/\mathrm{Div}_{\mathrm{P}}(X) \supseteq \mathrm{Div}_{\mathrm{C}}(X)/\mathrm{Div}_{\mathrm{P}}(X) =: \mathrm{Cl}\,\mathrm{Div}_{\mathrm{C}}(X).$$

Denoting by Rat_X the constant sheaf of rational functions of X, there is a bijection

$$\mathrm{Cl}\,\mathrm{Div}_{\mathrm{W}}(X) \ni \{D\} \overset{\Upsilon}{\longmapsto} \{\mathcal{O}_X(D)\} \in \left\{ \begin{array}{c} \text{reflexive subsheaves} \\ \text{of } Rat_X \text{ having rank 1} \end{array} \right\} \Big/ \, H^0(X, \mathcal{O}_X^*).$$

(A coherent sheaf of $\mathcal{O}_X$-modules is called *reflexive* if it is isomorphic to its bidual, cf. [**24**].) $\mathcal{O}_X(D)$ is defined by sending every non-empty open set U of X onto

$$U \longmapsto \mathcal{O}_X(D)(U) := \left\{ \phi \in \mathbb{C}(X)^* \;\mid\; (\mathrm{div}(\phi) + D)|_U \geq 0 \right\} \cup \{\mathbf{0}\}.$$

The restriction $\Upsilon|_{\mathrm{Cl}\,\mathrm{Div}_{\mathrm{C}}(X)}$ gives the group isomorphism $\mathrm{Cl}\,\mathrm{Div}_{\mathrm{C}}(X) \cong \mathrm{Pic}(X)$, where $\mathrm{Pic}(X)$ is the *Picard group* of X (i.e., the group of isomorphism classes of invertible sheaves on X, cf. [**23**, Proposition II.6.15, p. 145]). A $\mathbb{Q}$-Cartier divisor D on X is said to be *ample* if some multiple of it is an integral very ample Cartier divisor (that is, a hyperplane section for some immersion of X into a projective space). If X is compact, a necessary and sufficient condition for D to be ample is given by the so-called *Nakai's criterion* (see [**22**, Thm. 5.1, pp. 30-32]).

(iii) For a complex variety X, we denote by

$$\mathrm{Sing}(X) := \{x \in X \;\mid\; \mathcal{O}_{X,x} \text{ is a non-regular local ring}\}$$

its *singular locus*. Let now $\Omega_{\mathrm{Reg}(X)/\mathbb{C}}$ be the sheaf of Kähler differentials on the *regular locus* $\mathrm{Reg}(X) := X \smallsetminus \mathrm{Sing}(X) \overset{\iota}{\hookrightarrow} X$. If X is normal, the unique (up to "$\sim$") Weil divisor K_X, which maps under Υ to the *canonical divisorial sheaf*

$$\omega_X := \iota_* \left(\bigwedge^{\dim_{\mathbb{C}}(X)} \Omega_{\mathrm{Reg}(X)/\mathbb{C}} \right),$$

is called the *canonical divisor* of X. Such an X is said to be $\mathbb{Q}$-*Gorenstein* if K_X is a $\mathbb{Q}$-Cartier divisor.

(iv) For a $\mathbb{Q}$-Weil divisor D on a normal compact complex variety X, we define

$$\mathrm{kod}(X, D) := \begin{cases} \mathrm{trans.deg}_{\mathbb{C}}(R(X,D)) - 1, & \text{if } R(X,D) \neq \mathbb{C}, \\ -\infty, & \text{otherwise,} \end{cases}$$

as the D-*dimension* of X, where $R(X,D) := \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD))$. In particular, $\mathrm{kod}(X, K_X)$ (resp., $\mathrm{kod}(X, -K_X)$) is called the *Kodaira dimension* (resp., the *anti-Kodaira dimension*) of X.

(v) For a birational morphism $f : X \longrightarrow Y$ between complex varieties we denote by

$$\mathrm{Exc}(f) := \left\{ x \in X \;\mid\; f^{-1} \text{ is not a morphism at } f(x) \right\}$$

the *exceptional locus* of f (equipped with the reduced subscheme structure). By a *desingularization* (or *resolution of singularities*) of a singular X we mean a proper, surjective birational morphism $f : \widehat{X} \longrightarrow X$ with $f|_{\widehat{X} \smallsetminus \mathrm{Exc}(f)}$ an isomorphism and $\mathrm{Sing}(\widehat{X}) = \varnothing$. (Throughout this paper all birational morphisms will be assumed to be proper.)

(vi) A $\mathbb{Q}$-Gorenstein complex variety X is said to have at worst *log-terminal* (respectively, *canonical* / *terminal*) singularities if there exists a desingularization $f : \widehat{X} \longrightarrow X$ of X such that all coefficients $\eta_j \in \mathbb{Q}$ in the *discrepancy divisor*

$$K_{\widehat{X}} - f^* (K_X) = \sum_j \eta_j \ D_j \in \mathrm{Div_C}(X) \otimes_{\mathbb{Z}} \mathbb{Q},$$

w.r.t. f are > -1 (≥ 0 / > 0). (This property is independent of the choice of f, with f^* denoting the pull-back of $\mathbb{Q}$-Cartier divisors by f.) In particular, f is called *crepant* whenever $\eta_j = 0$ for all j.

1. Preliminaries from the theory of normal surfaces

In this preliminary section we recall some basic notions from the intersection theory and the singularity theory of compact normal surfaces, and we give the definition of *minimal* (resp., *canonical*) *model* of a *normal pair* in the sense of Sakai [**50**]. (Convention: We use the word *surface* to mean a two-dimensional complex variety. A *curve* on a surface will be an 1-dimensional reduced, complete subscheme of it.)

• **Intersection theory on nonsingular surfaces**. Let X be a nonsingular surface. The *intersection number* $D_1 \cdot D_2$ of two Cartier divisors D_1, D_2 on X can be always defined provided that the intersection of their supports is compact (see Fulton [**19**, 2.4.9, p. 40]). For instance, if x is an isolated intersection of two curves C_1, C_2 on X, with $f, g \in \mathcal{O}_{X,x}$ specifying their local equations, then

$$i_x(C_1, C_2) := \dim_{\mathbb{C}}(\mathcal{O}_{X,x}/(f_x, g_x))$$

is the *intersection multiplicity* of C_1 and C_2 at x, and

$$C_1 \cdot C_2 = \sum_{x \in C_1 \cap C_2} i_x(C_1, C_2),$$

whenever C_1, C_2 have no common irreducible component. On the other hand, to extend this definition in the general case, the *self-intersection number* C^2 of a curve $C \overset{\iota}{\hookrightarrow} X$ is introduced as follows: Let $\mathcal{I}_C := \mathrm{Ker}(\mathcal{O}_X \to \iota_* \mathcal{O}_C)$ be the *ideal sheaf* of C in X. We consider the sheaf of $\mathcal{O}_C$-modules $\mathcal{I}_C/\mathcal{I}_C^2 \cong \mathcal{O}_X(-C) \otimes \mathcal{O}_C$, we pass to its dual

$$\mathcal{N}_{X/C} := \underline{\mathrm{Hom}}(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C) \cong \mathcal{O}_X(C) \otimes \mathcal{O}_C$$

(the so-called *normal sheaf* of C in X), and then we define

$$C^2 := C \cdot C := \ \deg_C \left(\mathcal{N}_{X/C}\right). \tag{1.1}$$

• **Intersection theory on compact normal surfaces**. We recall the definition of intersection numbers of $\mathbb{Q}$-Weil divisors on compact normal surfaces, due to Mumford [**38**, pp. 17-18]. Let $f : Z \longrightarrow Y$ be a desingularization a compact normal surface Y with $\mathrm{Exc}(f) = \bigcup_{j=1}^s E_j$. The *inverse image* f^*D of a $\mathbb{Q}$-Weil divisor D is defined to be

$$f^* D = \overline{D} + \sum_{j=1}^{s} a_j E_j \in \mathrm{Div_C}(Z) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $\overline{D}$ is the strict transform of D via f, and the rational numbers $a_1, \ldots, a_s$ are uniquely determined by the equations:

$$\overline{D} \cdot E_{j'} + \left(\sum_{j=1}^{s} a_j E_j \right) \cdot E_{j'} = 0, \quad \forall j' \in \{1, \ldots, s\}.$$

In this manner, one constructs a group homomorphism

$$f^* : \mathrm{Div}_{\mathrm{W}}(Y) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathrm{Div}_{\mathrm{C}}(Z) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

(Note that, even if D is integral on Y, f^*D is in general a $\mathbb{Q}$-Cartier divisor on Z, and that $f_* \mathcal{O}_Z(f^*D) \cong \mathcal{O}_Y(D)$, where f_* is the direct image under f.) The (*fractional*) *intersection number* $D \cdot D'$ of two $\mathbb{Q}$-Weil divisors D, D' on Y is defined to be the image of the pair (D, D') under the symmetric, $\mathbb{Q}$-bilinear map

$$(\mathrm{Div}_{\mathrm{W}}(Y) \otimes_{\mathbb{Z}} \mathbb{Q}) \times (\mathrm{Div}_{\mathrm{W}}(Y) \otimes_{\mathbb{Z}} \mathbb{Q}) \ni (D, D') \longmapsto D \cdot D' := f^*D \cdot f^*D' \in \mathbb{Q},$$

and is to be computed by utilizing the usual intersection pairings on Z with coefficients taken from $\mathbb{Q}$. In addition, $D \cdot D'$ is well-defined, in the sense, that it does not depend on the particular choice of f (see Fulton [**19**, 7.1.16, p. 125]).

• **Contractibility criterion.** Let Y be a normal surface. The singularities of Y are isolated, because $\mathrm{codim}(\mathrm{Sing}(Y)) \geq 2$. Let $y \in Y$ be a singularity and $f : Z \longrightarrow Y$ a desingularization of Y. The set-theoretic inverse image $f^{-1}(y)$ of y consists of finitely many irreducible curves $E_1, \ldots, E_s$. By Zariski's Main Theorem (see [**23**, Corollary III.11.4, p. 280]) $E := \sum_{j=1}^{s} E_j$ is connected (as topological space). On the other hand, we say that a connected curve C on Y *is contracted to* x if there is a birational morphism $\varphi : Y \longrightarrow X$, with X normal, $x = \varphi(C) \in X$, such that $Y \smallsetminus C \cong X \smallsetminus \{x\}$.

THEOREM 1.1. *A connected curve C on a compact normal surface Y having $C_1, \ldots, C_s$ as its irreducible components can be contracted to a normal point if and only if the intersection matrix $(C_i \cdot C_j)_{1 \leq i,j \leq s}$ is negative definite.*

PROOF. It suffices to pass to an arbitrary desingularization of Y and to apply the classical Grauert's criterion [**20**, p. 367]; compare [**50**, Thm. 2.1., p. 878]. □

• **Minimal desingularization.** A desingularization $f : X' \longrightarrow X$ of a normal surface X is *minimal* if $\mathrm{Exc}(f)$ does not contain any curve with self-intersection number -1 or, equivalently, if for an arbitrary desingularization $g : X'' \longrightarrow X$ of X, there exists a unique morphism $h : X'' \longrightarrow X'$ with $g = f \circ h$. A desingularization of a normal surface is *good* if (i) the irreducible components of the exceptional locus are smooth curves, and (ii) the preimage of each singular point is a divisor with simple normal crossings. For the proof of the *uniqueness,* up to (biregular) isomorphism, of both minimal and good minimal desingularizations, see Brieskorn [**13**, Lemma 1.6, p. 81] and Laufer [**35**, Thm. 5.12, pp. 91-92].

• **Local canonical divisors.** Let Y be a projective normal surface and $f : \widetilde{Y} \longrightarrow Y$ its minimal desingularization. Let $E_y := \sum_{j=1}^{s_y} E_j^{(y)}$ denote the fibre over a point $y \in \mathrm{Sing}(Y)$. We define the *local canonical divisor*

$$K(E_y) = \sum_{j=1}^{s_y} c_j^{(y)} E_j^{(y)} \in \mathrm{Div}_{\mathrm{C}}(\widetilde{Y}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

of $\widetilde{Y}$ at y by the equations:

$$K(E_y) \cdot E_j^{(y)} = K_{\widetilde{Y}} \cdot E_j^{(y)}, \quad \forall j \in \{1, \ldots, s_y\}.$$

Hence, we get

$$\sum_{y \in \ \mathrm{Sing}(Y)} K(E_y) = K_{\widetilde{Y}} - f^* K_Y \in \mathrm{Div}_{\mathrm{C}}(\widetilde{Y}) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where all coefficients $\left\{ c_j^{(y)} \middle| \, y \in \ \mathrm{Sing}(Y), \ j \in \{1, \ldots, s_y\} \right\}$ are nonpositive rationals (because $K(E_y) \cdot E_j^{(y)} \geq 0$ for all $j \in \{1, \ldots, s_y\}$ and all $y \in \mathrm{Sing}(Y)$). If all $c_j^{(y)}$'s belong to the interval $(-1, 0]$, then Y has at worst log-terminal singularities which turn out to be *quotient singularities* (see, e.g., [**36**, Thm. 4.6.18, pp. 215-218] or [**37**, Lemma 4.1.1, pp. 117-118]). If all $c_j^{(y)}$'s are $= 0$, then Y has at worst canonical singularities, i.e., rational **A-D-E**-singularities (also called *Kleinian* or *Du Val singularities*), cf. [**36**, Thm. 4.6.7, pp. 197-212]. (Terminal *singularities* are not present in dimension 2).

• **Minimal models**. From now on we consider *normal pairs* (Y, D), i.e., pairs consisting of a projective normal surface Y and a $\mathbb{Q}$-Weil divisor D on Y. A *birational morphism* $f : (Y, D) \longrightarrow (Y', D')$ between two normal pairs is a birational morphism $f : Y \longrightarrow Y'$ (in the common sense) with $\mathrm{Exc}(f) = \bigcup_{j=1}^{s} E_j$ and $f_* \mathcal{O}_Y(D) = \mathcal{O}_{Y'}(D')$ or, equivalently,

$$D - f^* D' = \sum_{j=1}^{s} d_j E_j \in \mathrm{Div}_{\mathrm{W}}(Y) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{1.2}$$

Definition 1.2. An irreducible curve C on Y is called *exceptional curve of first kind* for (Y, D) if $C^2 < 0$ and $D \cdot C < 0$. We say that Y is *minimal with respect to* D if it does not contain any exceptional curve of the first kind for (Y, D). We say that a normal pair (Y', D') is a *minimal model* of (Y, D) if (i) Y' is minimal w.r.t. D' (in the above sense) and (ii) there is a birational morphism $f : (Y, D) \longrightarrow (Y', D')$ which is either an isomorphism or *totally discrepant*, that is, $\mathrm{Exc}(f) \neq \varnothing$ and all d_j's in (1.2) are strictly positive. (Such an f can be always factorized into a sequence of successive contractions of exceptional curves of the first kind w.r.t. D, cf. [**50**, Proposition 7.3., pp. 884-885.].)

Definition 1.3. Let D be a $\mathbb{Q}$-Weil divisor D on a projective normal surface Y. We say that D is *nef* (numerically effective) if $D \cdot C \geq 0$ for all irreducible curves C on Y, and that D is *pseudoeffective* if $D \cdot E \geq 0$ for all nef divisors E on Y.

Theorem 1.4. ([**50**, Thm. 7.4]) *Every normal pair* (Y, D) *has a minimal model. Furthermore, if* D *is pseudoeffective, then* (Y, D) *admits a unique minimal model* (Y', D'). *In this case,* D' *is nef.*

Remark 1.5. (a) In particular, we say that Y' is a *minimal model* of Y whenever $(Y', K_{Y'})$ is a minimal model of (Y, K_Y). Analogously, we say that Y' is an *antiminimal model* of Y whenever $(Y', -K_{Y'})$ is a minimal model of $(Y, -K_Y)$.

(b) If C is an exceptional curve of the first kind for (Y, K_Y), then $\overline{C}^2 = -1$, where $\overline{C}$ denotes the strict transform of C by the minimal desingularization $f : \widetilde{Y} \longrightarrow Y$ of Y (cf. [**51**, Lemma 1.1, p. 629]). If Y happens to be nonsingular, an exceptional curve of the first kind for (Y, K_Y) is, as usual, a (-1)-curve.

• **Canonical models**. A normal pair (Y, D) is *canonical* if Y contains no irreducible curves C with $C^2 < 0$ and $D \cdot C \leq 0$. We say that a normal pair (Y'', D'') is a *canonical model* of a normal pair (Y, D) if (i) (Y'', D'') is canonical and (ii) there is a birational morphism $g : (Y, D) \longrightarrow (Y'', D'')$ of normal pairs such that all coefficients of $D - g^*D''$ are nonnegative. Every normal pair has a canonical model. Furthermore, every canonical model (Y'', D'') of a normal pair (Y, D) factors through a minimal model (Y', D'):

$$\begin{array}{ccc} & (Y,D) & \\ {}^{f}\swarrow & \circlearrowleft & \searrow^{g} \\ (Y',D') & \xrightarrow[h]{} & (Y'',D'') \end{array}$$

where $D' = h^*D''$ and f is either an isomorphism or a totally discrepant birational morphism (cf. [**50**, p. 886]).

Note 1.6. In analogy to 1.5 (a), we simply say that Y'' is a *canonical model* of Y whenever $(Y'', K_{Y''})$ is a canonical model of (Y, K_Y), and that Y'' is an *anticanonical model* of Y whenever $(Y'', -K_{Y''})$ is a canonical model of $(Y, -K_Y)$.

Theorem 1.7. ([**50**, § 7]) *If* (Y, D) *is a normal pair with* D *pseudoeffective and* $\mathrm{kod}(Y, D) = 2$, *then* (Y, D) *admits a unique canonical model* (Y'', D''). *Furthermore, if the ring* $R(Y, D)$ *is finitely generated, then* D'' *is an ample* $\mathbb{Q}$*-Cartier divisor on* Y'', *and* $Y'' \cong \mathrm{Proj}(R(Y, D))$.

• **Rational surfaces**. These are surfaces birationally equivalent to the projective plane $\mathbb{P}^2_{\mathbb{C}}$, having Kodaira dimension $-\infty$. A characterization of the class of nonsingular rational surfaces with anti-Kodaira dimension 2 is given in the following result of Sakai:

Theorem 1.8. ([**49**, Thm. 4.3]) *Let* X *be a nonsingular rational surface with* $\mathrm{kod}(X, -K_X) = 2$. *Then the anticanonical ring* $R(X, -K_X)$ *is finitely generated and the anticanonical model* $X_{\mathrm{antican}} \cong \mathrm{Proj}(R(X, -K_X))$ *of* X *has the following properties*:

(i) X_{antican} *has at worst isolated rational singularities.*

(ii) $-K_{X_{\mathrm{antican}}}$ *is an ample* $\mathbb{Q}$*-Cartier divisor.*

Moreover, if Y *is a normal projective surface satisfying* (i) *and* (ii), *and if we denote by* $f : X \longrightarrow Y$ *its minimal desingularization, then* X *is a rational surface with* $\mathrm{kod}(X, -K_X) = 2$ *and* $Y \cong X_{\mathrm{antican}}$.

2. Preliminaries from toric geometry

Before we are going to deal exclusively with toric *surfaces*, we recall those fundamental notions and auxiliary results from the *general theory* of toric varieties which will be used substantially in the sequel. For further details the reader is referred to the books of Oda [**41**], [**42**], and Fulton [**18**].

• **Fundamental notions**. The *linear hull*, the *affine hull*, the *positive hull* and *the convex hull* of a set B of vectors of $\mathbb{R}^d$, $d \geq 1$, will be denoted by $\mathrm{lin}(B)$, $\mathrm{aff}(B)$, pos(B) (or $\mathbb{R}_{\geq 0}\, B$) and conv(B), respectively. The *dimension* dim(B) of a $B \subset \mathbb{R}^d$ is defined to be the dimension of aff(B).

Let N be a free $\mathbb{Z}$-module of rank $d \geq 1$. N can be regarded as a *lattice* within $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^d$. The *lattice determinant* det(N) of N is the d-volume of the parallelepiped spanned by a $\mathbb{Z}$-basis of it. An $\mathbf{n} \in N$ is called *primitive* if conv($\{\mathbf{0}, \mathbf{n}\}) \cap N$ contains no other points except $\mathbf{0}$ and $\mathbf{n}$.

Let N be as above, $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual, $N_{\mathbb{R}}$, $M_{\mathbb{R}}$ their real scalar extensions, and $\langle .,. \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ the natural $\mathbb{R}$-bilinear pairing. A subset σ of $N_{\mathbb{R}}$ is called a *convex polyhedral cone* (*c.p.c.*, for short) if there exist vectors $\mathbf{n}_1, \ldots, \mathbf{n}_k \in N_{\mathbb{R}}$ such that $\sigma = \mathrm{pos}(\{\mathbf{n}_1, \ldots, \mathbf{n}_k\})$. Its *relative interior* int(σ) is the usual topological interior of it and considered as a subset of $\mathrm{lin}(\sigma) = \sigma + (-\sigma)$. The *dual cone* $\sigma^{\vee}$ of a c.p.c. σ is a c.p. cone defined by

$$\sigma^{\vee} := \{\mathbf{y} \in M_{\mathbb{R}} \mid \langle \mathbf{y}, \mathbf{x} \rangle \geq 0, \ \forall \mathbf{x}, \ \mathbf{x} \in \sigma\} \ .$$

Note that $(\sigma^{\vee})^{\vee} = \sigma$ and

$$\dim(\sigma \cap (-\sigma)) + \dim(\sigma^{\vee}) = \dim(\sigma^{\vee} \cap (-\sigma^{\vee})) + \dim(\sigma) = d.$$

A subset τ of a c.p.c. σ is called a *face* of σ (notation: $\tau \prec \sigma$), if for some $\mathbf{m}_0 \in \sigma^{\vee}$ we have $\tau = \{\mathbf{x} \in \sigma \mid \langle \mathbf{m}_0, \mathbf{x} \rangle = 0\}$. 1-dimensional faces are called *rays*.

A c.p.c. $\sigma = \mathrm{pos}(\{\mathbf{n}_1, \ldots, \mathbf{n}_k\})$ is called *simplicial* (resp., *rational*) if $\mathbf{n}_1, \ldots, \mathbf{n}_k$ are $\mathbb{R}$-linearly independent (resp., if $\mathbf{n}_1, \ldots, \mathbf{n}_k \in N_{\mathbb{Q}}$, where $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$). If ϱ is a ray of a rational c.p.c. σ, we denote by $\mathbf{n}(\varrho) \in N \cap \varrho$ the unique primitive vector with $\varrho = \mathbb{R}_{\geq 0}\, \mathbf{n}(\varrho)$, and we define

$$\mathrm{Gen}(\sigma) := \{\mathbf{n}(\varrho) \mid \varrho \text{ rays of } \sigma\} \quad (\text{the } \textit{set of minimal generators} \text{ of } \sigma).$$

A *strongly convex polyhedral cone* (*s.c.p.c.*, for short) is a convex polyhedral cone σ for which $\sigma \cap (-\sigma) = \{\mathbf{0}\}$, i.e., for which $\dim(\sigma^{\vee}) = d$.

• **Hilbert basis**. If $\sigma \subset N_{\mathbb{R}}$ is a rational s.c.p.c., then the subsemigroup $\sigma \cap N$ of N is a monoid. $\sigma \cap N$ is finitely generated as an additive semigroup for every rational c.p.c. $\sigma \subset N_{\mathbb{R}}$. Moreover, if σ is strongly convex, then among all the systems of generators of $\sigma \cap N$, there is a system $\mathrm{Hilb}_N(\sigma)$ of *minimal cardinality*, which is uniquely determined (up to the ordering of its elements) by the following characterization:

$$\mathrm{Hilb}_N(\sigma) = \left\{ \mathbf{n} \in \sigma \cap (N \smallsetminus \{\mathbf{0}\}) \;\middle|\; \begin{array}{l} \mathbf{n} \text{ cannot be expressed} \\ \text{as sum of two other vectors} \\ \text{belonging to } \sigma \cap (N \smallsetminus \{\mathbf{0}\}) \end{array} \right\}. \tag{2.1}$$

$\mathrm{Hilb}_N(\sigma)$ is called *the Hilbert basis of* σ w.r.t. N.

• **Affine toric varieties**. For a free $\mathbb{Z}$-module N of rank d having M as its dual, we define an d-dimensional *algebraic torus* $\mathbb{T} \cong (\mathbb{C}^*)^d$ by setting $\mathbb{T} := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$. Every $\mathbf{m} \in M$ assigns a character $\mathbf{e}(\mathbf{m}) : \mathbb{T} \to \mathbb{C}^*$. Moreover, each $\mathbf{n} \in N$ determines a 1-parameter subgroup

$$\vartheta_{\mathbf{n}} : \mathbb{C}^* \to \mathbb{T} \quad \text{with} \quad \vartheta_{\mathbf{n}}(\lambda)(\mathbf{m}) := \lambda^{\langle \mathbf{m}, \mathbf{n} \rangle}, \quad \text{for} \quad \lambda \in \mathbb{C}^*, \ \mathbf{m} \in M \ .$$

We can therefore identify M with the character group of $\mathbb{T}$ and N with the group of 1-parameter subgroups of $\mathbb{T}$. On the other hand, for a rational s.c.p. cone σ with

$M \cap \sigma^{\vee} = \mathbb{Z}_{\geq 0}\ \mathbf{m}_1 + \cdots + \mathbb{Z}_{\geq 0}\ \mathbf{m}_{\nu}$, we associate to the finitely generated monoidal subalgebra

$$\mathbb{C}\left[M \cap \sigma^{\vee}\right] = \bigoplus_{\mathbf{m} \in M \cap \sigma^{\vee}} \mathbb{C}\,\mathbf{e}\left(\mathbf{m}\right)$$

of the $\mathbb{C}$-algebra $\mathbb{C}\left[M\right] = \bigoplus_{\mathbf{m} \in M} \mathbb{C}\,\mathbf{e}\left(\mathbf{m}\right)$ an *affine toric variety*

$$\boxed{U_{\sigma} := \mathrm{Spec}\left(\mathbb{C}\left[M \cap \sigma^{\vee}\right]\right).}$$

U_{σ} admits a canonical $\mathbb{T}$-action which extends the group multiplication of the algebraic torus $\mathbb{T} = U_{\{\mathbf{0}\}}$:

$$\mathbb{T} \times U_{\sigma} \ni (t, u) \longmapsto t \cdot u \in U_{\sigma} \tag{2.2}$$

where, for $u \in U_{\sigma}$, $(t \cdot u)\left(\mathbf{m}\right) := t\left(\mathbf{m}\right) \cdot u\left(\mathbf{m}\right),\ \forall \mathbf{m},\ \mathbf{m} \in M \cap \sigma^{\vee}$. The orbits w.r.t. the action (2.2) are parametrized by the set of all the faces of σ. For a $\tau \prec \sigma$, we denote by $\mathrm{orb}(\tau)$ the orbit which is associated to τ.

PROPOSITION 2.1 (Embedding by binomials). *In the algebraic category, U_{σ}, identified with its image under the injective map*

$$\left(\mathbf{e}\left(\mathbf{m}_1\right), \ldots, \mathbf{e}\left(\mathbf{m}_{\nu}\right)\right) : U_{\sigma} \hookrightarrow \mathbb{C}^{\nu},$$

can be regarded as an affine variety determined by a finite number of equations of the form (monomial) = (monomial). *U_{σ} is independent of the semigroup generators $\{\mathbf{m}_1, \ldots, \mathbf{m}_{\nu}\}$ and each map $\mathbf{e}\left(\mathbf{m}\right)$ on U_{σ} is a morphism. In particular, for $\tau \prec \sigma$, U_{τ} is an open algebraic subset of U_{σ}. Moreover, if* $\dim(\sigma) = d$ *and* $\#\left(\mathrm{Hilb}_M\left(\sigma^{\vee}\right)\right) = k\ \ (\leq \nu)$, *then k is nothing but the* embedding dimension *of U_{σ}, i.e., the* minimal *number of generators of the maximal ideal of the local $\mathbb{C}$-algebra $\mathcal{O}_{U_{\sigma}, \mathbf{0}}$.*

PROOF. See Oda [**42**, Propositions 1.2 and 1.3., pp. 4–7]. □

• **Algebraic properties**. The well-known hierarchy of Noetherian rings

$$(\text{regular}) \Longrightarrow (\text{Gorenstein}) \Longrightarrow (\text{Cohen-Macaulay})$$

is used to describe the punctual algebraic behaviour of affine toric varieties.

DEFINITION 2.2 (Multiplicities and basic cones). Let N be a free $\mathbb{Z}$-module of rank d and $\sigma \subset N_{\mathbb{R}}$ a simplicial, rational s.c.p.c. of dimension $d' \leq d$. The cone σ can be obviously written as $\sigma = \varrho_1 + \cdots + \varrho_{d'}$, for distinct rays $\varrho_1, \ldots, \varrho_{d'}$. The *multiplicity* $\mathrm{mult}(\sigma; N)$ of σ with respect to N is defined as

$$\mathrm{mult}\left(\sigma; N\right) := \frac{\det\left(\mathbb{Z}\,\mathbf{n}\left(\varrho_1\right) \oplus \cdots \oplus \mathbb{Z}\,\mathbf{n}\left(\varrho_{d'}\right)\right)}{\det\left(N_{\sigma}\right)},$$

where N_{σ} is the sublattice of N generated (as a subgroup) by the set $N \cap \mathrm{lin}(\sigma)$. If $\mathrm{mult}(\sigma; N) = 1$, then σ is called a *basic cone* w.r.t. N.

THEOREM 2.3 (Smoothness criterion). *The affine toric variety U_{σ} is nonsingular (i.e., the corresponding local rings $\mathcal{O}_{U_{\sigma},\ u}$ are regular at all points u of U_{σ}) iff σ is simplicial and basic w.r.t. N.*

PROOF. See Oda [**42**, Thm. 1.10, p. 15]. □

Next theorem describes a necessary and sufficient condition for U_{σ} to be Gorenstein (see Ishida [**30**, §7]).

THEOREM 2.4 (Gorenstein property). *Let N be a free $\mathbb{Z}$-module of rank d and σ a rational s.c.p. cone in $N_{\mathbb{R}}$ with* $\dim(\sigma) = d$. *Then the following conditions are equivalent*:

(i) U_σ *is Gorenstein.*

(ii) $\exists!$ *primitive* $\mathbf{m}_\sigma \in M \cap (\mathrm{int}\,(\sigma^\vee))$ *such that* $M \cap (\mathrm{int}\,(\sigma^\vee)) = \mathbf{m}_\sigma + (M \cap \sigma^\vee)$.

Finally, let us recall the most fundamental property of U_σ's:

THEOREM 2.5 (Normality and CM-property). *The affine toric varieties U_σ are always normal and Cohen-Macaulay, and have at worst rational singularities.*

PROOF. See Fulton [**18**, pages 29-31 and 76] and Oda [**42**, pp. 125-126]. □

• **Fans and general toric varieties**. A *fan* w.r.t. $N \cong \mathbb{Z}^d$ is a finite collection Δ of rational s.c.p. cones in $N_{\mathbb{R}}$ such that

(i) any face τ of $\sigma \in \Delta$ belongs to Δ, and

(ii) for $\sigma_1, \sigma_2 \in \Delta$, the intersection $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

The union $|\Delta| := \cup \{\sigma \mid \sigma \in \Delta\}$ is called the *support* of Δ. We say that Δ is *d-dimensional* if all of its maximal cones are d-dimensional. Furthermore, we define

$$\Delta(i) := \{\sigma \in \Delta \mid \dim(\sigma) = i\}\ ,\ \text{for } 0 \leq i \leq d\ ,\ \text{and } \mathrm{Gen}(\Delta) := \bigcup_{\sigma \in \Delta} \mathrm{Gen}(\sigma)\ .$$

The *toric variety* X_Δ associated to a fan Δ (w.r.t. N) is by definition the identification space

$$\boxed{X_\Delta := \left(\left(\bigcup_{\sigma \in \Delta} U_\sigma \right) \ / \ \simeq \right)} \tag{2.3}$$

with $U_{\sigma_1} \ni u_1 \simeq u_2 \in U_{\sigma_2}$ iff there is a $\tau \in \Delta$, such that $\tau \prec \sigma_1 \cap \sigma_2$ and $u_1 = u_2$ within U_τ. A canonical $\mathbb{T}$-action on X_Δ is established via (2.2) on each U_σ, $\sigma \in \Delta$.

NOTE 2.6. (a) By Theorem 2.5, X_Δ is normal, Cohen-Macaulay, and has at worst rational singularities. Moreover, algebraic properties like those described in Theorems 2.3 and 2.4 are also local, and they are transferred to X_Δ, provided that they are valid for all of its affine "building blocks".

(b) If the toric variety X_Δ is $\mathbb{Q}$-Gorenstein, then it has at worst log-terminal singularities (see [**6**, Corollary 4.2, p. 10]).

(c) As complex variety, X_Δ is compact iff Δ is a *complete fan*, i.e., iff $|\Delta| = N_{\mathbb{R}}$ (see [**42**, Thm. 1.11, p. 16]).

(d) The topological *Euler characteristic* $e(X_\Delta)$ of a d-dimensional toric variety X_Δ equals $e(X_\Delta) = \sharp(\Delta(d))$ (see [**18**, p. 59]).

• **Maps of fans**. A *map of fans* $\varpi : (N', \Delta') \to (N, \Delta)$ is a $\mathbb{Z}$-linear homomorphism $\varpi : N' \to N$ whose scalar extension $\varpi_{\mathbb{R}} : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ satisfies the property:

$$\forall \sigma',\ \sigma' \in \Delta' \quad \exists\, \sigma,\ \sigma \in \Delta \quad \text{with} \quad \varpi_{\mathbb{R}}(\sigma') \subset \sigma\,.$$

$\varpi \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{C}^*} : N' \otimes_{\mathbb{Z}} \mathbb{C}^* \to N \otimes_{\mathbb{Z}} \mathbb{C}^*$ is a homomorphism between the two algebraic tori and the scalar extension $\varpi^\vee \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{R}} : M_{\mathbb{R}} \to M'_{\mathbb{R}}$ of the dual map $\varpi^\vee : M \to M'$ of ϖ induces canonically an *equivariant morphism* $\varpi_* : X_{\Delta'} \longrightarrow X_\Delta$. This map is *proper* iff $\varpi^{-1}(|\Delta|) = |\Delta'|$. In particular, if $N = N'$ and Δ' is a refinement of Δ, then $\mathrm{id}_* : X_{\Delta'} \longrightarrow X_\Delta$ is *proper* and *birational* (cf. [**42**, Thm. 1.15, pp. 20-21, and Cor. 1.18, p. 23]).

• **Desingularization**. By Carathéodory's Theorem concerning convex polyhedral cones one can choose a refinement Δ' of any given fan Δ, so that Δ' becomes *simplicial*. Since further subdivisions of Δ' reduce the multiplicities of its cones, we may arrive (after finitely many subdivisions) at a fan Δ'' having only basic cones. Thus, for every toric variety X_Δ there exists a refinement Δ'' of Δ consisting of exclusively basic cones w.r.t. N, i.e., such that $f = \mathrm{id}_* : X_{\Delta''} \longrightarrow X_\Delta$ is an equivariant desingularization of X_Δ (by Theorem 2.3).

• **Divisors and support functions**. Let X_Δ be a toric variety associated to a fan Δ, and let $\mathrm{Div}^{\mathbb{T}}_{\mathrm{W}}(X_\Delta)$ (resp., $\mathrm{Div}^{\mathbb{T}}_{\mathrm{C}}(X_\Delta)$) denote the group of $\mathbb{T}$-invariant Weil (resp., Cartier) divisors on it, i.e., the subgroup of divisors remaining invariant under the canonical $\mathbb{T}$-action on $\mathrm{Div}_{\mathrm{W}}(X_\Delta)$ (resp., on $\mathrm{Div}_{\mathrm{C}}(X_\Delta)$). Then

$$\mathrm{Div}^{\mathbb{T}}_{\mathrm{W}}(X_\Delta) = \bigoplus_{\varrho \in \Delta(1)} \mathbb{Z}\,\mathbf{V}_\Delta(\varrho), \quad \text{where } \mathbf{V}_\Delta(\varrho) := \text{ the closure of } \mathrm{orb}(\varrho),$$

and a $D = \sum_{\varrho \in \Delta(1)} \lambda_\varrho \mathbf{V}_\Delta(\varrho) \in \mathrm{Div}^{\mathbb{T}}_{\mathrm{W}}(X_\Delta)$ is a Cartier divisor iff for all $\sigma \in \Delta$ there exists $\mathbf{m}(\sigma) \in M$ such that $\langle \mathbf{m}(\sigma), \mathbf{n}(\varrho) \rangle = -\lambda_\varrho$, $\forall \mathbf{n}(\varrho) \in \sigma$.

Example 2.7 ([**18**, pp. 85-89]). The canonical divisor K_{X_Δ} of any toric variety X_Δ equals

$$K_{X_\Delta} = -\sum_{\varrho \in \Delta(1)} \mathbf{V}_\Delta(\varrho).$$

Theorem 2.8 ([**41**, p. 27], [**18**, p. 63]). *For every d-dimensional compact toric variety X_Δ there are exact sequences*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\mathrm{div}(\mathbf{e}(.))} & \mathrm{Div}^{\mathbb{T}}_{\mathrm{W}}(X_\Delta) & \longrightarrow & \mathrm{Cl}\,\mathrm{Div}_{\mathrm{W}}(X_\Delta) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M & \xrightarrow{\mathrm{div}(\mathbf{e}(.))} & \mathrm{Div}^{\mathbb{T}}_{\mathrm{C}}(X_\Delta) & \longrightarrow & \mathrm{Pic}(X_\Delta) & \longrightarrow & 0
\end{array}$$

where the first arrows send $\mathbf{m} \in M$ to be mapped onto

$$\mathrm{div}(\mathbf{e}(\mathbf{m})) = \sum_{\varrho \in \Delta(1)} \langle \mathbf{m}, \mathbf{n}(\varrho) \rangle \; \mathbf{V}_\Delta(\varrho).$$

In particular,

$$\mathrm{Cl}\,\mathrm{Div}^{\mathbb{T}}_{\mathrm{C}}(X_\Delta) = \mathrm{Cl}\,\mathrm{Div}_{\mathrm{C}}(X_\Delta) \cong \mathrm{Pic}(X_\Delta) \subseteq \mathrm{Cl}\,\mathrm{Div}^{\mathbb{T}}_{\mathrm{W}}(X_\Delta) = \mathrm{Cl}\,\mathrm{Div}_{\mathrm{W}}(X_\Delta),$$

and $\mathrm{rank}(\mathrm{Cl}\,\mathrm{Div}_{\mathrm{W}}(X_\Delta)) = \sharp(\Delta(1)) - d$.

Corollary 2.9 ([**18**, p. 65]). *Let Δ be a d-dimensional complete fan. Then the following conditions are equivalent*:

(i) Δ *is simplicial.*

(ii) *Every Weil divisor on X_Δ is a $\mathbb{Q}$-Cartier divisor.*

(iii) $\mathrm{Pic}(X_\Delta) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{Cl}\,\mathrm{Div}_{\mathrm{W}}(X_\Delta) \otimes_{\mathbb{Z}} \mathbb{Q}$.

(iv) *The Picard number of X_Δ is* $\mathrm{rank}(\mathrm{Pic}(X_\Delta)) = \sharp(\Delta(1)) - d$.

Definition 2.10 (Δ-support functions). Let N be a free $\mathbb{Z}$-module of rank d, M its dual, and Δ a fan w.r.t. N. A function $\psi : |\Delta| \longrightarrow \mathbb{R}$ is called Δ-*support function* if $\psi(N \cap |\Delta|) \subset \mathbb{Z}$ and ψ is linear on each $\sigma \in \Delta$, i.e., there exists a $\mathbf{l}_\sigma \in M$ for each $\sigma \in \Delta$ such that $\psi(\mathbf{n}) = \langle \mathbf{l}_\sigma, \mathbf{n} \rangle$, and $\langle \mathbf{l}_\sigma, \mathbf{n} \rangle = \langle \mathbf{l}_\tau, \mathbf{n} \rangle$ whenever

$\mathbf{n}\in\tau\prec\sigma$. Note that every Δ-support function ψ is determined by its values $\psi(\mathbf{n}(\varrho))$, $\varrho\in\Delta(1)$, and $\mathbf{l}_\sigma$ is a solution in M of the system of equations

$$\{\langle\mathbf{l}_\sigma,\mathbf{n}(\varrho)\rangle=\psi(\mathbf{n}(\varrho))\ \mid\ \varrho\in\Delta(1),\ \varrho\prec\sigma\}.$$

REMARK 2.11. (a) Each Δ-support function ψ assigns a $\mathbb{T}$-invariant Cartier divisor

$$\psi\longmapsto D_\psi:=-\sum_{\varrho\in\Delta(1)}\psi(\mathbf{n}(\varrho))\,\mathbf{V}_\Delta(\varrho)\in\mathrm{Div}_{\mathbb{C}}^{\mathbb{T}}(X_\Delta).$$

(b) To work with $\mathbb{T}$-invariant $\mathbb{Q}$-Cartier divisors one considers *rational* Δ-support functions by replacing the condition of Definition 2.10 with $\psi(N_{\mathbb{Q}}\cap|\Delta|)\subset\mathbb{Q}$ (and M with $M_{\mathbb{Q}}$).

DEFINITION 2.12. A Δ-support function ψ is called *upper convex* if

$$\psi(\mathbf{n}+\mathbf{n}')\geq\psi(\mathbf{n})+\psi(\mathbf{n}'),\quad\forall\mathbf{n},\mathbf{n}'\in N.$$

We say that an upper convex Δ-support function ψ is *strictly upper convex* whenever the set $\{\mathbf{l}_\sigma\mid\sigma\in\Delta\}$ (as defined in 2.10) is uniquely determined by ψ.

THEOREM 2.13 ([**42**, Thm. 2.7, pp. 76-77]). *Let X_Δ be a d-dimensional compact toric variety. For every Δ-support function ψ,*

$$\mathcal{P}_\psi:=\{\mathbf{m}\in M_{\mathbb{R}}\ \mid\ \langle\mathbf{m},\mathbf{n}\rangle\geq\psi(\mathbf{n}),\ \forall\mathbf{n}\in N_{\mathbb{R}}\}$$

is a convex polytope, and the set $H^0(X_\Delta,\mathcal{O}_{X_\Delta}(D_\psi))$ of global sections of the sheaf $\mathcal{O}_{X_\Delta}(D_\psi)$ is a finite dimensional $\mathbb{C}$-vector space having $\{\mathbf{e}(\mathbf{m})\mid\mathbf{m}\in M\cap\mathcal{P}_\psi\}$ as a basis. Moreover, $\mathcal{O}_{X_\Delta}(D_\psi)$ is generated by its global sections *if and only if ψ is upper convex* (*or, equivalently, $\mathcal{P}_\psi=\mathrm{conv}(\{\mathbf{l}_\sigma\mid\sigma\in\Delta\})$.*)

THEOREM 2.14 ([**42**, Thm. 2.13, pp. 82-83]). *If $\mathcal{O}_{X_\Delta}(D_\psi)$, as in* Thm. 2.13, *is generated by its global sections, $M\cap\mathcal{P}_\psi=\{\mathbf{m}_0,\ldots,\mathbf{m}_k\}$, and $\mathfrak{f}_\psi:X_\Delta\longrightarrow\mathbb{P}_{\mathbb{C}}^k$ is defined by*

$$\mathfrak{f}_\psi(x):=[\mathbf{e}(\mathbf{m}_0)(x):\mathbf{e}(\mathbf{m}_1)(x):\ldots:\mathbf{e}(\mathbf{m}_k)(x)],\quad\forall x\in X_\Delta,$$

then D_ψ is very ample (*i.e., $\mathfrak{f}_\psi$ is a closed embedding*) *if and only if the following conditions are satisfied*:

(i) *ψ is strictly upper convex, and*

(ii) *for each $\sigma\in\Delta(d)$, the set $M\cap\mathcal{P}_\psi-\mathbf{l}_\sigma$ generates the semigroup $M\cap\sigma^\vee$.*

Moreover, condition (i) *is equivalent to the following*:

(i)′ *$\mathcal{P}_\psi$ is d-dimensional and has exactly $\{\mathbf{l}_\sigma\mid\sigma\in\Delta\}$ as the set of its vertices.*

REMARK 2.15. (a) As it follows from the proof of Theorem 2.14, D_ψ is *ample* if and only if condition (i) (or, equivalently, condition (i)′) is satisfied.

(b) In dimension $d=2$, D_ψ is very ample if and only if it is ample because condition (ii) is satisfied automatically (see [**32**, Lemma 1.6.3, p. 32]). This fails in higher dimensions for singular X_Δ's.

3. Two-dimensional toric singularities

Examining two-dimensional toric singularities "under the microscope" (which turn out to be cyclic quotient singularities) one discovers a peculiar algebro-geometric world endowed with a rich combinatorial structure. Viewed historically, everything begins with Hirzebruch-Jung continued fractions (i.e., negative-regular continued fraction expansions of specific rational numbers; see [**28**], [**3**, Ch. III, §5]).

• **General notation**. For $n \in \mathbb{N}$, $m \in \mathbb{Z}$, we denote by $[m]_n$ the (uniquely determined) integer for which $0 \leq [m]_n < n$, $m \equiv [m]_n \,(\mathrm{mod}\; n)$. If $x \in \mathbb{Q}$, we define $\lceil x \rceil$ (resp., $\lfloor x \rfloor$) to be the least integer number $\geq x$ (resp., the greatest integer number $\leq x$), $\langle x \rangle := x - \lfloor x \rfloor$ the fractional part of x, and $((x))$ the *sawtooth function*:

$$((x)) := \begin{cases} \langle x \rangle - \frac{1}{2}, & \text{if} \quad x \notin \mathbb{Z}, \\ 0, & \text{if} \quad x \in \mathbb{Z}. \end{cases} \tag{3.1}$$

"gcd" and "lcm" will be abbreviations for greatest common divisor and least common multiple. Furthermore, for an integer $n \geq 2$, we denote by $\zeta_n := \exp(\frac{2\pi\sqrt{-1}}{n})$ the "first" n-th primitive root of unity.

• **Finite continued fractions**. The use of finite continued fractions enables convenient rational approximations to the minimal generators of two-dimensional rational s.c.p. cones. For this reason it becomes the most important tool in the theory of two-dimensional toric singularities.

Let κ and λ be two given relatively prime positive integers. Suppose $\frac{\kappa}{\lambda}$ can be written as

$$\frac{\kappa}{\lambda} = a_1 + \cfrac{\varepsilon_1}{a_2 + \cfrac{\varepsilon_2}{a_3 + \cfrac{\varepsilon_3}{\ddots \; a_{\nu-1} + \cfrac{\varepsilon_{\nu-1}}{a_\nu}}}} \tag{3.2}$$

The right-hand side of (3.2) is called *semi-regular continued fraction* for $\frac{\kappa}{\lambda}$ (and ν its *length*) if it has the following properties:

(i) a_j is an integer for all j, $1 \leq j \leq \nu$,

(ii) $\varepsilon_j \in \{-1, 1\}$ for all j, $1 \leq j \leq \nu - 1$,

(iii) $a_j \geq 1$ for $j \geq 2$ and $a_\nu \geq 2\,(!)$, and

(iv) if $a_j = 1$ for some j, $1 < j < \nu$, then $\varepsilon_j = 1$.

In particular, if $\varepsilon_j = 1$ (resp., $\varepsilon_j = -1$) for *all* j, $1 \leq j < \nu$, then we write

$$\frac{\kappa}{\lambda} = [a_1, a_2, \ldots, a_\nu] \qquad \left(\text{resp., } \frac{\kappa}{\lambda} = [\![a_1, a_2, \ldots, a_\nu]\!]\right).$$

This is the *regular* (resp., the *negative-regular*) *continued fraction expansion* of $\frac{\kappa}{\lambda}$. These two expansions always exist, are unique (in this form), and can be obtained by the usual and the modified euclidean algorithm, respectively, depending on the choice of the kind of the associated remainders (see [**14**, 3.1 and 3.5]).

Proposition 3.1. *If* $\lambda, \kappa \in \mathbb{Z}$ *with* $1 < \lambda < \kappa$, $\gcd(\lambda, \kappa) = 1$, *and*

$$\frac{\kappa}{\lambda} = [a_1, a_2, \ldots, a_t] = [\![b_1, b_2, \ldots, b_s]\!]$$

are the regular and negative-regular continued fraction expansions of $\frac{\kappa}{\lambda}$, *respectively, then for* $t \geq 2$ *the ordered* s*-tuple* $(b_1, b_2, \ldots, b_s)$ *equals*

$$\begin{cases} (a_1+1, \underbrace{2,...,2}_{(a_2-1)\text{-times}}, a_3+2, \underbrace{2,...,2}_{(a_4-1)\text{-times}}, \ldots, a_{t-1}+2, \underbrace{2,...,2}_{(a_t-1)\text{-times}}), & \text{if } t \text{ even} \\ (a_1+1, \underbrace{2,...,2}_{(a_2-1)\text{-times}}, a_3+2, \underbrace{2,...,2}_{(a_4-1)\text{-times}}, \ldots, \underbrace{2,...,2}_{(a_{t-1}-1)\text{-times}}, a_t+1), & \text{if } t \text{ odd} \end{cases}$$

Proof. See [**14**, Proposition 3.6, pp. 219-220]. □

Corollary 3.2. *The length* s *of the negative-regular continued fraction expansion of* $\frac{\kappa}{\lambda}$ *equals*

$$s = \begin{cases} \sum_{i=1}^{t/2} a_{2i}, & \text{if } t \text{ is even,} \\ (\sum_{i=1}^{(t+1)/2} a_{2i}) + 1, & \text{if } t \text{ is odd.} \end{cases} \tag{3.3}$$

• **Dedekind sums**. Let p, q be two integers with $q > 0$ and $\gcd(p, q) = 1$. The (classical) *Dedekind sum* $\mathrm{DS}(p, q)$ of p and q is defined to be

$$\mathrm{DS}(p,q) := \sum_{j=1}^{q-1} \left(\!\!\left(\frac{j}{q} \right)\!\!\right) \left(\!\!\left(\frac{pj}{q} \right)\!\!\right) \tag{3.4}$$

It satisfies $\mathrm{DS}(-p, q) = -\mathrm{DS}(p, q)$, and the reciprocity law:

$$\mathrm{DS}(p,q) + \mathrm{DS}(q,p) = -\tfrac{1}{4} + \tfrac{1}{12}\left(\tfrac{p}{q} + \tfrac{q}{p} + \tfrac{1}{pq} \right) .$$

The sums $\mathrm{DS}(p, q)$ arose for the first time in Dedekind's investigations on the logarithm of the eta-function (see [**16**]). The book [**46**] contains further references and details on the history of Dedekind sums. A well-known formula for $\mathrm{DS}(p, q)$ (cf. [**46**, p. 18] or [**29**, p. 100]) is that one given by the trigonometrical expression

$$\mathrm{DS}(p,q) = \tfrac{1}{4q} \sum_{j=1}^{q-1} \left[\cot\left(\tfrac{jp\pi}{q} \right) \cdot \cot\left(\tfrac{j\pi}{q} \right) \right]. \tag{3.5}$$

If $0 \leq p < q$, another elegant identity, showing the relationship between $\mathrm{DS}(p, q)$ and the negative-regular continued fraction expansion $\frac{q}{q-p} = [\![b_1, \ldots, b_s]\!]$, can be derived by Myerson's results (see [**39**, p. 421], [**44**, p. 12]):

$$\mathrm{DS}(p,q) = \frac{1}{12} \left(\sum_{j=1}^{s} (3 - b_j) + \frac{1}{q}(p + \widehat{p}) - 2 \right). \tag{3.6}$$

Here $\widehat{p}$ denotes the uniquely determined integer, so that $0 \leq \widehat{p} < q$, and

$$p\,\widehat{p} \equiv 1 (\mathrm{mod}\ q), \quad (\text{i.e., } [p\,\widehat{p}]_q = 1) .$$

$\widehat{p}$ is often called *the socius* of p. If $p \neq 0$ (which means that $q \neq 1$), then using a formula due to Voronoi (cf. [**53**, p. 183]), $\widehat{p}$ can be written as

$$\widehat{p} = \left[3 - 2p + 6 \left(\sum_{j=1}^{p-1} \left(\left\lfloor \tfrac{jq}{p} \right\rfloor \right)^2 \right) \right]_q .$$

• **Two-dimensional cones**. Up to lattice automorphisms, the "lattice geometry" of two-dimensional rational s.c.p. cones is completely describable by means of just two integers ("parameters").

LEMMA 3.3. *Let N be a free $\mathbb{Z}$-module of rank 2 and $\sigma \subset N_{\mathbb{R}}$ a two-dimensional rational s.c.p. cone with* $\mathrm{Gen}(\sigma) = \{\mathbf{n}_1, \mathbf{n}_2\}$. *Then there is a $\mathbb{Z}$-basis $\{\mathfrak{y}_1, \mathfrak{y}_2\}$ of N and two integers $p = p_\sigma,\ q = q_\sigma \in \mathbb{Z}_{\geq 0}$ with $0 \leq p < q$, and $\gcd(p,q) = 1$ for $p \neq 0$, such that*

$$\mathbf{n}_1 = \mathfrak{y}_1, \quad \mathbf{n}_2 = p\,\mathfrak{y}_1 + q\,\mathfrak{y}_2, \quad q = \mathrm{mult}\,(\sigma; N) = \frac{\det(\mathbb{Z}\ \mathbf{n}_1 \oplus \mathbb{Z}\ \mathbf{n}_2)}{\det(N)}\ .$$

PROOF. See [**14**, Lemma 3.9, p. 221]. □

DEFINITION 3.4. If N is a free $\mathbb{Z}$-module of rank 2 and $\sigma \subset N_{\mathbb{R}}$ a two-dimensional rational s.c.p. cone with $\mathrm{Gen}(\sigma) = \{\mathbf{n}_1, \mathbf{n}_2\}$, then we call σ a (p,q)-*cone w.r.t. the basis* $\{\mathfrak{y}_1, \mathfrak{y}_2\}$, if $p = p_\sigma,\ q = q_\sigma$ as in Lemma 3.3. (To avoid confusion, we should stress at this point that saying "w.r.t. the basis $\{\mathfrak{y}_1, \mathfrak{y}_2\}$" we just indicate the choice of one suitable $\mathbb{Z}$-basis of N among *all* its $\mathbb{Z}$-bases in order to apply Lemma 3.3 for σ; but, of course, if $\{\mathfrak{y}_1, \mathfrak{y}_2'\}$ were a $\mathbb{Z}$-basis of N having the *same* property, i.e., $\mathbf{n}_2 = p'\,\mathfrak{y}_1 + q'\,\mathfrak{y}_2'$, $0 \leq p' < q'$, $\gcd(p', q') = 1$, then obviously $p' = p$ and $q' = q$, i.e., $\mathfrak{y}_2' = \mathfrak{y}_2$!)

PROPOSITION 3.5. *Let N be a free $\mathbb{Z}$-module of rank 2 and $\sigma, \tau \subset N_{\mathbb{R}}$ two 2-dimensional rational s.c.p. cones. Then the following conditions are equivalent*

(i) *There exists a torus-equivariant isomorphism $U_\sigma \cong U_\tau$ mapping* $\mathrm{orb}(\sigma)$ *onto* $\mathrm{orb}(\tau)$.

(ii) *There exists a $\mathbb{Z}$-module automorphism $\varpi : N \longrightarrow N$ of N whose scalar extension $\varpi_{\mathbb{R}} : N_{\mathbb{R}} \longrightarrow N_{\mathbb{R}}$ has the property: $\varpi(\sigma) = \tau$.*

(iii) *For the numbers p_σ, p_τ, q_σ, q_τ associated to σ, τ w.r.t. a basis $\{\mathfrak{y}_1, \mathfrak{y}_2\}$ (as in* Lemma 3.3) *we have*

$$q_\tau = q_\sigma \quad \text{and} \quad \begin{cases} \text{either} & p_\tau = p_\sigma \\ \text{or} & p_\tau \neq 0,\ p_\sigma \neq 0 \ \text{ and } \ p_\tau = \widehat{p}_\sigma \end{cases}$$

PROOF. See [**14**, Proposition 3.12, pp. 222-223]. □

REMARK 3.6. Up to replacement of p by its socius $\widehat{p}$ (which corresponds just to the interchange of the coordinates), these two numbers $p = p_\sigma$ and $q = q_\sigma$ parametrize uniquely the isomorphism class of the germ $(U_\sigma,\ \mathrm{orb}\,(\sigma))$.

LEMMA 3.7. *Let N be a free $\mathbb{Z}$-module of rank 2, $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual and $\sigma \subset N_{\mathbb{R}}$ a two-dimensional (p,q)-cone w.r.t. a $\mathbb{Z}$-basis $\{\mathfrak{y}_1, \mathfrak{y}_2\}$ of N. Denoting by $\{\mathfrak{m}_1, \mathfrak{m}_2\}$ the dual basis of $\{\mathfrak{y}_1, \mathfrak{y}_2\}$ in M, the cone $\sigma^\vee \subset M_{\mathbb{R}}$ is a $(q-p, q)$-cone w.r.t. $\{\mathfrak{m}_2, \mathfrak{m}_1 - \mathfrak{m}_2\}$.*

PROOF. See [**14**, Lemma 3.14, p. 223]. □

▷ From now on, and for the rest of the present section, we fix a free $\mathbb{Z}$-module N of rank 2, its dual M, a *nonbasic* two-dimensional (p,q)-cone $\sigma \subset N_{\mathbb{R}} \cong \mathbb{R}^2$ w.r.t. a $\mathbb{Z}$-basis $\{\mathfrak{y}_1, \mathfrak{y}_2\}$ of N, $\mathrm{Gen}(\sigma) = \{\mathbf{n}_1, \mathbf{n}_2\}$, the dual basis $\{\mathfrak{m}_1, \mathfrak{m}_2\}$ of $\{\mathfrak{y}_1, \mathfrak{y}_2\}$ in M, and the dual cone $\sigma^\vee \subset M_{\mathbb{R}}$ of σ. Moreover, we consider the negative-regular continued fraction expansion of both rationals $\frac{q}{q-p}$ and $\frac{q}{p}$:

$$\frac{q}{q-p} = [\![b_1, b_2, \ldots, b_s]\!]\,, \quad \frac{q}{p} = \frac{q}{q-(q-p)} = [\![b_1^\vee, b_2^\vee, \ldots, b_t^\vee]\!]\,. \tag{3.7}$$

NOTE 3.8. (a) As it is known (cf. [**42**, p. 29]),

$$(b_1 + b_2 + \cdots + b_s) - s = (b_1^{\vee} + b_2^{\vee} + \cdots + b_t^{\vee}) - t = s + t - 1\,.$$

(b) Replacing p by its socius $\widehat{p}$ in the rationals (3.7), and passing to their negative-regular continued fraction expansions (cf. [**28**, p.20]), we get

$$\frac{q}{q-\widehat{p}} = [\![b_s, b_{s-1}, \ldots, b_2, b_1]\!]\,, \quad \frac{q}{\widehat{p}} = [\![b_t^{\vee}, b_{t-1}^{\vee}, \ldots, b_2^{\vee}, b_1^{\vee}]\!]\,. \tag{3.8}$$

DEFINITION 3.9. For any integer $s \geq 1$ and any s-tuple $(x_1, x_2, \ldots, x_s) \in \mathbb{R}^s$ we define the symmetric $(s \times s)$-matrix $\mathbf{L}_s\,(x_1, x_2, \ldots, x_s)$ as follows:

$$\mathbf{L}_s\,(x_1, x_2, \ldots, x_s) := \begin{pmatrix} x_1 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & x_2 & -1 & \cdots & \cdots & 0 \\ 0 & -1 & x_3 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & & \ddots & \\ 0 & \cdots & \cdots & 0 & -1 & x_s \end{pmatrix}.$$

The matrix $\mathbf{L}_s(b_1, \ldots, b_s)$, for $(b_1, \ldots, b_s) \in \mathbb{Z}^s$ the s-tuple in (3.7), has determinant

$$\det\left(\mathbf{L}_s(b_1, \ldots, b_s)\right) = q = \prod_{j=1}^{s} [\![b_j, b_2, \ldots, b_s]\!] \tag{3.9}$$

$$= \left(\prod_{j=1}^{s} b_i\right)\left(1 - \sum_{1 \leq k \leq s-1} \tfrac{1}{b_k b_{k+1}} + \sum_{1 \leq k < l \leq s-2} \tfrac{1}{b_k b_{k+1}} \tfrac{1}{b_l b_{l+1}} - \cdots\right),$$

which is the "highest" *continuant* of the fraction $\frac{q}{q-p}$ (cf. Perron [**43**, Ch. I, §2-§4]).

DEFINITION 3.10 (Gammas and Deltas). Having this continuant as our starting point we define two sequences of integers $(\gamma_j)_{0 \leq j \leq s+1}$ and $(\delta_j)_{0 \leq j \leq s+1}$ ("minor continuants" of $\frac{q}{q-p}$) by setting

$$\gamma_j := \det\left(\mathbf{L}_{s-j}(b_{j+1}, \ldots, b_s)\right), \quad \forall j \in \{0, 1, \ldots, s-1\},$$

with $\gamma_s := 1$, $\gamma_{s+1} := 0$ (as its final values), and

$$\delta_j := \det\left(\mathbf{L}_{j-1}(b_1, \ldots, b_{j-1})\right), \quad \forall j \in \{2, 3, \ldots, s+1\},$$

with $\delta_0 := 0$, $\delta_1 := 1$ (as its initial values). It is an exercise of linear algebra to show that

$$\gamma_{j-1} + \gamma_{j+1} = b_j \gamma_j, \quad \delta_{j-1} + \delta_{j+1} = b_j \delta_j, \quad \forall j \in \{1, \ldots, s\}, \tag{3.10}$$

and

$$\gamma_{j-1}\delta_j - \gamma_j \delta_{j-1} = q, \quad \forall j \in \{1, 2, \ldots, s+1\}. \tag{3.11}$$

REMARK 3.11. By (3.9) and the definition given above, $\gamma_0 = q$, and

$$\tfrac{q}{q-p} = b_1 - \frac{1}{[\![b_2, b_3, \ldots, b_s]\!]} \Longrightarrow \gamma_1 = \det\left(\mathbf{L}_{s-1}(b_2, \ldots, b_s)\right) = q - p.$$

Moreover, comparing the negative-regular continued fraction expansions (3.7) and (3.8) of $\frac{q}{q-p}$ and $\frac{q}{q-\widehat{p}}$, respectively, we see that the gammas for the one become the deltas for the other (and vice versa). For this reason, $\delta_s = q - \widehat{p}$ and $\delta_{s+1} = q$.

Next Lemmas will be useful for several technical computations in sections 4-7.

LEMMA 3.12. *If $b_1, \ldots, b_s$ are the integers defined in* (3.7)*, then the symmetric* $(s \times s)$*-matrix* $\mathbf{L}_s(b_1, \ldots, b_s)$ *is positive definite.*

PROOF. For every $\mathbf{x} = (x_1, \ldots, x_s) \in \mathbb{R}^s$ we define the set

$$\Lambda_{\mathbf{x}} := \{(j,k) \in \{1, \ldots, s\} \times \{1, \ldots, s\} \mid \; j < k \text{ and } x_j = x_k\}.$$

For all $(x_1, \ldots, x_s) \in \mathbb{R}^s \smallsetminus \{(0, \ldots, 0)\}$ we have

$$(x_1, \ldots, x_s)\; \mathbf{L}_s(b_1, \ldots, b_s)\; (x_1, \ldots, x_s)^T = \sum_{j=1}^{s} b_j x_j^2 - 2 \sum_{1 \le j < k \le s} x_j x_k$$

$$= \sum_{j=1}^{s} (b_j - 2)\, x_j^2 + \left(x_1^2 + \sum_{\substack{1 \le j < k \le s \\ (j,k) \notin \Lambda_{\mathbf{x}}}} (x_j - x_k)^2 + x_s^2 \right)$$

which is > 0 because $b_j \ge 2$ for all $j \in \{1, \ldots, s\}$ (by Proposition 3.1). □

LEMMA 3.13. *Suppose that* $(y_1, \ldots, y_s) \in \mathbb{R}^s$*,* $s \ge 1$*, and that the integers* $b_1, \ldots, b_s$ *are those defined in* (3.7)*. Then the linear system*

$$\mathbf{L}_s(b_1, \ldots, b_s)\,(\xi_1, \ldots, \xi_s)^T = (y_1, \ldots, y_s)^T$$

has a unique solution $(\xi_1, \ldots, \xi_s) \in \mathbb{R}^s$ *with coordinates given by the formulae*

$$\xi_j = \frac{1}{q} \left(\sum_{1 \le k < j} \gamma_j \delta_k y_k + \gamma_j \delta_j y_j + \sum_{1 \le j < k \le s} \gamma_k \delta_j y_k \right), \quad \forall j \in \{1, \ldots, s\}.$$

PROOF. Since $\det(\mathbf{L}_s(b_1, \ldots, b_s)) = \gamma_0 = q \ne 0$, the uniqueness is obvious. On the other hand, it is easy to prove (by (3.10) and (3.11)) that $(-1)^{k+j}$ times the determinant of the (k,j)-minor of $\mathbf{L}_s(b_1, .., b_s)$ equals

$$\left(\begin{array}{c} \text{the } j\text{-th coordinate of the vector} \\ (\mathbf{L}_s(b_1, .., b_s))^{-1}(0, \ldots, 0, \underbrace{1}_{k-\text{th pos.}}, 0, \ldots, 0)^T \end{array} \right) = \begin{cases} \gamma_j \delta_k, & \text{if } 1 \le k < j, \\ \gamma_j \delta_j, & \text{if } k = j, \\ \gamma_k \delta_j, & \text{if } 1 \le j < k \le s. \end{cases}$$

Hence, to determine ξ_j, it sufficies to apply Cramer's rule. □

DEFINITION 3.14. (i) In $N_{\mathbb{R}}$ we define $s+2$ vectors $(\mathbf{u}_j)_{0 \le j \le s+1}$ as follows:

$$\mathbf{u}_j := \frac{\gamma_j}{q}\,\mathbf{n}_1 + \frac{\delta_j}{q}\,\mathbf{n}_2 = \beta_j \mathfrak{y}_1 + \delta_j \mathfrak{y}_2, \quad \forall j \in \{0, 1, \ldots, s+1\},$$

where $\beta_j := \frac{1}{q}(\gamma_j + p\delta_j)$. Since $\beta_0 = 1$, $\beta_1 = 1$, and $\beta_j = b_j \beta_{j-1} - \beta_{j-2}$, for all $j \in \{2, \ldots, s+1\}$, the β_j's are integers and therefore the $\mathbf{u}_j$'s belong to N. Note that $\mathbf{u}_0 = \mathbf{n}_1 = \mathfrak{y}_1$, $\mathbf{u}_1 = \mathfrak{y}_1 + \mathfrak{y}_2$,

$$(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{s-1}, \mathbf{u}_s)^T = \mathbf{L}_s(b_1, \ldots, b_s)^{-1}(\mathbf{n}_1, 0, 0, \ldots, 0, 0, \mathbf{n}_2)^T,$$

(as vectorial matrix multiplication) and $\mathbf{u}_{s+1} = \mathbf{n}_2$.
(ii) Analogously, we define $t+2$ vectors $(\mathbf{u}_j^{\vee})_{0 \le j \le t+1}$ belonging to M by setting $\mathbf{u}_0^{\vee} := \mathfrak{m}_2$,

$$(\mathbf{u}_1^{\vee}, \mathbf{u}_2^{\vee}, \ldots, \mathbf{u}_t^{\vee})^T := \mathbf{L}_t(b_1^{\vee}, \ldots, b_t^{\vee})^{-1}(\mathfrak{m}_1, 0, \ldots, 0, (q-p)\,\mathfrak{m}_2 + q\,(\mathfrak{m}_1 - \mathfrak{m}_2))^T,$$

and $\mathbf{u}_{t+1}^{\vee} := (q-p)\,\mathfrak{m}_2 + q\,(\mathfrak{m}_1 - \mathfrak{m}_2)$.

PROPOSITION 3.15. *If we define*

$$\Theta_\sigma := \operatorname{conv}(\sigma \cap (N \smallsetminus \{\mathbf{0}\})) \subset N_{\mathbb{R}}, \text{ resp., } \Theta_{\sigma^\vee} := \operatorname{conv}(\sigma^\vee \cap (M \smallsetminus \{\mathbf{0}\})) \subset M_{\mathbb{R}},$$

and denote by $\partial\Theta_\sigma^{\mathbf{cp}}$ *(resp., by* $\partial\Theta_{\sigma^\vee}^{\mathbf{cp}}$*) the part of the boundary* $\partial\Theta_\sigma$ *(resp.,* $\partial\Theta_{\sigma^\vee}$*) containing only its compact edges, then the Hilbert bases* (2.1) *of the cones* σ (w.r.t. N) *and* $\sigma^\vee$ (w.r.t. M) *are equal to*

$$\begin{aligned} \operatorname{Hilb}_N(\sigma) &= \partial\Theta_\sigma^{\mathbf{cp}} \cap N = \{\mathbf{u}_j \mid 0 \leq j \leq s+1\}, \\ \operatorname{Hilb}_M(\sigma^\vee) &= \partial\Theta_{\sigma^\vee}^{\mathbf{cp}} \cap M = \left\{\mathbf{u}_j^\vee \mid 0 \leq j \leq t+1\right\}. \end{aligned}$$

(See Figure 1.)

PROOF. It follows from [**42**, pp. 26-29] and [**14**, Thm. 3.16, pp. 226-228]. □

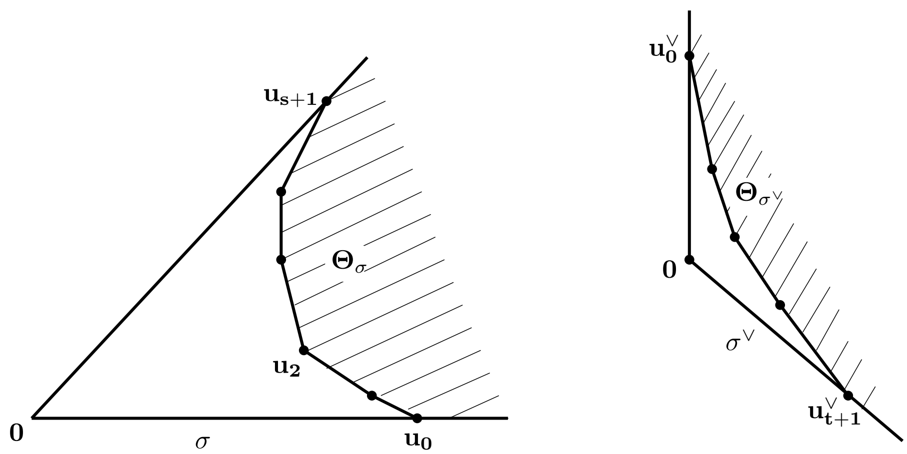


FIGURE 1.

• **Quotient structure and defining equations**. $U_\sigma = \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ has only one singular point, namely $\operatorname{orb}(\sigma)$, which is a quotient singularity. More precisely, we have the following:

PROPOSITION 3.16. $\operatorname{orb}(\sigma) \in U_\sigma$ *is a cyclic quotient singularity. In particular,* $U_\sigma \cong \mathbb{C}^2/G = \operatorname{Spec}(\mathbb{C}[z_1, z_2]^G)$*, with* $G \subset \mathrm{GL}(2, \mathbb{C})$ *denoting the cyclic group* G *of order* q *which is generated by* $\operatorname{diag}(\zeta_q^{-p}, \zeta_q)$ *and acts on* $\mathbb{C}^2 = \operatorname{Spec}(\mathbb{C}[z_1, z_2])$ *linearly and effectively.*

PROOF. See Fulton [**18**, § 2.2, pp. 32-34]. □

In fact, since we know the Hilbert basis $\operatorname{Hilb}_M(\sigma^\vee)$ by Proposition 3.15 explicitly, it is also possible to find the polynomial equations whose zero locus contains the singularity $\operatorname{orb}(\sigma) \in U_\sigma$ at the origin after having embedded U_σ into $\mathbb{C}^{t+2}$ (see Proposition 2.1).

THEOREM 3.17 (Defining equations). $U_\sigma \cong \operatorname{Spec}(\mathbb{C}[z_0, z_1, \ldots, z_{t+1}]/\mathcal{I})$*, where* $\mathcal{I}$ *is the ideal generated by the set of the* $\frac{1}{2}t\,(t+1)$ *polynomials*

$$\{z_{j-1}z_{k+1} - F_{jk}(z_1, z_2, \ldots, z_t) \mid 1 \leq j \leq k \leq t\},$$

with

$$F_{jk}(z_1,z_2,\ldots,z_t) := \begin{cases} z_j^{b_j^\vee}, & \text{if } j=k, \\ z_j^{b_j^\vee-1}\, z_{j+1}^{b_{j+1}^\vee-2} \cdots z_{k-1}^{b_{k-1}^\vee-2}\, z_k^{b_k^\vee-1}, & \text{if } j<k. \end{cases}$$

Proof. $\mathcal{I}$ is the kernel of the $\mathbb{C}$-algebra epimorphism

$$\mathbb{C}[z_0,z_1,\ldots,z_{t+1}] \longrightarrow \mathbb{C}[\sigma^\vee \cap M],$$

sending the variable z_j to $\mathbf{e}(\mathbf{u}_j^\vee)$, for all $j \in \{0,1,\ldots,t,t+1\}$. The reader is referred to Riemenschneider [**48**, §2, pp. 217-220] for details of the computation. □

Remark 3.18. $\mathrm{orb}(\sigma) \in U_\sigma$ is a hypersurface singularity if and only if

$$t=1 \Longleftrightarrow (p=1 \text{ and } q\geq 2) \Longleftrightarrow U_\sigma \cong \mathrm{Spec}(\mathbb{C}[z_0,z_1,z_2]/\,(z_0z_2 - z_1^q)).$$

In this case, $\mathrm{orb}(\sigma)$ is analytically isomorphic to the *Kleinian singularity* or *Du Val singularity* of type $\mathbf{A}_{q-1}$. Moreover, using Theorem 2.4, it is easy to see that all two-dimensional *Gorenstein* toric singularities (or, equivalently, all two-dimensional *canonical* toric singularities) are necessarily of this sort.

Note 3.19. For a methodical study of the behaviour of local differentials around the singular point $\mathrm{orb}(\sigma) \in U_\sigma$ under its minimal resolution one introduces the so-called *local index*

$$\mathrm{lind}\,(U_\sigma, \mathrm{orb}(\sigma)) := \min\left\{ k \in \mathbb{N} \;\middle|\; k\left(1 - \tfrac{\gamma_j+\delta_j}{q}\right) \in \mathbb{Z}\ ,\ \forall j \in \{1,\ldots,s\}\right\}$$

of U_σ *at* $\mathrm{orb}(\sigma)$. (See below formula (4.13) and Note 4.5 (b).) By (3.10) and (3.11) it is easy to express this auxiliary positive integer in terms of the parameters $p=p_\sigma$ and $q=q_\sigma$ of σ as follows:

$$\mathrm{lind}\,(U_\sigma, \mathrm{orb}(\sigma)) = \frac{q}{\gcd\,(q, q-p+1)} = \frac{q}{\gcd\,(q, p-1)}. \tag{3.12}$$

Geometrically, setting $F_\sigma := \mathrm{conv}(\{\mathbf{u}_0, \mathbf{u}_{s+1}\})$, $L_\sigma := \mathrm{aff}(\{\mathbf{u}_0, \mathbf{u}_{s+1}\})$ = the line determined by F_σ, and $L'_\sigma :=$ the line passing through $\mathbf{0}$ and being parallel to L_σ, $\mathrm{lind}(U_\sigma, \mathrm{orb}(\sigma))$ equals

$$\sharp \left\{ \begin{array}{c} \text{lines passing through at least one lattice point, belonging to the interior} \\ \text{of the strip bounded by } L_\sigma \text{ and } L'_\sigma \text{ and being parallel to them} \end{array} \right\} + 1.$$

(Convention: We may extend the notion of local index even if $\mathrm{orb}(\sigma)$ is assumed to be a *nonsingular* point, by defining $\mathrm{lind}(U_\sigma, \mathrm{orb}(\sigma)) := 1$. In this case, equality (3.12) remains true for $p=0$ and $q=1$.)

• **Minimal desingularization**. To construct the minimal desingularization of $U_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ one has to subdivide σ into $s+1$ smaller basic cones by using all the elements of $\mathrm{Hilb}_N(\sigma)$ as minimal generators of the new rays.

Theorem 3.20 (Toric version of Hirzebruch's desingularization). *The refinement* $\widetilde{\Delta}_\sigma := \{\{\mathbb{R}_{\geq 0}\,\mathbf{u}_j + \mathbb{R}_{\geq 0}\,\mathbf{u}_{j+1} \mid 1 \leq j \leq s+1\}$ together with their faces$\}$ *of* $\Delta_\sigma := \{\sigma$ together with its faces$\}$ *consists of basic cones, is the coarsest refinement of* Δ_σ *with this property, and gives rise to the construction of the* (good, in the sense of §1) minimal equivariant resolution $f = \mathrm{id}_* : X_{\widetilde{\Delta}_\sigma} \longrightarrow X_{\Delta_\sigma} = U_\sigma$ *of the singular point* $\mathrm{orb}(\sigma) \in U_\sigma$. *Moreover, for* $j \in \{1,\ldots,s\}$, *each exceptional prime divisor* $\mathbf{V}_{\widetilde{\Delta}_\sigma}(\mathbb{R}_{\geq 0}\,\mathbf{u}_j)$ *w.r.t.* f *is isomorphic to the projective line* $\mathbb{P}^1_{\mathbb{C}}$. (Figure 2 illustrates $\widetilde{\Delta}_\sigma$ for a singularity of this kind with $p=4$ and $q=11$.)

Proof. See Hirzebruch [**28**, pp. 15-20] who constructs $X_{\widetilde{\Delta}_\sigma}$ by resolving the unique singularity lying over $\mathbf{0} \in \mathbb{C}^3$ in the normalization of the hypersurface

$$\left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 \middle| \ z_1^q - z_2 z_3^{q-p} = 0 \right\},$$

and Oda [**42**, pp. 24-30] for a proof which uses only the tools of toric geometry. □

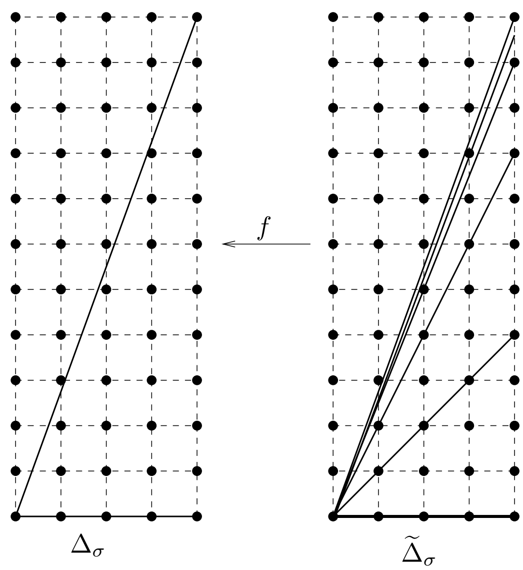


Figure 2.

4. Combinatorial data and invariants of compact toric surfaces

The geometric properties and the invariants of compact toric surfaces depend on the parametrization of each of the cones of their defining complete fans (in the sense of 3.6) and can be studied systematically by means of their *combinatorial data* (see below Definition 4.2).

• **Two-dimensional complete fans**. Let N be a free $\mathbb{Z}$-module of rank 2 and Δ an arbitrary complete fan of two-dimensional s.c.p. cones in $N_\mathbb{R}$ with

$$\Delta(1) = \{\tau_1, \tau_2, \ldots, \tau_\nu\}, \quad \Delta(2) = \{\sigma_1, \sigma_2, \ldots, \sigma_\nu\}, \ \ \nu \geq 3,$$

and

$$\text{Gen}(\Delta) = \{\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_\nu\}, \quad \tau_i = \mathbb{R}_{\geq 0}\, \mathbf{n}_i, \ \ \forall i \in \{1, 2, \ldots, \nu\},$$

with $\sigma_i := \tau_i + \tau_{i+1}$. We assume that the minimal generators $\mathbf{n}_1, \ldots, \mathbf{n}_\nu$ of Δ go *anticlockwise* around the origin exactly once in this order (see Figure 3). Moreover, we set $\mathbf{n}_{\nu+1} := \mathbf{n}_1$ and $\mathbf{n}_0 := \mathbf{n}_\nu$. (In definitions and formulae involving enumerated sets of numbers or vectors in which the index set $\{1, \ldots, \nu\}$ is meant as a cycle, we shall read the indices i "mod ν", even if it is not mentioned explicitly.)
Now we set $M := \text{Hom}_\mathbb{Z}(N, \mathbb{Z})$ and denote by X_Δ the toric surface obtained by gluing together the affine varieties

$$U_{\sigma_i} := \text{Spec}\left(\mathbb{C}[\sigma_i^\vee \cap M]\right), \ \ i \in \{1, 2, \ldots, \nu\},$$

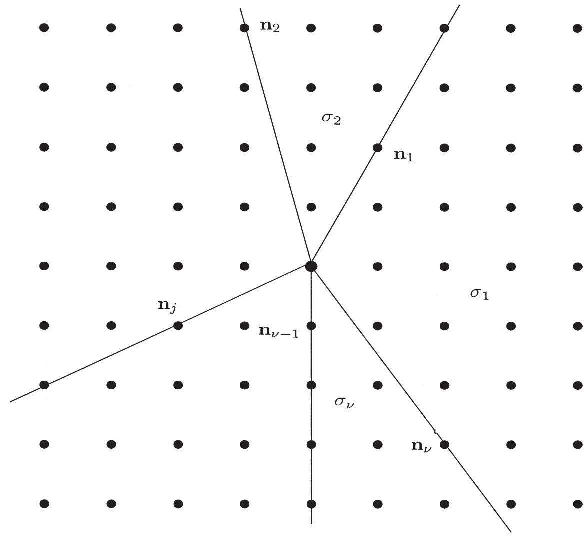


FIGURE 3.

as in (2.3). X_Δ is necessarily projective (cf. [**41**, Proposition 8.1, pp. 51-52]), the group of its $\mathbb{T}$-invariant Weil divisors is

$$\mathrm{Div}_{\mathrm{W}}^{\mathbb{T}}(X_\Delta) = \bigoplus_{i=1}^{\nu} \mathbb{Z}\, C_i, \quad \text{where } C_i := \mathbf{V}_\Delta\left(\mathbb{R}_{\geq 0}\, \mathbf{n}_i\right),$$

and its topological Euler characteristic equals $e(X_\Delta) = \nu$ (see Note 2.6 (d)). Moreover, since Δ is necessarily *simplicial*, every Weil divisor on X_Δ is a $\mathbb{Q}$-Cartier divisor (by Corollary 2.9). Next, we assume that σ_i is a (p_i, q_i)-cone (in the sense of 3.4, w.r.t. a suitable $\mathbb{Z}$-basis of N) and introduce the notation

$$I_\Delta := \{\, i \in \{1, \ldots, \nu\} \mid q_i > 1 \,\}, \quad J_\Delta := \{\, i \in \{1, \ldots, \nu\} \mid q_i = 1 \,\}, \tag{4.1}$$

to separate the indices corresponding to nonbasic from those corresponding to basic cones. By Theorem 2.3 and Note 2.6 (a) we have obviously

$$\mathrm{Sing}(X_\Delta) = \{\, \mathrm{orb}(\sigma_i) \mid i \in I_\Delta \,\}.$$

For all $i \in I_\Delta$ we write

$$\frac{q_i}{q_i - p_i} = \left[\!\left[b_1^{(i)}, b_2^{(i)}, \ldots, b_{s_i}^{(i)} \right]\!\right] \tag{4.2}$$

and, in accordance with what is already mentioned for a single nonbasic 2-dimensional rational s.c.p. cone in §3, we define for each $j \in \{0, 1, \ldots, s_i + 1\}$ integers $\gamma_j^{(i)}$, $\delta_j^{(i)}$ as follows:

$$\begin{cases} \gamma_j^{(i)} := \det\left(\mathbf{L}_{s_i - j}(b_{j+1}^{(i)}, \ldots, b_{s_i}^{(i)})\right), \quad \forall j \in \{0, 1, \ldots, s_i - 1\}, \\ \gamma_{s_i}^{(i)} := 1, \quad \gamma_{s_i+1}^{(i)} := 0, \\ \delta_0^{(i)} := 0, \quad \delta_1^{(i)} := 1, \\ \delta_j^{(i)} := \det\left(\mathbf{L}_{j-1}(b_1^{(i)}, \ldots, b_{j-1}^{(i)})\right), \quad \forall j \in \{2, 3, \ldots, s_i + 1\}. \end{cases} \tag{4.3}$$

Notice that

$$\gamma_0^{(i)} = q_i, \quad \gamma_1^{(i)} = q_i - p_i, \tag{4.4}$$

and

$$\begin{cases} \gamma_{j-1}^{(i)} + \gamma_{j+1}^{(i)} = b_j^{(i)} \gamma_j^{(i)}, & \forall j \in \{0, 1, \ldots, s_i\}, \\ \delta_{j-1}^{(i)} + \delta_{j+1}^{(i)} = b_j^{(i)} \delta_j^{(i)}, & \forall j \in \{0, 1, \ldots, s_i\}, \\ \gamma_{j-1}^{(i)} \delta_j^{(i)} - \gamma_j^{(i)} \delta_{j-1}^{(i)} = q_i, & \forall j \in \{1, 2, \ldots, s_i + 1\}, \end{cases} \tag{4.5}$$

and finally,

$$\delta_{s_i}^{(i)} = q_i - \widehat{p}_i, \quad \delta_{s_i+1}^{(i)} = q_i. \tag{4.6}$$

• **The minimal desingularization of** X_Δ. Maintaining the above notation and setting

$$\mathbf{u}_j^{(i)} := \frac{\gamma_j^{(i)}}{q_i} \mathbf{n}_i + \frac{\delta_j^{(i)}}{q_i} \mathbf{n}_{i+1}, \quad \forall j \in \{0, 1, \ldots, s_i + 1\},$$

we get

$$\mathbf{u}_{j-1}^{(i)} + \mathbf{u}_{j+1}^{(i)} = b_j^{(i)} \mathbf{u}_j^{(i)}, \quad \forall j \in \{1, \ldots, s_i\}, \tag{4.7}$$

and we can define the two-dimensional complete fan

$$\widetilde{\Delta} := \left\{ \begin{array}{c} \text{the cones } \{\sigma_i \mid i \in J_\Delta\} \text{ and} \\ \left\{ \pi_j^{(i)} := \mathbb{R}_{\geq 0}\, \mathbf{u}_j^{(i)} + \mathbb{R}_{\geq 0}\, \mathbf{u}_{j+1}^{(i)} \;\middle|\; i \in I_\Delta,\ j \in \{0, 1, \ldots, s_i\} \right\}, \\ \text{together with their faces} \end{array} \right\}.$$

By construction,

$$f = \text{ id}_* : X_{\widetilde{\Delta}} \longrightarrow X_\Delta \tag{4.8}$$

is the *(good) minimal desingularization* of X_Δ (as we just patch together the (good) minimal desingularizations of U_{σ_i}'s established by Theorem 3.20). Defining

$$\begin{cases} E_j^{(i)} := \mathbf{V}_{\widetilde{\Delta}}(\mathbb{R}_{\geq 0}\, \mathbf{u}_j^{(i)}), & \forall i \in I_\Delta \text{ and } \forall j \in \{1, 2, \ldots, s_i\}, \\ \overline{C}_i := \mathbf{V}_{\widetilde{\Delta}}(\mathbb{R}_{\geq 0}\, \mathbf{n}_i), & \forall i \in \{1, 2, \ldots, \nu\}, \end{cases}$$

we observe that $\overline{C}_i$ is the strict transform of C_i w.r.t. f,

$$E^{(i)} := \sum_{j=1}^{s_i} E_j^{(i)}$$

the exceptional divisor replacing the singular point $\text{orb}(\sigma_i)$ via f, and

$$\text{Div}_{\text{W}}^{\mathbb{T}}(X_{\widetilde{\Delta}}) \otimes_{\mathbb{Z}} \mathbb{Q} = \text{Div}_{\text{C}}^{\mathbb{T}}(X_{\widetilde{\Delta}}) \otimes_{\mathbb{Z}} \mathbb{Q} = \left(\bigoplus_{i=1}^{\nu} \mathbb{Q}\, \overline{C}_i \right) \oplus \left(\bigoplus_{i \in I_\Delta} \bigoplus_{j=1}^{s_i} \mathbb{Q}\, E_j^{(i)} \right).$$

Moreover, the topological Euler characteristic of $X_{\widetilde{\Delta}}$ equals $e(X_\Delta) = \nu + \sum_{i \in I_\Delta} s_i$, and the discrepancy divisor w.r.t. f is

$$K_{X_{\widetilde{\Delta}}} - f^* K_{X_\Delta} = \sum_{i \in I_\Delta} K(E^{(i)}) \tag{4.9}$$

with each of the $K(E^{(i)})$'s a $\mathbb{Q}$-Cartier divisor supported in $\bigcup_{j=1}^{s_i} E_j^{(i)}$ and having coefficients which will be described precisely in Proposition 4.4.

Definition 4.1 (The additional characteristic numbers r_i). For every index $i \in \{1, 2, \ldots, \nu\}$ we introduce integers r_i *uniquely determined* by the conditions:

$$r_i \mathbf{n}_i = \begin{cases} \mathbf{u}^{(i-1)}_{s_{i-1}} + \mathbf{u}^{(i)}_1, & \text{if } i \in I'_\Delta, \\ \mathbf{n}_{i-1} + \mathbf{u}^{(i)}_1, & \text{if } i \in I''_\Delta, \\ \mathbf{u}^{(i-1)}_{s_{i-1}} + \mathbf{n}_{i+1}, & \text{if } i \in J'_\Delta, \\ \mathbf{n}_{i-1} + \mathbf{n}_{i+1}, & \text{if } i \in J''_\Delta, \end{cases} \tag{4.10}$$

where

$$I'_\Delta := \{ i \in I_\Delta \mid q_{i-1} > 1 \}, \quad I''_\Delta := \{ i \in I_\Delta \mid q_{i-1} = 1 \},$$

and

$$J'_\Delta := \{ i \in J_\Delta \mid q_{i-1} > 1 \}, \quad J''_\Delta := \{ i \in J_\Delta \mid q_{i-1} = 1 \},$$

with I_Δ, J_Δ as in (4.1). As we shall see below in Lemma 4.3, the integer $-r_i$ is nothing but the self-intersection number of $\overline{C}_i$ on $X_{\widetilde{\Delta}}$.

Definition 4.2. The integers ν,

$$p_i, \quad \widehat{p}_i, \quad q_i, \quad r_i, \tag{4.11}$$

for all $i \in \{1, 2, \ldots, \nu\}$, together with the sets of integers

$$s_i, \left\{ b^{(i)}_j \;\middle|\; 1 \le j \le s_i \right\}, \left\{ \gamma^{(i)}_j \;\middle|\; 0 \le j \le s_i + 1 \right\}, \left\{ \delta^{(i)}_j \;\middle|\; 0 \le j \le s_i + 1 \right\}, \tag{4.12}$$

for all $i \in I_\Delta$, which were introduced above, will be called *the combinatorial data* of the surface X_Δ. These data describe completely its algebro-geometric and topological properties. In particular, if X_Δ is nonsingular, (4.12) are not present, $p_i = \widehat{p}_i = 0$, $q_i = 1$, $\forall i \in \{1, \ldots, \nu\}$, and therefore the only nontrivial data are the integers $r_i, i \in \{1, \ldots, \nu\}$.

Lemma 4.3. *The intersection numbers of any pair of generators of the group* $\mathrm{Div}^{\mathbb{T}}_{\mathrm{C}}(X_{\widetilde{\Delta}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ *are the following*:

$$E^{(k)}_j \cdot E^{(i)}_{j'} = \begin{cases} 1, & \text{if } k = i \text{ and } j - j' = \pm 1, \\ -b^{(i)}_j, & \text{if } k = i \text{ and } j = j', \\ 0, & \text{otherwise}, \end{cases}$$

for all $k, i \in I_\Delta$ and all $j \in \{1, \ldots, s_k\}$, $j' \in \{1, \ldots, s_i\}$.

$$E^{(k)}_j \cdot \overline{C}_i = \begin{cases} 1, & \text{if } j = 1 \text{ and } k = i, \\ 1, & \text{if } j = s_{i-1} \text{ and } k = i - 1, \\ 0, & \text{otherwise}, \end{cases}$$

for all $k \in I_\Delta$ and all $j \in \{1, \ldots, s_k\}$, $i \in \{1, \ldots, \nu\}$.

$$\overline{C}_i \cdot \overline{C}_{i'} = \begin{cases} -r_i, & \text{if } i = i', \\ 1, & \text{if } \begin{cases} \text{either } i' = i + 1 \text{ and } i \in J_\Delta, \\ \text{or } i' = i - 1 \text{ and } i - 1 \in J_\Delta, \end{cases} \\ 0, & \text{otherwise}, \end{cases}$$

for all $i, i' \in \{1, \ldots, \nu\}$.

Proof. It is easy to show that the two prime divisors defined by the closures of the orbits of the rays of $\pi_j^{(i)} = \mathbb{R}_{\geq 0}\,\mathbf{u}_j^{(i)} + \mathbb{R}_{\geq 0}\,\mathbf{u}_{j+1}^{(i)}$ (resp., of $\sigma_i = \tau_i + \tau_{i+1}$, $i \in J_\Delta$) intersect transversely at one point, namely at $\mathrm{orb}(\pi_j^{(i)})$ (resp., at $\mathrm{orb}(\sigma_i)$) with multiplicity 1, and therefore their intersection number equals 1. The remaining pairs of distinct generators of $\mathrm{Div}_{\mathbb{C}}^{\mathbb{T}}(X_{\widetilde{\Delta}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ have intersection number 0 because they arise from rays of $\widetilde{\Delta}$ which are not adjacent. Next, let us determine $(E_j^{(i)})^2$. Setting $z_j^{(i)} := \mathbf{e}(\mathbf{u}_j^{(i)})$, we get

$$\begin{cases} U_{\pi_{j-1}^{(i)}} = \ \mathrm{Spec}\left(\mathbb{C}[z_{j-1}^{(i)}, z_j^{(i)}]\right), \\ U_{\pi_j^{(i)}} = \ \mathrm{Spec}\left(\mathbb{C}[z_j^{(i)}, z_{j+1}^{(i)}]\right) = \mathrm{Spec}\left(\mathbb{C}[(z_{j-1}^{(i)})^{-1}, (z_{j-1}^{(i)})^{b_j^{(i)}}\, z_j^{(i)}]\right), \\ U_{\pi_{j-1}^{(i)} \cap \pi_j^{(i)}} = \ \mathrm{Spec}\left(\mathbb{C}[(z_{j-1}^{(i)})^{\pm 1}, (z_{j-1}^{(i)})^{b_j^{(i)}}\, z_j^{(i)}]\right), \end{cases}$$

(by (4.7)), with

$$\begin{cases} E_j^{(i)} \cap U_{\pi_{j-1}^{(i)}} = \mathrm{Spec}\left(\mathbb{C}[z_{j-1}^{(i)}]\right), \\ E_j^{(i)} \cap U_{\pi_j^{(i)}} = \mathrm{Spec}\left(\mathbb{C}[(z_{j-1}^{(i)})^{-1}]\right), \\ E_j^{(i)} \cap U_{\pi_{j-1}^{(i)} \cap \pi_j^{(i)}} = \ \mathrm{Spec}\left(\mathbb{C}[(z_{j-1}^{(i)})^{\pm 1}]\right), \end{cases}$$

and the conormal sheaf $\mathcal{I}_{E_j^{(i)}}/\mathcal{I}^2_{E_j^{(i)}} = \mathcal{O}_{X_{\widetilde{\Delta}}}(-E_j^{(i)})$ on $X_{\widetilde{\Delta}}$, viewed as a sheaf of $\mathcal{O}_{E_j^{(i)}}$-modules, is invertible on $E_j^{(i)}$ (where $\mathcal{I}_{E_j^{(i)}}$ denotes the ideal sheaf of $E_j^{(i)}$ in $X_{\widetilde{\Delta}}$). The line bundle on $E_j^{(i)}$ corresponding to $\mathcal{I}_{E_j^{(i)}}/\mathcal{I}^2_{E_j^{(i)}}$ is constructed by the identification

$$\begin{array}{ccccc} (E_j^{(i)} \cap U_{\pi_{j-1}^{(i)}}) \times \mathbb{C} & & & & (E_j^{(i)} \cap U_{\pi_j^{(i)}}) \times \mathbb{C} \\ \cup & & & & \cup \\ (E_j^{(i)} \cap U_{\pi_{j-1}^{(i)} \cap \pi_j^{(i)}}) \times \mathbb{C} & \ni (z_{j-1}^{(i)}, \lambda) & \leftrightsquigarrow & (z_j^{(i)}, (z_{j-1}^{(i)})^{b_j^{(i)}} \lambda) \ \in & (E_j^{(i)} \cap U_{\pi_{j-1}^{(i)} \cap \pi_j^{(i)}}) \times \mathbb{C}, \end{array}$$

and has $z_j^{(i)} \longmapsto (z_{j-1}^{(i)})^{b_j^{(i)}}$ as its transition function. But the same line bundle corresponds also to the Cartier divisor $b_j^{(i)}\{0\}$ on $E_j^{(i)}$, where $0 \in E_j^{(i)} \cap U_{\pi_{j-1}^{(i)}} \cong \mathbb{C}$ denotes the origin. Hence, $\mathcal{O}_{E_j^{(i)}}(b_j^{(i)}\{0\}) \cong \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(b_j^{(i)})$, and

$$\mathcal{N}_{X_{\widetilde{\Delta}}/E_j^{(i)}} \cong \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-b_j^{(i)}) \Longrightarrow (E_j^{(i)})^2 = \deg_{E_j^{(i)}}(\mathcal{N}_{X_{\widetilde{\Delta}}/E_j^{(i)}}) = -b_j^{(i)} \quad \text{(by (1.1))}.$$

The proof of the equality $\overline{C}_i^2 = -r_i$ is similar (and uses (4.10) instead of (4.7)). $\square$

Proposition 4.4. *The $\mathbb{Q}$-Cartier divisor $K(E^{(i)})$, for an $i \in I_\Delta$, is expressed as rational linear combination of the exceptional rational curves $E_j^{(i)}$, $j = 1, \ldots, s_i$, as follows*:

$$\boxed{K(E^{(i)}) = \sum_{j=1}^{s_i} \left(\frac{\gamma_j^{(i)} + \delta_j^{(i)}}{q_i} - 1 \right) E_j^{(i)}} \tag{4.13}$$

PROOF. For $i \in I_\Delta$, let the local canonical divisor at $\mathrm{orb}(\sigma_i)$ be

$$K(E^{(i)}) = \sum_{j=1}^{s_i} \xi_j^{(i)} E_j^{(i)}.$$

In order to find the rational coefficients $\xi_j^{(i)}$ we have to solve the linear system

$$K(E^{(i)}) \cdot E_{j'}^{(i)} = K_{X_{\widetilde{\Delta}}} \cdot E_{j'}^{(i)}, \quad j' \in \{1, \ldots, s_i\},$$

i.e., the system

$$\begin{aligned}\left(\sum_{j=1}^{s_i} \xi_j^{(i)} E_j^{(i)}\right) \cdot E_{j'}^{(i)} &= \left(-\sum_{i=1}^{\nu} \overline{C}_i - \sum_{k \in I_\Delta} \sum_{j=1}^{s_k} E_j^{(k)}\right) \cdot E_{j'}^{(i)} \\ &= \left(-\overline{C}_i - \overline{C}_{i+1} - \sum_{j=1}^{s_i} E_j^{(i)}\right) \cdot E_{j'}^{(i)}, \quad j' \in \{1, \ldots, s_i\}.\end{aligned}$$

By Lemma 4.3, this is equivalent to the following:

$$\left(\mathbf{L}_{s_i}(b_1^{(i)}, \ldots, b_{s_i}^{(i)})\right) (\xi_1^{(i)}, \ldots, \xi_{s_i}^{(i)})^T = (2 - b_1^{(i)}, \ldots, 2 - b_{s_i}^{(i)})^T.$$

Using Lemma 3.13 and formulae (4.3), (4.5), we compute

$$\xi_j^{(i)} = \tfrac{1}{q_i} \left(\sum_{1 \le k < j} \gamma_j^{(i)} \delta_k^{(i)} (2 - b_k^{(i)}) + \gamma_j^{(i)} \delta_j^{(i)} (2 - b_j^{(i)}) + \sum_{1 \le j < k \le s_i} \gamma_k^{(i)} \delta_j^{(i)} (2 - b_k^{(i)}) \right)$$

$$= \tfrac{1}{q_i} \left(\gamma_j^{(i)} \sum_{1 \le k < j} (2\delta_k^{(i)} - b_k^{(i)} \delta_k^{(i)}) + \gamma_j^{(i)} \delta_j^{(i)} (2 - b_j^{(i)}) + \delta_j^{(i)} \sum_{1 \le j < k \le s_i} (2\gamma_k^{(i)} - b_k^{(i)} \gamma_k^{(i)}) \right)$$

$$= \frac{1}{q_i} \left(\begin{array}{c} \gamma_j^{(i)} \sum_{1 \le k < j} \left(2\delta_k^{(i)} - \delta_{k-1}^{(i)} - \delta_{k+1}^{(i)}\right) + \gamma_j^{(i)} \delta_j^{(i)} \left(2 - b_j^{(i)}\right) \\ + \delta_j^{(i)} \sum_{1 \le j < k \le s_i} \left(2\gamma_k^{(i)} - \gamma_{k-1}^{(i)} - \gamma_{k+1}^{(i)}\right) \end{array} \right)$$

$$= \frac{1}{q_i} \left(\begin{array}{c} \gamma_j^{(i)} \left(\delta_1^{(i)} - \delta_0^{(i)} + \delta_{j-1}^{(i)} - \delta_j^{(i)}\right) + \gamma_j^{(i)} \delta_j^{(i)} \left(2 - b_j^{(i)}\right) \\ + \delta_j^{(i)} \left(\gamma_{j+1}^{(i)} - \gamma_j^{(i)} + \gamma_{s_i}^{(i)} - \gamma_{s_i+1}^{(i)}\right) \end{array} \right)$$

$$= \frac{1}{q_i} \left(\gamma_j^{(i)} + \delta_j^{(i)} - q_i \right).$$

Thus, (4.13) is true. □

NOTE 4.5. (a) In the literature related to cyclic quotient singularities, a formula equivalent to

$$q_i K(E^{(i)}) = \sum_{j=1}^{s_i} (\gamma_j^{(i)} + \delta_j^{(i)} - q_i) E_j^{(i)},$$

was first mentioned in Knöller's article [**31**, §3.1, p. 211]. This alternative proof is based on the existence of a *unique* effective Cartier divisor $Z_i \in \mathrm{Div}_{\mathrm{C}}(f^{-1}(U_i))$ supported in $\bigcup_{j=1}^{s_i} E_j^{(i)}$, such that $Z_i \cdot E_j^{(i)} = \kappa_j^{(i)} q_i$, for all $j \in \{1, \ldots, s_i\}$, where $\kappa_1^{(i)}, \ldots, \kappa_{s_i}^{(i)}$ are given non-positive integers (see [**31**, Lemma of p. 207]).

(b) Obviously, for $i \in I_\Delta$, the local index $l = \mathrm{lind}(X_\Delta, \mathrm{orb}\,(\sigma_i))$ introduced in 3.19 is the smallest positive integer for which $-l\,K(E^{(i)})$ is a Cartier divisor on $X_{\widetilde{\Delta}}$.

COROLLARY 4.6. *The self-intersection number of $K(E^{(i)})$ equals*

$$\boxed{K(E^{(i)})^2 = -\left(\tfrac{q_i-p_i+1}{q_i} + \tfrac{q_i-\widehat{p}_i+1}{q_i}\right) + 2 + \sum_{j=1}^{s_i}(2-b_j^{(i)})} \tag{4.14}$$

PROOF. By (4.13) we have

$$\begin{aligned} K(E^{(i)})^2 &= K(E^{(i)}) \cdot K(E^{(i)}) \\ &= K(E^{(i)}) \cdot \left(\sum_{j=1}^{s_i} \left(\tfrac{\gamma_j^{(i)}+\delta_j^{(i)}}{q_i} - 1 \right) E_j^{(i)} \right) \\ &= \sum_{j=1}^{s_i} \left(\tfrac{\gamma_j^{(i)}+\delta_j^{(i)}}{q_i} - 1 \right) K(E^{(i)}) \cdot E_j^{(i)}. \end{aligned}$$

Since each of $E_j^{(i)}$'s is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$, adjunction formula (cf. [**3**, p. 85] or [**23**, Ch. V, Proposition 1.5, p. 361]) and Lemma 4.3 give

$$K(E^{(i)}) \cdot E_j^{(i)} = -2 - (E_j^{(i)})^2 = b_j^{(i)} - 2.$$

Hence,

$$\begin{aligned} K(E^{(i)})^2 &= \sum_{j=1}^{s_i} \left(\tfrac{\gamma_j^{(i)}+\delta_j^{(i)}}{q_i} - 1 \right) (b_j^{(i)} - 2) \\ &= \sum_{j=1}^{s_i} \left(\tfrac{\gamma_j^{(i)}+\delta_j^{(i)}}{q_i} \right) (b_j^{(i)} - 2) + \sum_{j=1}^{s_i}(2-b_j^{(i)}) \\ &= \sum_{j=1}^{s_i} \left(\tfrac{\gamma_j^{(i)}+\delta_j^{(i)}}{q_i} \right) b_j^{(i)} - 2\sum_{j=1}^{s_i} \left(\tfrac{\gamma_j^{(i)}+\delta_j^{(i)}}{q_i} \right) + \sum_{j=1}^{s_i}(2-b_j^{(i)}). \end{aligned}$$

Now taking into account (4.5), we have

$$(\gamma_j^{(i)} + \delta_j^{(i)}) b_j^{(i)} = \gamma_{j-1}^{(i)} + \gamma_{j+1}^{(i)} + \delta_{j-1}^{(i)} + \delta_{j+1}^{(i)},$$

which means that

$$K(E^{(i)})^2 = \tfrac{1}{q_i}((\gamma_0^{(i)} + \gamma_{s_i+1}^{(i)} + \delta_0^{(i)} + \delta_{s_i+1}^{(i)}) - (\gamma_1^{(i)} + \gamma_{s_i}^{(i)} + \delta_1^{(i)} + \delta_{s_i}^{(i)})) + \sum_{j=1}^{s_i}(2-b_j^{(i)}).$$

After simple evaluation of the “extreme” gammas and deltas by (4.3), (4.4), and (4.6), we obtain formula (4.14). □

Lemma 4.7. *The (fractional) intersection numbers of any pair* $C_i, C_{i'}$ *of generators of* $\mathrm{Div}_{\mathrm{W}}^{\mathbb{T}}(X_\Delta)$ *(with* $i, i' \in \{1, \ldots, \nu\}$*) are the following*:

$$C_i \cdot C_{i'} = \begin{cases} \frac{1}{q_i}, & \text{if } i' = i+1, \\ \frac{1}{q_{i-1}}, & \text{if } i' = i-1, \\ -r_i + \frac{q_{i-1} - \widehat{p}_{i-1}}{q_{i-1}} + \frac{q_i - p_i}{q_i}, & \text{if } i' = i \text{ and } i \in I'_\Delta, \\ -r_i + \frac{q_i - p_i}{q_i}, & \text{if } i' = i \text{ and } i \in I''_\Delta, \\ -r_i + \frac{q_{i-1} - \widehat{p}_{i-1}}{q_{i-1}}, & \text{if } i' = i \text{ and } i \in J'_\Delta, \\ -r_i, & \text{if } i' = i \text{ and } i \in J''_\Delta, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. If $i - i' \notin \{0, \pm 1\}$, then the intersection of the supports of C_i and $C_{i'}$ is empty, and therefore $C_i \cdot C_{i'} = 0$. On the other hand, for every $i \in \{1, \ldots, \nu\}$, it is easy (by appropriate use of Lemma 3.13) to verify that

$$f^* C_i = \begin{cases} \overline{C}_i + \frac{1}{q_{i-1}} \sum_{j=1}^{s_{i-1}} \delta_j^{(i-1)} E_j^{(i-1)} + \frac{1}{q_i} \sum_{j=1}^{s_i} \gamma_j^{(i)} E_j^{(i)}, & \text{if } i \in I'_\Delta, \\ \overline{C}_i + \frac{1}{q_i} \sum_{j=1}^{s_i} \gamma_j^{(i)} E_j^{(i)}, & \text{if } i \in I''_\Delta, \\ \overline{C}_i + \frac{1}{q_{i-1}} \sum_{j=1}^{s_{i-1}} \delta_j^{(i-1)} E_j^{(i-1)} & \text{if } i \in J'_\Delta, \\ \overline{C}_i, & \text{if } i \in J''_\Delta. \end{cases}$$

For every $i \in I'_\Delta$, we compute C_i^2 as follows:

$$C_i^2 = \overline{C}_i^2 + 2\overline{C}_i \cdot \left(\frac{1}{q_{i-1}} \delta_{s_{i-1}}^{(i-1)} E_{s_{i-1}}^{(i-1)} + \frac{1}{q_i} \gamma_1^{(i)} E_1^{(i)} \right) + \frac{1}{q_{i-1}^2} \left(\sum_{j=1}^{s_{i-1}} \delta_j^{(i-1)} E_j^{(i-1)} \right)^2 + \frac{1}{q_i^2} \left(\sum_{j=1}^{s_i} \gamma_j^{(i)} E_j^{(i)} \right)^2,$$

where $\overline{C}_i^2 = -r_i$, and

$$2\overline{C}_i \cdot \left(\tfrac{1}{q_{i-1}} \delta_{s_{i-1}}^{(i-1)} E_{s_{i-1}}^{(i-1)} + \tfrac{1}{q_i} \gamma_1^{(i)} E_1^{(i)} \right) = 2 \left(\tfrac{q_{i-1} - \widehat{p}_{i-1}}{q_{i-1}} + \tfrac{q_i - p_i}{q_i} \right),$$

by Lemma 4.3 and (4.4), (4.6). Moreover, by (4.4) and (4.5), we have

$$\begin{aligned} \left(\sum_{j=1}^{s_i} \gamma_j^{(i)} E_j^{(i)} \right)^2 &= -\sum_{j=1}^{s_i} b_j^{(i)} \left(\gamma_j^{(i)} \right)^2 + 2(\gamma_1^{(i)} \gamma_2^{(i)} + \cdots + \gamma_{s_i}^{(i)} \gamma_1^{(i)}) \\ &= -\sum_{j=1}^{s_i} \left(\gamma_{j-1}^{(i)} + \gamma_{j+1}^{(i)} \right) \gamma_j^{(i)} + 2(\gamma_1^{(i)} \gamma_2^{(i)} + \cdots + \gamma_{s_i}^{(i)} \gamma_1^{(i)}) \\ &= -\gamma_0^{(i)} \gamma_1^{(i)} + \gamma_{s_i}^{(i)} \gamma_{s_i+1}^{(i)} = -q_i \left(q_i - p_i \right), \end{aligned}$$

and analogously,

$$\left(\sum_{j=1}^{s_{i-1}} \delta_j^{(i-1)} E_j^{(i-1)} \right)^2 = -q_{i-1} \left(q_{i-1} - \widehat{p}_{i-1} \right).$$

Consequently,

$$C_i^2 = -r_i + \frac{q_{i-1} - \widehat{p}_{i-1}}{q_{i-1}} + \frac{q_i - p_i}{q_i}.$$

The computation of the self-intersection number C_i^2 in the other cases is similar. Next, let us determine $C_i \cdot C_{i+1}$. By Lemma 4.3, $C_i \cdot C_{i+1} = 1$, whenever $i \in J_\Delta$, and

$$C_i \cdot C_{i+1} = \left(\overline{C}_i + \frac{1}{q_i}\sum_{j=1}^{s_i}\gamma_j^{(i)}E_j^{(i)}\right)\left(\overline{C}_{i+1} + \frac{1}{q_i}\sum_{j=1}^{s_i}\delta_j^{(i)}E_j^{(i)}\right)$$

$$= \overline{C}_i \cdot \left(\frac{1}{q_i}\delta_1^{(i)}E_1^{(i)}\right) + \overline{C}_{i+1} \cdot \left(\frac{1}{q_i}\gamma_s^{(i)}E_s^{(i)}\right) + \frac{1}{q_i^2}\left(\sum_{j=1}^{s_i}\gamma_j^{(i)}E_j^{(i)}\right) \cdot \left(\sum_{j=1}^{s_i}\delta_j^{(i)}E_j^{(i)}\right)$$

$$= \frac{1}{q_i} + \frac{1}{q_i}$$

$$+ \frac{1}{q_i^2}\left(\gamma_1^{(i)}(\delta_2^{(i)} - b_1^{(i)}\delta_1^{(i)}) + \sum_{j=2}^{s_i-1}\gamma_j^{(i)}(\delta_{j-1}^{(i)} - b_j^{(i)}\delta_j^{(i)} + \delta_{j+1}^{(i)}) + \gamma_s^{(i)}(\delta_{s-1}^{(i)} - b_s^{(i)}\delta_s^{(i)})\right)$$

whenever $i \in I_\Delta$. In the second case, since $\delta_{j-1}^{(i)} - b_j^{(i)}\delta_j^{(i)} + \delta_{j+1}^{(i)} = 0$ for all indices $j \in \{2, \ldots, s_i - 1\}$ (by (4.5)), and

$$\delta_2^{(i)} - b_1^{(i)}\delta_1^{(i)} = \delta_2^{(i)} - \delta_0^{(i)} - \delta_2^{(i)} = 0, \quad \gamma_s^{(i)}(\delta_{s-1}^{(i)} - b_s^{(i)}\delta_s^{(i)}) = -\gamma_s^{(i)}\delta_{s+1}^{(i)} = -q_i,$$

we get $C_i \cdot C_{i+1} = \frac{1}{q_i}$. □

Proposition 4.8 (Direct computation of $K_{X_\Delta}^2$). *The self-intersection number of the canonical divisor of X_Δ equals*

$$\boxed{K_{X_\Delta}^2 = \sum_{i=1}^{\nu}\left(\tfrac{2}{q_i} - r_i\right) + \sum_{i \in I_\Delta}\left(\tfrac{q_i - p_i}{q_i} + \tfrac{q_i - \widehat{p}_i}{q_i}\right)} \tag{4.15}$$

Proof. Since

$$K_{X_\Delta}^2 = (-\sum_{i=1}^{\nu}C_i)^2 = \sum_{i=1}^{\nu}C_i^2 + 2\sum_{1 \le i < j \le \nu}C_i \cdot C_j,$$

using Lemma 4.7 we get

$$K_{X_\Delta}^2 = \sum_{i=1}^{\nu}(\tfrac{2}{q_i} - r_i) + \left(\sum_{i \in I'_\Delta}(\tfrac{q_{i-1} - \widehat{p}_{i-1}}{q_{i-1}} + \tfrac{q_i - p_i}{q_i}) + \sum_{i \in I''_\Delta}\tfrac{q_i - p_i}{q_i} + \sum_{i \in J'_\Delta}\tfrac{q_{i-1} - \widehat{p}_{i-1}}{q_{i-1}}\right).$$

Note that the second summand (in the big parenthesis) equals the sum of the two rational numbers $\frac{q_i - p_i}{q_i}$ and $\frac{q_i - \widehat{p}_i}{q_i}$ over *all* $i \in I_\Delta$, because each of them is counted *once* for every singular point of X_Δ, and therefore $K_{X_\Delta}^2$ can be written in the form (4.15). □

• **Computing $K_{X_\Delta}^2$ via Noether's formula**. On $X_{\widetilde{\Delta}}$ the *usual Noether's formula* for rational nonsingular compact complex surfaces gives

$$\tfrac{1}{12}(K_{X_{\widetilde{\Delta}}}^2 + e(X_{\widetilde{\Delta}})) = \chi(\mathcal{O}_{X_{\widetilde{\Delta}}}) = 1,$$

i.e.,

$$K_{X_{\widetilde{\Delta}}}^2 = 12 - e(X_{\widetilde{\Delta}}) = 12 - \nu - \sum_{i \in I_\Delta}s_i.$$

This equality, combined with (4.9) and (4.14), leads to *generalized Noether's formulae* which are valid for the (not necessarily nonsingular) toric surface X_Δ.

Proposition 4.9 (First version of Noether's formula). *The self-intersection number of the canonical divisor of* X_Δ *equals*

$$\boxed{K_{X_\Delta}^2 = 12 - \nu + \sum_{i \in I_\Delta} \left(\tfrac{q_i - p_i + 1}{q_i} + \tfrac{q_i - \widehat{p}_i + 1}{q_i} - 2 + \sum_{j=1}^{s_i} (b_j^{(i)} - 3) \right)} \tag{4.16}$$

Proof. We have

$$\begin{aligned} K_{X_\Delta}^2 &= K_{X_{\widetilde{\Delta}}}^2 - \left(\sum_{i \in I_\Delta} K(E^{(i)}) \cdot \sum_{i \in I_\Delta} K(E^{(i)}) \right) \\ &= 12 - \nu - \sum_{i \in I_\Delta} s_i - \left(\sum_{i \in I_\Delta} K(E^{(i)}) \cdot \sum_{i \in I_\Delta} K(E^{(i)}) \right) \\ &= 12 - \nu - \sum_{i \in I_\Delta} s_i - \sum_{i \in I_\Delta} \left(K(E^{(i)}) \cdot K(E^{(i)}) \right), \end{aligned}$$

where the latter equality follows from the fact that the intersection of the supports of the divisors $K(E^{(i_1)})$ and $K(E^{(i_2)})$, for every pair $i_1, i_2 \in I_\Delta$ with $i_1 \neq i_2$, is empty. By (4.14),

$$\begin{aligned} K_{X_\Delta}^2 &= 12 - \nu - \sum_{i \in I_\Delta} s_i - \sum_{i \in I_\Delta} K(E^{(i)})^2 \\ &= 12 - \nu - \sum_{i \in I_\Delta} s_i + \sum_{i \in I_\Delta} \left(\tfrac{q_i - p_i + 1}{q_i} + \tfrac{q_i - \widehat{p}_i + 1}{q_i} - 2 + \sum_{j=1}^{s_i} (b_j^{(i)} - 2) \right) \end{aligned}$$

which can be written in the form (4.16). □

Corollary 4.10 (Second version of Noether's formula). *The self-intersection number of the canonical divisor of* X_Δ *equals*

$$\boxed{K_{X_\Delta}^2 = 12 - \nu + 2 \sum_{i \in I_\Delta} \left(\frac{1}{q_i} - 6 \ \mathrm{DS}\,(p_i, q_i) - 1 \right)} \tag{4.17}$$

(*and therefore can be calculated by means of sawtooth functions or, alternatively, by cotangent functions, cf.* (3.1), (3.4) *and* (3.5)).

Proof. For every $i \in I_\Delta$ it sufficies to express $\sum_{j=1}^{s_i} (b_j^{(i)} - 3)$ within (4.16) in terms of the Dedekind sum $\mathrm{DS}(p_i, q_i)$ by utilizing formula (3.6). □

Remark 4.11. (a) Since

$$\sum_{i=1}^{\nu} \left(\tfrac{2}{q_i} - r_i \right) = - \sum_{i=1}^{\nu} r_i + \sum_{i \in I_\Delta} \tfrac{2}{q_i} + 2\,(\nu - \sharp(I_\Delta))$$

the equalities (4.15), (4.16) and (4.17) give

$$\sum_{i=1}^{\nu} r_i = 3\nu - 12 - \sum_{i\in I_\Delta}\left(\sum_{j=1}^{s_i}(b_j^{(i)}-3)\right) \tag{4.18}$$

$$= 3\nu - 12 - \sum_{i\in I_\Delta}\left(\tfrac{p_i+\widehat{p}_i}{q_i} - 12\ \mathrm{DS}\,(p_i,q_i) - 2\right)$$

Formula (4.18) generalizes the well-known formula for *nonsingular* X_Δ's (see Oda [**42**, formula (1), p. 45] or Fulton [**18**, equation $(**)$, p. 44]).

(b) If X_Δ is Gorenstein (see Remark 3.18), then

$$K_{X_\Delta}^2 = 12 - \nu - \sum_{i\in I_\Delta} s_i \quad \text{and} \quad \sum_{i=1}^{\nu} r_i = 3\nu - 12 + \sum_{i\in I_\Delta} s_i.$$

5. Classification of compact toric surfaces

Compact toric surfaces are to be classified *up to isomorphism* by means of specially designed "weighted" plane graphs which have the combinatorial data (4.11) as their weights. Our presentation uses a generalization of the $\mathbb{Z}$-weighted circular graphs introduced by Oda in [**41**, Ch. I, § 8, pp. 50-58], [**42**, pp. 42-46], for the study of *nonsingular* compact toric surfaces, and related results of Koelman [**32**, § 1.2].

Lemma 5.1. *Let N be a free $\mathbb{Z}$-module of rank 2, and Δ, Δ' two 2-dimensional fans with $|\Delta| = |\Delta'| = N_\mathbb{R}$. Assume that*

$$\begin{cases} \Delta(1) = \{\tau_1,\ldots,\tau_\nu\}, & \Delta(2) = \{\sigma_1,\ldots,\sigma_\nu\},\ \nu \geq 3, \\ \Delta'(1) = \{\tau'_1,\ldots,\tau'_{\nu'}\}, & \Delta'(2) = \{\sigma'_1,\ldots,\sigma'_{\nu'}\},\ \nu' \geq 3, \end{cases}$$

with $\sigma_i := \tau_i + \tau_{i+1}$, $\sigma'_i := \tau'_i + \tau'_{i+1}$, and τ_i's (resp., τ'_i's) going anticlockwise around the origin once (w.r.t. the given enumeration of the indices), and that

$$p_i,\quad \widehat{p}_i,\quad q_i,\quad r_i;\quad p'_i,\quad \widehat{p}'_i,\quad q'_i,\quad r'_i,$$

are the combinatorial data (4.11) of X_Δ and $X_{\Delta'}$, respectively. If we denote by $\mathrm{GL}(N,\mathbb{Z})$ *the automorphism group of N, and if we define*

$$\mathrm{GL}_+(N,\mathbb{Z}) := \{\varpi \in \mathrm{GL}(N,\mathbb{Z}) \mid \det(\varpi) = 1\},$$
$$\mathrm{GL}_-(N,\mathbb{Z}) := \{\varpi \in \mathrm{GL}(N,\mathbb{Z}) \mid \det(\varpi) = -1\},$$

then the following conditions are equivalent:

(i) *There exists a $\varpi \in \mathrm{GL}_+(N,\mathbb{Z})$ (resp., a $\varpi \in \mathrm{GL}_-(N,\mathbb{Z})$) such that $\varpi_\mathbb{R}(\Delta) = \Delta'$.*

(ii) *We have $\nu = \nu'$, and there exist ordering preserving permutations $\vartheta, \vartheta' \in \mathfrak{S}_\nu$ (i.e., $i_1 < i_2 \Rightarrow \vartheta(i_1) < \vartheta(i_2)$ and $\vartheta'(i_1) < \vartheta'(i_2)$, w.r.t. the usual cyclic ordering "$<$"), such that for all $i \in \{1,\ldots,\nu\}$ the following equalities hold true*:

$$p_{\vartheta(i)} = p'_{\vartheta'(i)},\quad q_{\vartheta(i)} = q'_{\vartheta'(i)},\quad r_{\vartheta(i)} = r'_{\vartheta'(i)}$$
$$(\text{resp., } p_{\vartheta(i)} = \widehat{p}'_{[\nu-\vartheta'(i)+1]_\nu},\ q_{\vartheta(i)} = q'_{[\nu-\vartheta'(i)+1]_\nu},\ r_{\vartheta(i)} = r'_{[\nu-\vartheta'(i)+2]_\nu}, (r'_0 := r'_\nu)).$$

Proof. See Proposition 3.5 and Koelman [**32**, Lemma 1.2.27, pp. 16-17]. □

DEFINITION 5.2. Let $\mathfrak{G}$ be a plane graph (i.e., a drawing of a planar graph in the plane with no crossings) and $\mathrm{Vert}(\mathfrak{G})$, $\mathrm{Edg}(\mathfrak{G})$, the set of its vertices and the set of its edges, respectively. $\mathfrak{G}$ is called *circular graph* if its vertices are points on a circle and its edges are the corresponding arcs (on this circle, each of which connects two consecutive vertices). We say that a circular graph $\mathfrak{G}$ is $\mathbb{Z}$-*weighted at its vertices* and *double* $\mathbb{Z}$-*weighted at its edges* (and call it WVE^{2}C-*graph*, for short) if it is accompanied by two maps $\mathrm{Vert}(\mathfrak{G}) \overset{\mathfrak{w}}{\longmapsto} \mathbb{Z}$, $\mathrm{Edg}(\mathfrak{G}) \overset{\mathfrak{w}'}{\longmapsto} \mathbb{Z}^2$, assigning to each vertex an integer and to each edge a pair of integers, respectively.

DEFINITION 5.3. We say that two WVE^{2}C-graphs $\mathfrak{G}_1$ and $\mathfrak{G}_2$ having

$$\mathrm{Vert}(\mathfrak{G}_k) \overset{\mathfrak{w}_k}{\longmapsto} \mathbb{Z}, \quad \mathrm{Edg}(\mathfrak{G}_k) \overset{\mathfrak{w}'_k}{\longmapsto} \mathbb{Z}^2, \quad k = 1, 2,$$

as weighting maps are *isomorphic* (and we use the notation $\mathfrak{G}_1 \underset{\text{gr.}}{\cong} \mathfrak{G}_2$) if there exists a bijection $\theta : \mathrm{Vert}(\mathfrak{G}_1) \longrightarrow \mathrm{Vert}(\mathfrak{G}_2)$, such that

(i) for each edge $\overline{\mathbf{v}\mathbf{w}}$ of $\mathfrak{G}_1$ with vertices $\mathbf{v}$ and $\mathbf{w}$, $\overline{\theta(\mathbf{v})\theta(\mathbf{w})}$ is an edge of $\mathfrak{G}_2$,

(ii) $\mathfrak{w}_1(\mathbf{v}) = \mathfrak{w}_2(\theta(\mathbf{v}))$, $\forall \mathbf{v} \in \mathrm{Vert}(\mathfrak{G}_1)$, and

(iii) $\mathfrak{w}'_1(\overline{\mathbf{v}\mathbf{w}}) = \mathfrak{w}'_2(\overline{\theta(\mathbf{v})\theta(\mathbf{w})})$.

DEFINITION 5.4. A WVE^{2}C-graph $\mathfrak{G}$ is said to be *anticlockwise* (resp., *clockwise*) *directed* if its reference circle (on which the vertices are located) is viewed as a cycle equipped with the anticlockwise (resp., clockwise) direction.

DEFINITION 5.5. Let N be a free $\mathbb{Z}$-module of rank 2 and Δ a two-dimensional fan with $|\Delta| = N_{\mathbb{R}}$. Using the combinatorial data (4.11) of X_Δ we associate to Δ an anticlockwise directed WVE^{2}C-graph $\mathfrak{G}_\Delta$ with

$$\mathrm{Vert}(\mathfrak{G}_\Delta) = \{\mathbf{v}_1, \ldots, \mathbf{v}_\nu\} \quad \text{and} \quad \mathrm{Edg}(\mathfrak{G}_\Delta) = \{\overline{\mathbf{v}_1\mathbf{v}_2}, \ldots, \overline{\mathbf{v}_\nu\mathbf{v}_1}\},$$

by defining its "weights" as follows:

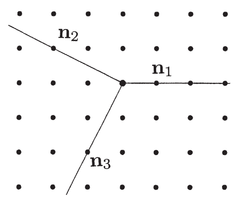
$$\mathrm{Vert}(\mathfrak{G}_\Delta) \ni \mathbf{v}_i \longmapsto -r_i, \quad \mathrm{Edg}(\mathfrak{G}_\Delta) \ni \overline{\mathbf{v}_i\mathbf{v}_{i+1}} \longmapsto (p_i, q_i), \quad \forall i \in \{1, \ldots, \nu\}.$$

The *reverse graph* $\mathfrak{G}_\Delta^{\mathrm{rev}}$ of $\mathfrak{G}_\Delta$ is defined to be the directed WVE^{2}C-graph which is obtained by changing the double weight (p_i, q_i) of the edge $\overline{\mathbf{v}_i\mathbf{v}_{i+1}}$ into $(\widehat{p}_i, q_i)$ and reversing the initial anticlockwise direction of $\mathfrak{G}_\Delta$ into clockwise direction.

NOTE 5.6 (Conventions for the drawings). (a) In the drawing of directed WVE^{2}C-graphs $\mathfrak{G}_\Delta$ in the plane we shall attach only the weight $-r_i$ at the vertex $\mathbf{v}_i$ (without mentioning $\mathbf{v}_i$ itself), for $i \in \{1, \ldots, \nu\}$, and the double weight (p_i, q_i) at the edge $\overline{\mathbf{v}_i\mathbf{v}_{i+1}}$, for $i \in I_\Delta$, and leave edges $\overline{\mathbf{v}_i\mathbf{v}_{i+1}}$, for $i \in J_\Delta$, without any decoration (or, in other words, with the blank space around an edge meaning always the double weight $(0, 1)$), in order to switch to the notation introduced in [**41**, pp. 50-58], [**42**, pp. 42-46] (for the study of nonsingular X_Δ's). Let us furthermore note that the choice of $-r_i$, instead of r_i, as the weight of the vertex $\mathbf{v}_i$, is more natural because it indicates the self-intersection number of the corresponding irreducible rational curve which occurs in the minimal desingularization of X_Δ, and is again adopted from [**41**], [**42**].

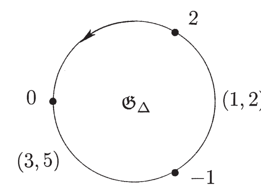
(b) In practice, having definition 5.3 in hand, to decide if two given directed WVE^{2}C-graphs $\mathfrak{G}_1$ and $\mathfrak{G}_2$ (which possess the same mumber of vertices) are isomorphic (or not), we may travel on their reference circles (following the prescribed directions) and find out if there exists a suitable bijection sending the weights of $\mathfrak{G}_1$ to equal weights of $\mathfrak{G}_2$ (or not), without insisting on the use of enumerations of the vertices.

EXAMPLE 5.7. Let $N = \mathbb{Z}^2$ be the standard rectangular lattice within $\mathbb{R}^2$ and Δ the fan with $\mathrm{Gen}(\Delta) = \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ given in Figure 4 (a). Computing the combinatorial data (4.11) of X_Δ, we get the WVE^{2}C-graph $\mathfrak{G}_\Delta$ which is depicted in Figure 4 (b). X_Δ is isomorphic to the weighted projective plane $\mathbb{P}^2_{\mathbb{C}}(5,2,1)$.



(a)

i	$\mathbf{n}_i$	p_i	q_i	r_i
1	$(1,0)$	0	1	-2
2	$(-2,1)$	3	5	0
3	$(-1,-2)$	1	2	1



(b)

FIGURE 4.

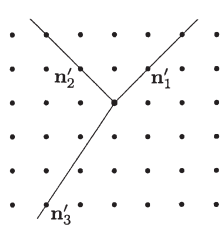
THEOREM 5.8 (Classification Theorem I). *Let N be a free $\mathbb{Z}$-module of rank 2, and Δ, Δ' two 2-dimensional fans with $|\Delta| = |\Delta'| = N_{\mathbb{R}}$. Then the following conditions are equivalent*:

(i) *The compact toric surfaces X_Δ and $X_{\Delta'}$ are isomorphic.*

(ii) *Either $\mathfrak{G}_{\Delta'} \underset{\mathrm{gr.}}{\cong} \mathfrak{G}_\Delta$ or $\mathfrak{G}_{\Delta'} \underset{\mathrm{gr.}}{\cong} \mathfrak{G}_\Delta^{\mathrm{rev}}$.*

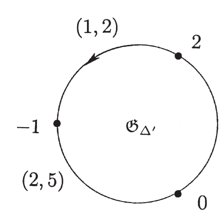
PROOF. Obviously, X_Δ and $X_{\Delta'}$ are isomorphic if and only if there exists an automorphism $\varpi \in \mathrm{GL}(N,\mathbb{Z})$ such that $\varpi_{\mathbb{R}}(\Delta) = \Delta'$. By Lemma 5.1, $\mathfrak{G}_{\Delta'} \underset{\mathrm{gr.}}{\cong} \mathfrak{G}_\Delta$ whenever $\varpi \in \mathrm{GL}_+(N,\mathbb{Z})$, and $\mathfrak{G}_{\Delta'} \underset{\mathrm{gr.}}{\cong} \mathfrak{G}_\Delta^{\mathrm{rev}}$ whenever $\varpi \in \mathrm{GL}_-(N,\mathbb{Z})$. $\square$

EXAMPLE 5.9. If Δ' is the fan with $\mathrm{Gen}(\Delta') = \{\mathbf{n}'_1, \mathbf{n}'_2, \mathbf{n}'_3\}$ given in Figure 5 (a), then we see that the WVE^{2}C-graph $\mathfrak{G}_{\Delta'}$ of $X_{\Delta'}$ (depicted in Figure 5 (b)) is isomorphic to the reverse graph $\mathfrak{G}_\Delta^{\mathrm{rev}}$ of $\mathfrak{G}_\Delta$, where Δ denotes the fan defined in Example 5.7. Obviously, $p'_i = \widehat{p}_{4-i}$, $r'_i = r_{[5-i]_3}$, for $i = 1,2,3$, and $X_{\Delta'}$ is isomorphic to X_Δ.



(a)

i	$\mathbf{n}'_i$	p'_i	q'_i	r'_i
1	$(1,1)$	1	2	-2
2	$(-1,1)$	2	5	1
3	$(-2,-3)$	0	1	0



(b)

FIGURE 5.

THEOREM 5.10 (Classification Theorem II). *Let* $\mathfrak{G}$ *be a* WVE^{2}C*-graph and assume that* $\mathrm{Vert}(\mathfrak{G}) = \{\mathbf{v}_i \,|\, 1 \leq i \leq \nu\}$, $\nu \geq 3$, *having for each* $i \in \{1, \ldots, \nu\}$ *an integer number* $-r_i$ *as the weight of its vertex* $\mathbf{v}_i$, *and* $(p_i, q_i) \in \mathbb{Z}^2$ *with* $0 \leq p_i \leq q_i$, $\gcd(p_i, q_i) = 1$, *as the double weight of the edge* $\overline{\mathbf{v}_i \mathbf{v}_{i+1}}$. *Suppose (without loss of generality) that* $\mathfrak{G}$ *is anticlockwise directed. Then the following are equivalent*:
(i) *There exists a free* $\mathbb{Z}$*-module* N *of rank* 2 *and a two-dimensional fan* Δ, *such that* $|\Delta| = N_{\mathbb{R}}$, *with* X_Δ *having combinatorial data* (4.11), (4.12), *and* $\mathfrak{G}_\Delta \underset{\text{gr.}}{\cong} \mathfrak{G}$.
(ii) *Both conditions* (4.18) *and*

$$\prod_{i=1}^{\nu} (\mathbf{S}(r_i)\ \mathbf{B}_i) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.1}$$

are satisfied, where

$$\mathbf{S}(k) := \begin{pmatrix} k & 1 \\ -1 & 0 \end{pmatrix},\ \forall k \in \mathbb{Z}, \quad \mathbf{B}_i := \begin{cases} \prod_{k=1}^{s_i} \mathbf{S}(b_k^{(i)}), & \text{if } i \in I_\Delta, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } i \in J_\Delta, \end{cases}$$

I_Δ, J_Δ *as defined by* (4.1), *and* $\left\{ b_j^{(i)} \;\middle|\; 1 \leq j \leq s_i \right\}$, $i \in I_\Delta$, *determined by* (4.2).

PROOF. The implication (i)⇒(ii) follows from 4.11 (a), and the equations (4.7) and (4.10). To verify the inverse implication (ii)⇒(i) we may w.l.o.g. work with the standard rectangular lattice $N = \mathbb{Z}^2$ within $\mathbb{R}^2$. Besides that, it is convenient to extend the definition of s_i for all $i \in \{1, \ldots, \nu\}$ by setting $s_i = 0$, $\forall i \in J_\Delta$. Let $\{\mathfrak{e}_1, \mathfrak{e}_2\}$ be the basis of $\mathbb{Z}^2$ consisting of the unit vectors. If we define

$$\begin{cases} \mathbf{x}_0^{(1)} := \mathfrak{e}_1,\ \mathbf{x}_1^{(1)} := \mathfrak{e}_1 + \mathfrak{e}_2, & \text{and} \\ \mathbf{x}_j^{(1)} := b_j^{(1)} \mathbf{x}_{j-1}^{(1)} - \mathbf{x}_{j-2}^{(1)}, & \forall j \in \{2, \ldots, s_1 + 1\} \\ & \text{(i.e., if } 1 \in I_\Delta), \end{cases}$$

and

$$\begin{cases} \mathbf{x}_0^{(2)} := \mathbf{x}_{s_1+1}^{(1)},\ \mathbf{x}_1^{(2)} := r_2 \mathbf{x}_0^{(2)} - \mathbf{x}_{s_1}^{(1)}, & \text{and} \\ \mathbf{x}_j^{(2)} := b_j^{(2)} \mathbf{x}_{j-1}^{(2)} - \mathbf{x}_{j-2}^{(2)}, & \forall j \in \{2, \ldots, s_2 + 1\} \\ & \text{(i.e., if } 2 \in I_\Delta), \end{cases}$$

and continue this procedure (with $\mathbf{x}_j^{(i)}$'s going anticklockwise around the origin) until we arrive to $\mathbf{x}_{s_\nu+1}^{(\nu)}$, then we construct $\nu + \sum_{i \in I_\Delta} s_i$ distinct vectors

$$\left\{ \mathbf{x}_j^{(i)} \;\middle|\; 1 \leq i \leq \nu,\ \ 0 \leq j \leq s_i + 1 \right\} \quad (\text{because } \mathbf{x}_0^{(i)} = \mathbf{x}_{s_{i-1}+1}^{(i-1)}),$$

with $\{\mathbf{x}_j^{(i)}, \mathbf{x}_{j+1}^{(i)}\}$ a $\mathbb{Z}$-basis of $\mathbb{Z}^2$. Condition (5.1) guarantees that $\mathbf{x}_{s_\nu+1}^{(\nu)} = \mathbf{x}_0^{(1)}$ and $\mathbf{x}_1^{(1)} = r_1 \mathbf{x}_0^{(1)} - \mathbf{x}_{s_\nu}^{(\nu)} = \mathfrak{e}_1 + \mathfrak{e}_2$. Furthermore, (4.18) can be written as

$$\sum_{i=1}^{\nu} r_i + \sum_{i \in I_\Delta} (\sum_{j=1}^{s_i} b_j^{(i)}) = 3(\nu + \sum_{i \in I_\Delta} s_i) - 12,$$

and is exactly the condition which ensures that the above vectors $\mathbf{x}_j^{(i)}$ go around the origin *only once*. Thus, we can define a complete fan

$$\Delta_{\text{basic}} := \left\{ \begin{array}{c} \text{the cones } \left\{ \mathbb{R}_{\geq 0}\mathbf{x}_j^{(i)} + \mathbb{R}_{\geq 0}\mathbf{x}_{j+1}^{(i)} \;\middle|\; 1 \leq i \leq \nu, \quad 0 \leq j \leq s_i \right\} \\ \text{together with their faces} \end{array} \right\}$$

consisting of basic cones. Since for $i \in I_\Delta$ the matrix $-\mathbf{L}_{s_i}(b_1^{(i)}, \ldots, b_{s_i}^{(i)})$ is negative definite (see Lemma 3.12), the irreducible curves $\{\mathbf{V}_{\Delta_{\text{basic}}}(\mathbb{R}_{\geq 0}\,\mathbf{x}_j^{(i)}) \,|\, 1 \leq j \leq s_i\}$ can be contracted to a normal point (by Theorem 1.1). Consider the birational morphism $X_{\Delta_{\text{basic}}} \longrightarrow X_\Delta$ contracting all these curves for all $i \in I_\Delta$. By construction, X_Δ has (4.11) and (4.12) as its combinatorial data, $X_{\Delta_{\text{basic}}}$ is isomorphic to $X_{\widetilde{\Delta}}$, and $\mathfrak{G}_\Delta \underset{\text{gr.}}{\cong} \mathfrak{G}$. □

REMARK 5.11. The graph-theoretic interpretation of what happens by passing from a *singular* compact toric surface X_Δ (with combinatorial data (4.11) and (4.12)) to its minimal desingularization $f : X_{\widetilde{\Delta}} \longrightarrow X_\Delta$ is illustrated in Figure 6 (in which we assume, for simplification's sake, that $I_\Delta = \{1, \ldots, \nu\}$).

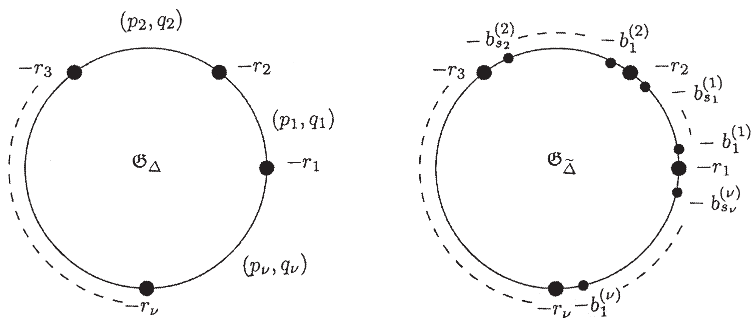


FIGURE 6.

6. Minimal, antiminimal and anticanonical models

In the present section we explain how one can make use of the general theory of §1 to obtain minimal models of normal pairs (X_Δ, D), and then we turn our attention to the antiminimal and anticanonical models of *nonsingular* X_Δ's.

• **Exceptional curves and minimal models**. Maintaining the notation introduced in §4, let D be a $\mathbb{Q}$-Weil divisor on a compact toric surface X_Δ with

$$D \sim \sum_{i=1}^{\nu} \lambda_i C_i \in \text{Div}_{\text{W}}^{\mathbb{T}}(X_\Delta) \otimes_{\mathbb{Z}} \mathbb{Q}, \quad \text{(for suitable } \lambda_1, \ldots, \lambda_\nu \in \mathbb{Q}, \text{ cf. Thm. 2.8).}$$

LEMMA 6.1. *The irreducible curve C_j (with $j \in \{1, \ldots, \nu\}$) is an exceptional curve of the first kind for the normal pair (X_Δ, D) (in the sense of §1) if and only* *if the following conditions are satisfied*:

$$\left\{\begin{array}{ll}\left\{\begin{array}{l}\frac{q_{j-1}-\widehat{p}_{j-1}}{q_{j-1}}+\frac{q_j-p_j}{q_j}<r_j, \text{ and}\\ \frac{\lambda_{j-1}}{q_{j-1}}+\lambda_j\left(\frac{q_{j-1}-\widehat{p}_{j-1}}{q_{j-1}}+\frac{q_j-p_j}{q_j}-r_j\right)+\frac{\lambda_{j+1}}{q_{j+1}}<0\end{array}\right\}, & \forall j\in I'_\Delta,\\ \left\{\begin{array}{l}\frac{q_j-p_j}{q_j}<r_j, \text{ and}\\ \lambda_{j-1}+\lambda_j\left(\frac{q_j-p_j}{q_j}-r_j\right)+\frac{\lambda_{j+1}}{q_{j+1}}<0\end{array}\right\}, & \forall j\in I''_\Delta,\\ \left\{\begin{array}{l}\frac{q_{j-1}-\widehat{p}_{j-1}}{q_{j-1}}<r_j, \text{ and}\\ \frac{\lambda_{j-1}}{q_{j-1}}+\lambda_j\left(\frac{q_{j-1}-\widehat{p}_{j-1}}{q_{j-1}}-r_j\right)+\lambda_{j+1}<0\end{array}\right\}, & \forall j\in J'_\Delta,\\ r_j>0 \text{ and } \lambda_{j-1}-\lambda_j r_j+\lambda_{j+1}<0, & \forall j\in J''_\Delta.\end{array}\right\} \tag{6.1}$$

PROOF. C_j is an exceptional curve of the first kind for the normal pair (X_Δ, D) iff both C_j^2 and $D\cdot C_j=(\sum_{i=1}^{\nu}\lambda_i C_i)\cdot C_j=\lambda_{j-1}(C_{j-1}\cdot C_j)+\lambda_j C_j^2+\lambda_{j+1}(C_j\cdot C_{j+1})$ are negative. Applying Lemma 4.7, we get the above inequalities. □

THEOREM 6.2. *Suppose that C_j is an exceptional curve of the first kind for (X_Δ, D) (i.e., that conditions* (6.1) *are satisfied). Let $(X_\Delta, D)\xrightarrow{\varphi_1}(X_{\Delta_1}, D_1)$ be the contraction of the curve C_j (with $|\Delta|=|\Delta_1|$, $\Delta_1(1)=\Delta(1)\smallsetminus\{\tau_j\}$, and $D_1=(\varphi_1)_*(D)$). Then φ_1 is totally discrepant. Moreover, there exists a finite sequence of birational morphisms*

$$(X_\Delta, D)\xrightarrow{\varphi_1}(X_{\Delta_1}, D_1)\xrightarrow{\varphi_2}(X_{\Delta_2}, D_2)\xrightarrow{\varphi_3}\cdots\xrightarrow{\varphi_\mu}(X_{\Delta_\mu}, D_\mu) \tag{6.2}$$

of normal pairs such that (X_{Δ_μ}, D_μ) is a minimal model of (X_Δ, D).

PROOF. By (1.2) we get $D=\varphi_1^*(D_1)+\left(\frac{D\cdot C_j}{C_j^2}\right)C$, with $\left(\frac{D\cdot C_j}{C_j^2}\right)>0$. If (X_{Δ_1}, D_1) is a minimal model of (X_Δ, D), then we stop; otherwise, we consider the contraction φ_2 of an exceptional curve of the first kind for the normal pair (X_{Δ_1}, D_1) (which is again totally discrepant) and repeat the same procedure until we arrive at a minimal model of (X_Δ, D). For this, we need only a finite sequence (6.2) of birational morphisms because in each step the number of the (finitely many) irreducible components of the exceptional set is reduced by one. □

REMARK 6.3. (a) Setting $\lambda_1=\cdots=\lambda_\nu=-1$ (resp., $\lambda_1=\cdots=\lambda_\nu=1$) we obtain by Theorem 6.2 a minimal model (resp., an antiminimal model) of X_Δ in the usual sense (see 1.5 (a)). In particular, minimal models with non-nef canonical divisor *either* admit a $\mathbb{P}^1_{\mathbb{C}}$-fibration and have Picard number ≥ 2 *or* have numerically ample anticanonical divisor and Picard number 1 (see [**51**, Thm. 4.9, p. 639]).

(b) If D is not pseudoeffective, there may be different choices to construct minimal models. For instance, even if X_Δ is *nonsingular*, it does not admit a uniquely determined minimal model (i.e., for $D=K_{X_\Delta}$), cf. [**2**, Remark 10.23, p. 156]. In fact, in this case, the set of all possible minimal models consists of the projective plane $\mathbb{P}^2_{\mathbb{C}}$ together with the *Hirzebruch surfaces*

$$\mathbb{F}_\kappa := \left\{([z_0:z_1:z_2],[t_1:t_2])\in\mathbb{P}^2_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}} \;\middle|\; z_1t_1^\kappa=z_2t_2^\kappa\right\},$$

where κ is an integer with $0\leq\kappa\neq 1$ (see [**27**], [**2**, Ch. 12], [**41**, Thm 8.2, pp. 52-56]). $\mathbb{F}_\kappa$ can be viewed as the rational scroll $\varpi:\mathbb{P}(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}\oplus\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\kappa))\longrightarrow\mathbb{P}^1_{\mathbb{C}}$ with twisting number κ, on the one hand, and as the toric surface having $-\kappa, 0, \kappa, 0$ as weights at the four vertices of its circular graph, on the other. Obviously, $\mathbb{F}_\kappa$ is isomorphic to $\mathbb{F}_{\kappa'}$ only if $\kappa = \kappa'$. Nevertheless, one can pass from the Hirzebruch surface $\mathbb{F}_\kappa$ to $\mathbb{F}_{\kappa+1}$ (and vice versa) by blowing up a $\mathbb{T}$-fixed point $\mathfrak{p}$ of $\mathbb{F}_\kappa$, and then contracting the strict transform of the fiber $\varpi^{-1}(\varpi(\mathfrak{p}))$ (which is a (-1)-curve) to another $\mathbb{T}$-fixed point $\mathfrak{p}'$, i.e., by *an elementary transformation*, as it is shown via the weighted circular graphs of Figure 7.

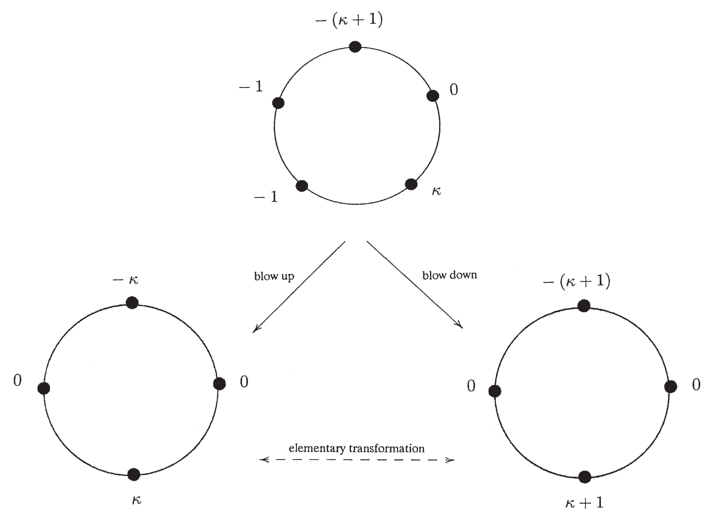


FIGURE 7.

• **Antiminimal and anticanonical models of nonsingular** X_Δ**'s**. Every nonsingular compact toric surface X_Δ admits a unique antiminimal model $X_{\Delta_{\mathrm{antim}}}$ and a unique anticanonical model $X_{\Delta_{\mathrm{antican}}}$ because $\mathrm{kod}(X_\Delta, -K_{X_\Delta}) = 2$ (cf. [**49**, §7.6]), which means, in particular, that $-K_{X_\Delta}$ is pseudoeffective (by [**49**, Lemma 3.1, p. 396]) and therefore one can apply Theorems 1.4 and 1.7.

DEFINITION 6.4. A nonsingular projective surface X is called *Del Pezzo surface* if its anticanonical divisor $-K_X$ is ample. Correspondingly, a normal projective surface X with at worst log-terminal singularities is called *log Del Pezzo surface* if $-K_X$ is a $\mathbb{Q}$-Cartier ample divisor. The *index* $\mathrm{ind}(X)$ of a log Del Pezzo surface X is defined to be the smallest positive integer ℓ for which ℓK_X is a Cartier divisor.

THEOREM 6.5. *The anticanonical model* $X_{\Delta_{\mathrm{antican}}} \cong \mathrm{Proj}(R(X_\Delta, -K_{X_\Delta}))$ *of any nonsingular compact toric surface* X_Δ *is a toric log Del Pezzo surface. Moreover, every toric log Del Pezzo surface is the anticanonical model of the surface obtained by its minimal desingularization.*

PROOF. Since $\mathrm{kod}(X_\Delta, -K_{X_\Delta}) = 2$, $R(X_\Delta, -K_{X_\Delta})$ is finitely generated and the assertion is true by Theorem 1.8. □

NOTE 6.6. (a) The anticanonical model $X_{\Delta_{\mathrm{antican}}}$ of a nonsingular compact toric surface X_Δ is constructed by considering the so-called *Zariski decomposition* of $-K_{X_\Delta} = (-K_{X_\Delta})^{(+)} + (-K_{X_\Delta})^{(-)}$ and contracting the (finitely many) irreducible curves C on X_Δ for which $(-K_{X_\Delta})^{(+)} \cdot C = 0$ (see [**2**, Ch. 14] and [**50**, p. 886]).
(b) To extract the antiminimal model $X_{\Delta_{\mathrm{antim}}}$ of a nonsingular compact toric surface

X_Δ from $X_{\Delta_{\rm antican}}$ it suffices to resolve (minimally) all Gorenstein singular points of $X_{\Delta_{\rm antican}}$ by a single birational *crepant* morphism.

► **Open problem**. The classification of all toric log Del Pezzo surfaces up to isomorphism remains an open combinatorial problem. Though there exist *finitely many* isomorphy classes of toric log Del Pezzo surfaces of given index[1] $\ell \geq 1$, the examination of those having high indices or (even worse) high Picard numbers seems to demand rather tricky techniques, at least from the computational point of view. In fact, for given index ℓ, there are two main problems: the *local* one, i.e., to classify all possible types of the available cones, and the *global* one, i.e., to check which combinations of admissible cones fit together to give the required fans.

REMARK 6.7 (A geometric reformulation of the classification problem). If X_Δ is a compact toric surface, then the $\mathbb{Q}$-Cartier divisor $-K_{X_\Delta}$ is defined by a rational Δ-support function taking the value 1 at every $\mathbf{n} \in \mathrm{Gen}(\Delta)$ (see 2.7 and 2.11). By Theorem 2.14 and Remark 2.15 $-K_{X_\Delta}$ is ample if and only if this function is strictly upper convex, which means that all elements of $\mathrm{Gen}(\Delta)$ are vertices of a lattice polygon. Thus, in geometric terms, the classification of toric log Del Pezzo surfaces X_Δ of a given index $\ell \geq 1$ (up to isomorphism) is equivalent to the classification (up to unimodular transformation) of lattice polygons $\mathcal{Q} \subset N_{\mathbb{R}}$ with $\mathbf{0} \in \mathrm{int}(\mathcal{Q})$ and $\{ F_\sigma | \ \sigma \in \Delta(2)\}$ as edge-set (in the notation of 3.19), such that ℓ equals the lcm of $\{ \mathrm{lind}\,(U_\sigma, \mathrm{orb}\,(\sigma)) | \, \sigma \in \Delta(2)\}$. (See below Lemma 6.8.) For such $\mathcal{Q}$'s, we have necessarily[2]

$$\mathrm{int}(\tfrac{1}{\ell}\mathcal{Q}) \cap N = \{\mathbf{0}\}. \tag{6.3}$$

In particular, the *finiteness* of the classes of lattice polygons fulfilling the equality (6.3) follows from results of Hensley [**25**] and Lagarias & Ziegler [**34**].

LEMMA 6.8. *Using the notation of* §4, *the index* $\ell = \mathrm{ind}(X_\Delta)$ *of a toric log Del Pezzo surface* X_Δ *equals*

$$\mathrm{ind}\,(X_\Delta) = \begin{cases} \mathrm{lcm}\,\{ \mathrm{lind}\,(U_{\sigma_i}, \mathrm{orb}\,(\sigma_i)) \ | \ i \in I_\Delta \}, & \text{if} \ \ I_\Delta \neq \varnothing, \\ 1, & \text{if} \ \ I_\Delta = \varnothing, \end{cases}$$

where lind *denotes the local index introduced in* 3.19.

PROOF. If X_Δ is nonsingular, then obviously $\mathrm{ind}(X_\Delta) = 1$. Otherwise, we have $I_\Delta \neq \varnothing$, and $\ell = \mathrm{ind}(X_\Delta)$ is the smallest positive integer for which

$$f^* \ell\, K_{X_\Delta} = \ell\, K_{X_{\widetilde{\Delta}}} - \sum_{i \in I_\Delta} \ell\, K(E^{(i)}) = \ell\, K_{X_{\widetilde{\Delta}}} + \sum_{i \in I_\Delta} \sum_{j=1}^{s_i} \ell \left(1 - \tfrac{\gamma_j^{(i)} + \delta_j^{(i)}}{q_i} \right) E_j^{(i)}$$

is a Cartier divisor on the surface $X_{\widetilde{\Delta}}$ obtained by the minimal desingularization (4.8) of X_Δ. By 4.5 (b) we have $\ell = \mathrm{lcm}\,\{ \mathrm{lind}\,(U_{\sigma_i}, \mathrm{orb}\,(\sigma_i)) \ | \ i \in I_\Delta \}$. □

We shall henceforth deal only with toric Del Pezzo surfaces X_Δ with $\mathrm{ind}(X_\Delta) \leq 2$.

LEMMA 6.9. *Let* $\sigma \subset N_{\mathbb{R}}$ *be a two dimensional* (p, q)*-cone* (*w.r.t. a suitable* $\mathbb{Z}$*-basis of* N, *in the sense of* 3.4). *Then*

$$\mathrm{lind}\,(U_\sigma, \mathrm{orb}\,(\sigma)) = 1 \Longleftrightarrow \begin{cases} \text{either } p = 0 \ \textit{and} \ q = 1, \\ \text{or } p = 1 \ \textit{and} \ q \geq 2, \end{cases} \tag{6.4}$$

[1]This follows, e.g., by more general results of A. and L. Borisov [**11**].

[2]Condition (6.3) is also *sufficient* (for X_Δ to be log-Del Pezzo) only for $\ell = 1$.

and

$$\operatorname{lind}(U_\sigma, \operatorname{orb}(\sigma)) = 2 \Longleftrightarrow \quad (q = 2(p-1) \ \text{ and } \ p \text{ is odd } \ \geq 3)\,. \tag{6.5}$$

PROOF. By (3.12) $\operatorname{lind}(U_\sigma, \operatorname{orb}(\sigma)) = 1$ means that $q = \gcd(q, q-p+1)$, and therefore $q \mid p-1$. Since $p - 1 < p < q$, p, q satisfy (6.4). The converse is obvious. If $\operatorname{lind}(U_\sigma, \operatorname{orb}(\sigma)) = 2$, then $q = 2\,\gcd(q, q-p+1)$. Thus, q is even, $q \nmid p-1$, and

$$\left.\begin{array}{r} \frac{q}{2} \mid p-1 \Longrightarrow \exists \lambda \in \mathbb{N} : 2(p-1) = \lambda q \\ p - 1 < p < q \Longrightarrow 1 \leq \frac{2(p-1)}{q} \leq 2, \end{array}\right\} \Longrightarrow \lambda = 1 \Longrightarrow q = 2(p-1).$$

p is odd (because otherwise $\gcd(p, q) \geq 2$). The converse is obvious. □

THEOREM 6.10. *Up to isomorphism, there exist exactly* 16 *toric log Del Pezzo surfaces of index* $\ell = 1$. *Their* WVE^{2}C*-graphs are illustrated in* Figures 8-10.

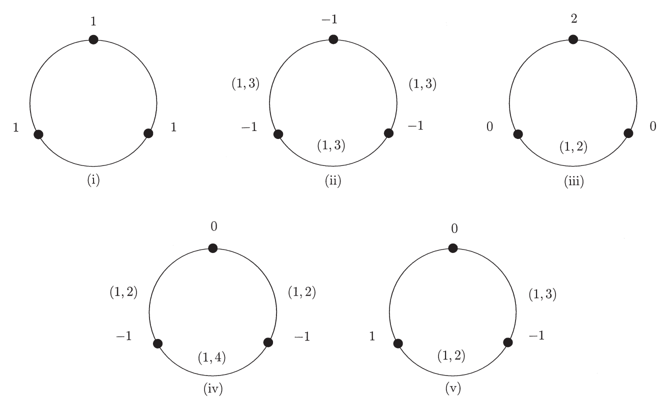


FIGURE 8.

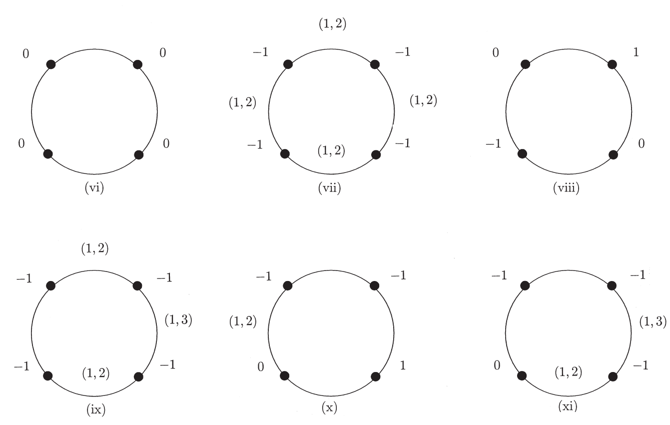


FIGURE 9.

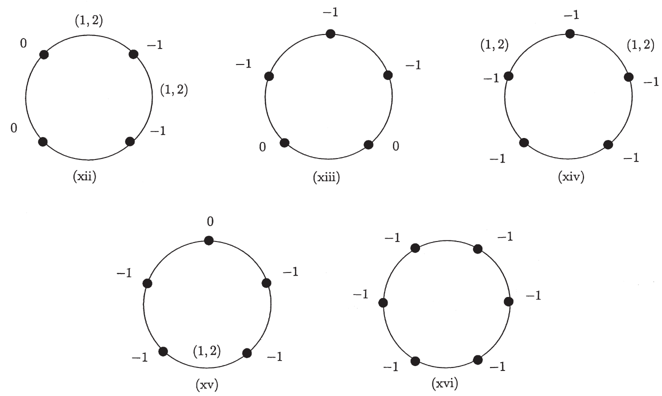


FIGURE 10.

SKETCH OF A FIRST PROOF. Let X_Δ be a toric log Del Pezzo surface of this kind (with combinatorial data (4.11) and (4.12)). Since $-K_{X_\Delta}$ is ample, Nakai's criterion [**42**, Thm. 2.18, pp. 86-87] and Lemma 4.7 give

$$-K_{X_\Delta} \cdot C_i = C_{i-1} \cdot C_i + C_i^2 + C_i \cdot C_{i+1} = -r_i + 2 > 0 \Longrightarrow r_i \leq 1, \tag{6.6}$$

for all $i \in \{1, \ldots, \nu\}$. On the other hand, using (6.4), (4.18) can be written as

$$\sum_{i=1}^{\nu} r_i = 3\nu - 12 + \sum_{i \in I_\Delta} s_i = 3\nu - 12 + \sum_{i \in I_\Delta} (q_i - 1). \tag{6.7}$$

From (6.6) and (6.7) we conclude that

$$3 \leq \nu \leq 6 - \frac{1}{2} \sum_{i \in I_\Delta} s_i \leq 6. \tag{6.8}$$

The "classical" toric Del Pezzo surfaces (with $I_\Delta = \varnothing$) are 5, namely those corresponding to the WVE^{2}C-graphs (i), (vi), (viii), (xiii) and (xvi) of Figures 8, 9, and 10, and have been classified in [**4**, Proposition 6, p. 22], [**42**, Proposition 2.21, pp. 88-89], and [**55**, Proposition 2.7, pp. 40-41]. For the singular toric log Del Pezzos we have obviously $\nu \in \{3, 4, 5\}$. If $\nu = 5$, then there are either one or two singular points (coming necessarily from cones of type $(p, q) = (1, 2)$; cf. (6.4) and (6.8)). By (6.7) and the fact that $X_{\widetilde{\Delta}}$ must be contractible either to $\mathbb{P}^2_{\mathbb{C}}$ or to an $\mathbb{F}_\kappa$ after blowing down (at most 4) (-1)-curves (see 6.3 (b)), we infer that *either* one of the r_i's equals 0 and the others $= 1$ *or* all r_i's are equal to 1. Now having the main constituents of all possible WVE^{2}C-graphs in hand (i.e., the weights (p_i, q_i), $-r_i$, and $b_k^{(i)}$'s which are $= 2$), it is enough to test via (5.1) which of these graphs can be realized as WVE^{2}C-graphs of a complete fan (specifying automatically the ordering of the 5 available two-dimensional cones). As it turns out, only the circular graphs (xiv) and (xv) of Figure 10 "survive" (up to "$\underset{\text{gr.}}{\cong}$") after having performed this test. The admissible WVE^{2}C-graphs for $\nu \in \{3, 4\}$ can be found similarly.

SKETCH OF A SECOND PROOF. In [**5**] Batyrev proved that there exist exactly 16 lattice lattice polygons satisfying condition (6.3) for $\ell = 1$. (These are actually the so-called *reflexive polygons.*) Several alternative proofs of this fact are given in [**32**, Thm. 4.2.3, p. 86], [**45**], and [**40**, Prop. 3.4.1, pp. 55-57]. Drawing the rays which begin from the origin and pass through the vertices of each of these lattice polygons one constructs the corresponding fans and recognizes the type of the two-dimensional rational s.c.p. cones, and where the singular points are located; afterwards, calculating the r_i's, it is easy to build up the required WVE^{2}C-graphs.□

REMARK 6.11. (a) Table 1 contains a more precise description of the 16 toric log Del Pezzo surfaces of index 1 as projective varieties. In cases (viii), (xi), (xiii), (xv) and (xvi) the centers of blow-ups are smooth $\mathbb{T}$-fixed points. In cases (ii), (iii), (iv), (v), (vii), (ix), (x), (xii) and (xiv) we indicate the embedding of the surface X_Δ into $\mathbb{P}_{\mathbb{C}}^{(K^2_{X_\Delta})}$ induced by the global sections of the sheaf $\mathcal{O}_{X_\Delta}(-K_{X_\Delta})$.

(b) The surface obtained by the minimal desingularization of X_Δ in cases (ii), (iv), (v), (vii) and (ix)-(xvi) is isomorphic to $\mathbb{P}^2_{\mathbb{C}}$ blown up at $9 - K^2_{X_\Delta}$ points which are in almost general position (see Hidaka & Watanabe [**26**, Thm. 3.4, p. 325]).

(c) All possible types of singularities which can occur in the (not necessarily toric) log Del Pezzos of index $\ell = 1$ are to be found in the "long lists" contained in [**1, 56**].

Nr.	X_Δ	Nr.	X_Δ
(i)	$\mathbb{P}^2_{\mathbb{C}}$	(ix)	(realized as a surface of degree 4 in $\mathbb{P}^4_{\mathbb{C}}$)
(ii)	$\mathbb{P}^2_{\mathbb{C}}/(\mathbb{Z}/3\mathbb{Z})$ (this can be realized as the cubic surface $\{ [z_0 : .. : z_3] \in \mathbb{P}^3_{\mathbb{C}} \ z_0^3 = z_1 z_2 z_3 \}$)	(x)	(realized as a surface of degree 7 in $\mathbb{P}^7_{\mathbb{C}}$)
(iii)	$\mathbb{P}^2_{\mathbb{C}}(1,1,2)$ (realized as the cone over a quadric in $\mathbb{P}^2_{\mathbb{C}}$ obtained by contracting the minimal section of an $\mathbb{F}_2$)	(xi)	$\mathbb{P}^2_{\mathbb{C}}(1,2,3)$ blown up at one point
(iv)	$\mathbb{P}^2_{\mathbb{C}}(1,1,2)/(\mathbb{Z}/2\mathbb{Z})$ (realized as a surface of degree 4 in $\mathbb{P}^4_{\mathbb{C}}$)	(xii)	(realized as a surface of degree 7 in $\mathbb{P}^7_{\mathbb{C}}$)
(v)	$\mathbb{P}^2_{\mathbb{C}}(1,2,3)$ (realized as a surface of degree 6 in $\mathbb{P}^6_{\mathbb{C}}$)	(xiii)	$\mathbb{P}^2_{\mathbb{C}}$ blown up at two points
(vi)	$\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$	(xiv)	(realized as a surface of degree 5 in $\mathbb{P}^5_{\mathbb{C}}$)
(vii)	(realized as a surface of degree 4 in $\mathbb{P}^4_{\mathbb{C}}$)	(xv)	$\mathbb{P}^2_{\mathbb{C}}(1,1,2)$ blown up at two points
(viii)	$\mathbb{P}^2_{\mathbb{C}}$ blown up at one point ($\cong \mathbb{F}_1$)	(xvi)	$\mathbb{P}^2_{\mathbb{C}}$ blown up at three points

TABLE 1.

THEOREM 6.12. *Up to isomorphism, there are only* 7 *toric log Del Pezzo surfaces of index* $\ell = 2$ *with Picard number* 1, *namely those whose* WVE^{2}C*-graphs are illustrated in* Figure 11, *and whose structure (as weighted projective planes or quotients thereof) is described in the last column of* Table 4.

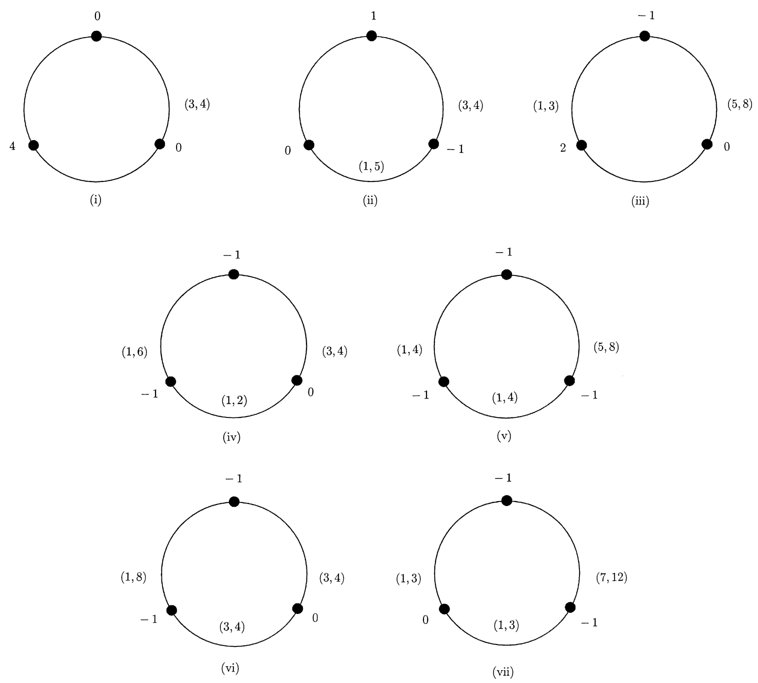


FIGURE 11.

PROOF. Let X_Δ be a toric log Del Pezzo surface of this kind. By Lemma 6.8 Δ contains necessarily at least one cone σ with $\mathrm{lind}(U_\sigma, \mathrm{orb}(\sigma)) = 2$. Without loss of generality, we may work with the standard rectangular lattice $\mathbb{Z}^2$ within $\mathbb{R}^2$ and assume that $\Delta(2)$ consists of the cones

$$\sigma_i = \mathbb{R}_{\geq 0}\,\mathbf{n}_i + \mathbb{R}_{\geq 0}\,\mathbf{n}_{i+1},\ \forall i \in \{1,2,3\},\ \text{with } \mathbf{n}_1 = (1,0)\,, \mathbf{n}_2 = (p_1, 2(p_1-1))\,,$$

(p_1 odd ≥ 3, cf. (6.5)), and with p_i, q_i, r_i denoting the combinatorial data (4.11) of X_Δ. The third minimal generator $\mathbf{n}_3$ of Δ belongs necessarily to the set

$$\mathcal{M} := \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \tfrac{2(p_1-1)}{p_1} x < y < 0 \right\} \cap \mathbb{Z}^2.$$

Let us now define

$$\mathcal{L}_{\sigma_2} := \{ (x,y) \in \mathcal{M} \mid 2(p_1-1)x - p_1 y = -1 \ \},$$
$$\mathcal{L}'_{\sigma_2} := \left\{ (x,y) \in \mathcal{M} \;\middle|\; \begin{array}{c} x = p_1 - \lambda q_2, \quad y = 2(p_1-1) - \mu q_2, \\ \text{for some } \lambda, \mu \in \mathbb{Z} \ \text{ with } \ \mu x - \lambda y = \pm 1 \end{array} \right\},$$
$$\mathcal{L}''_{\sigma_2} := \left\{ (x,y) \in \mathcal{M} \;\middle|\; \begin{array}{c} x = p_1 p_2 + \lambda q_2, \quad y = 2(p_1-1)p_2 + \mu q_2, \\ \text{for some } \lambda, \mu \in \mathbb{Z} \ \text{ with } \ \mu p_1 - 2\lambda(p_1-1) = \pm 1 \end{array} \right\},$$

and

$$\begin{aligned} \mathcal{L}_{\sigma_3} &:= \{ (x,y) \in \mathcal{M} \mid y = -1 \}, \\ \mathcal{L}'_{\sigma_3} &:= \{ (x,y) \in \mathcal{M} \mid y = -q_3,\ x = \kappa q_3 + 1, \text{ for some } \kappa \in \mathbb{Z} \}, \\ \mathcal{L}''_{\sigma_3} &:= \left\{ (x,y) \in \mathcal{M} \;\middle|\; \begin{array}{c} p_3 x + 2\lambda(p_3-1) = 1,\ p_3 y + 2\mu(p_3-1) = 0, \\ \text{for some } \lambda, \mu \in \mathbb{Z} \text{ with } \mu x - \lambda y = \pm 1 \end{array} \right\}. \end{aligned}$$

To determine all possible values of the coordinates of $\mathbf{n}_3$ one has to examine (for symmetry reasons) only the six cases indicated in Table 2.

Case	p_2	q_2	p_3	q_3	Condition for $\mathbf{n}_3$	$r_1 + r_2 + r_3$ (by (4.18))
(a)	0	1	0	1	$\mathbf{n}_3 \in \mathcal{L}_{\sigma_2} \cap \mathcal{L}_{\sigma_3}$	$\frac{p_1-1}{2} - 5$
(b)	0	1	1	≥ 2	$\mathbf{n}_3 \in \mathcal{L}_{\sigma_2} \cap \mathcal{L}'_{\sigma_3}$	$\frac{p_1-1}{2} + q_3 - 6$
(c)	1	≥ 2	1	≥ 2	$\mathbf{n}_3 \in \mathcal{L}'_{\sigma_2} \cap \mathcal{L}'_{\sigma_3}$	$\frac{p_1-1}{2} + q_2 + q_3 - 7$
(d)	0	1	≥ 3	$2(p_3-1)$	$\mathbf{n}_3 \in \mathcal{L}_{\sigma_2} \cap \mathcal{L}''_{\sigma_3}$	$\frac{p_1-1}{2} + \frac{p_3-1}{2} - 7$
(e)	1	≥ 2	≥ 3	$2(p_3-1)$	$\mathbf{n}_3 \in \mathcal{L}'_{\sigma_2} \cap \mathcal{L}''_{\sigma_3}$	$\frac{p_1-1}{2} + \frac{p_3-1}{2} + q_2 - 8$
(f)	≥ 3	$2(p_2-1)$	≥ 3	$2(p_3-1)$	$\mathbf{n}_3 \in \mathcal{L}''_{\sigma_2} \cap \mathcal{L}''_{\sigma_3}$	$\sum_{i=1}^{3} (\frac{p_i-1}{2}) - 9$

Table 2.

Since $-2K_{X_\Delta}$ is an ample Cartier divisor, Nakai's criterion informs us that

$$-2K_{X_\Delta} \cdot C_i > 0 \Longrightarrow (C_1 + C_2 + C_3) \cdot C_i > 0, \quad \forall i \in \{1,2,3\}, \tag{6.9}$$

(where $C_i := \mathbf{V}_\Delta(\mathbb{R}_{\geq 0}\mathbf{n}_i)$). Hence, using (6.9) and Lemma 4.7, we find concrete upper bounds for r_1, r_2 and r_3, leading to further restrictions on p_i, q_i, which are summarized in Table 3.

Case	r_1	r_2	r_3	Restrictions on p_i, q_i
(a)	≤ 1	≤ 1	≤ 1	$3 \leq p_1 \leq 17$ (p_1 odd ≥ 3)
(b)	≤ 1	≤ 1	≤ 1	$7 \leq p_1 + 2q_3 \leq 19$ (p_1 odd ≥ 3)
(c)	≤ 1	≤ 1	≤ 1	$11 \leq p_1 + 2(q_2 + q_3) \leq 21$, ($p_1$ odd ≥ 3)
(d)	≤ 0	≤ 1	≤ 1	$6 \leq p_1 + p_3 \leq 20$ (p_1, p_3 odd ≥ 3)
(e)	≤ 0	≤ 1	≤ 1	$10 \leq p_1 + 2q_2 + p_3 \leq 22$ (p_1, p_3 odd ≥ 3)
(f)	≤ 0	≤ 0	≤ 0	$9 \leq p_1 + p_2 + p_3 \leq 21$ (p_1, p_2, p_3 odd ≥ 3)

Table 3.

Taking into account the conditions for $\mathbf{n}_3$, these inequalities have only one solution in case (a) (see below (i) in Table 4), two solutions in case (b) (namely (ii) and (iii)), three solutions in case (c) (namely (iv), (v), and (vii)), one solution in case (e) (namely (vi)), whereas they have no solution in cases (d) and (f)!

Nr.	p_1	q_1	p_2	q_2	p_3	q_3	$\mathbf{n}_3$	r_1	r_2	r_3	X_Δ
(i)	3	4	0	1	0	1	$(-1,-1)$	0	0	-4	$\mathbb{P}^2_{\mathbb{C}}(1,1,4)$
(ii)	3	4	0	1	1	5	$(-4,-5)$	1	-1	0	$\mathbb{P}^2_{\mathbb{C}}(1,4,5)$
(iii)	5	8	0	1	1	3	$(-2,-3)$	0	1	-2	$\mathbb{P}^2_{\mathbb{C}}(1,3,8)$
(iv)	3	4	1	6	1	2	$(-3,-2)$	0	1	1	$\mathbb{P}^2_{\mathbb{C}}(1,2,3)/(\mathbb{Z}/2\mathbb{Z})$
(v)	5	8	1	4	1	4	$(-3,-4)$	1	1	1	$\mathbb{P}^2_{\mathbb{C}}(1,1,2)/(\mathbb{Z}/4\mathbb{Z})$
(vi)	3	4	1	8	3	4	$(-5,-4)$	0	1	1	$\mathbb{P}^2_{\mathbb{C}}(1,2,1)/(\mathbb{Z}/4\mathbb{Z})$
(vii)	7	12	1	3	1	3	$(-2,-3)$	1	1	0	$\mathbb{P}^2_{\mathbb{C}}(1,1,4)/(\mathbb{Z}/3\mathbb{Z})$

Table 4.

Having found $\mathbf{n}_3$'s (and consequently r_i's), we determine both the precise structure of X_Δ's (see last column) and the WVE^{2}C-graphs of Figure 11. □

Remark 6.13. Alexeev and Nikulin proved in [**1**, Thm. 4.2, pp. 105-106] that, up to isomorphism, there exist exactly 18 (not necessarily toric) log Del Pezzo surfaces of index 2 with Picard number = 1. Among them there are 14 having only cyclic quotient singularities. By Theorem 6.12 we see that only 7 out of these 14 surfaces are toric.

7. Riemann-Roch formula

The Euler-Poincaré characteristic

$\chi(\mathcal{O}_X(D)) := \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(D)) - \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X(D)) + \dim_{\mathbb{C}} H^2(X, \mathcal{O}_X(D))$

of the coherent sheaf $\mathcal{O}_X(D)$ associated to a divisor D on a *nonsingular* projective surface X is given by the well-known *Riemann-Roch formula*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2} D \cdot (D - K_X), \tag{7.1}$$

(see [**3**, formula (6), p. 26], [**21**, p. 472], or [**23**, Ch. V, Thm. 1.6, p. 362]). To generalize (7.1) in the category of *normal* projective surfaces (say, with mild singularities) in the case in which D is a Weil non-Cartier divisor one has to add to the right-hand side certain "correction terms" due to the contribution of singularities. For compact toric surfaces we recall briefly the purely combinatorial method for the computation of $\chi(\mathcal{O}_{X_\Delta}(D))$ whenever D is a $\mathbb{T}$-invariant Cartier divisor with $\mathcal{O}_{X_\Delta}(D)$ generated by the global sections, and then we pass to Blache's RR-formula (7.3) to deal with the general case.

• **Traditional computation of** $\chi(\mathcal{O}_{X_\Delta}(D))$ **by toric tools**. Let X_Δ be a compact toric surface as in §4. Consider the $\mathbb{T}$-invariant Cartier divisor D_ψ on X_Δ associated to an upper convex Δ-support function ψ. Since $H^j(X_\Delta, \mathcal{O}_{X_\Delta}(D_\psi))$ vanishes for $j \geq 1$ (see [**42**, pp. 76-77]), we have

$$\chi(\mathcal{O}_{X_\Delta}(D_\psi)) = \dim_{\mathbb{C}} H^0(X_\Delta, \mathcal{O}_{X_\Delta}(D_\psi)) = \sharp(\mathcal{P}_\psi \cap M) \tag{7.2}$$

by Theorem 2.13, and we can calculate $\chi(\mathcal{O}_{X_\Delta}(D_\psi))$ by Pick's formula giving the number of lattice points of the integral convex polygon $\mathcal{P}_\psi$ in terms of its area and its lattice points on the boundary.

EXAMPLE 7.1. If Δ is the complete fan in $\mathbb{R}^2$ (w.r.t. the lattice $N = \mathbb{Z}^2$) with $\text{Gen}(\Delta) = \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$, where $\mathbf{n}_1 = (2,-1)$, $\mathbf{n}_2 = (-1,-1)$, $\mathbf{n}_3 = (-1,3)$, and ψ takes the values $\psi(\mathbf{n}_1) = 1$, $\psi(\mathbf{n}_2) = -14$, $\psi(\mathbf{n}_3) = 2$, then

$$\mathcal{P}_\psi = \text{ conv}\left(\{(1,1),(5,9),(10,4)\}\right)$$

is the lattice triangle of Figure 12, and (7.2) gives

$$\begin{aligned}\chi\left(\mathcal{O}_{X_\Delta}\left(D_\psi\right)\right) &= \text{area}(\mathcal{P}_\psi) + \frac{1}{2}(\sharp\left(\partial\mathcal{P}_\psi \cap M\right)) + 1 \\ &= \frac{1}{2}\left|\det\begin{pmatrix} 1 & 5 & 10 \\ 1 & 9 & 4 \\ 1 & 1 & 1\end{pmatrix}\right| + \frac{1}{2}(12) + 1 = 37.\end{aligned}$$

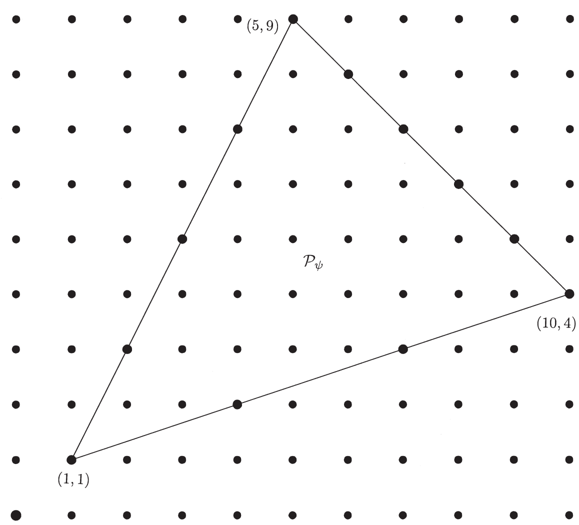


FIGURE 12.

• **Generalized Riemann-Roch formula**. Now let X be a projective surface having at worst *quotient singularities*. It is possible to compute $\chi\left(\mathcal{O}_X\left(D\right)\right)$ for an *arbitrary* Weil divisor D on X by the *generalized Riemann-Roch formula*:

$$\chi\left(\mathcal{O}_X\left(D\right)\right) = \chi\left(\mathcal{O}_X\right) + \frac{1}{2}D\cdot(D-K_X) + \mathfrak{Y}_X(D) \tag{7.3}$$

(see Blache [**10**, Thm. 1.2, pp. 312-313]), where the contribution

$$\mathfrak{Y}_X(D) = \sum_{x\in\text{Sing}(X)} \mathfrak{Y}_{X,x}(D) \tag{7.4}$$

of the singular set of X to (7.3) is given by uniquely determined maps

$$\mathfrak{Y}_{X,x} : \text{Div}_{\text{W}}(X,x)/\text{Div}_{\text{C}}(X,x) \longrightarrow \mathbb{Q}$$

for each analytic germ (X,x) with $x \in \text{Sing}(X)$. In fact, it can be shown that if one considers a desingularization $f : \widetilde{X} \longrightarrow X$ of X, then

$$\mathfrak{Y}_{X,x}(D) = -\frac{1}{2}\left(\left\langle f^*D - \overline{D}\right\rangle \cdot \left(\lfloor f^*D\rfloor - K_{\widetilde{X}}\right)\right), \tag{7.5}$$

where $\overline{D}$ is the strict transform of D. (Here, by "$\lfloor\ \rfloor$" and "$\langle\ \rangle$" is meant the integer part and the fractional part, respectively, of a $\mathbb{Q}$-Weil divisor). $\mathfrak{Y}_{X,x}(D)=0$ if and only if D is a Cartier divisor, and the right-hand side of (7.5) is well-defined over $x\in \mathrm{Sing}(X)$ at which D is not Cartier. Moreover, the rational number $\mathfrak{Y}_{X,x}(D)$ does not depend on the particular choice of f. (Formula (7.3) generalizes results from [**12**, Prop. 2, pp. 302-304], [**19**, 18.3.4, pp. 360-361] and [**47**, Thm. 9.1, pp. 409-411].) Let us apply (7.3) for a compact toric surface $X=X_\Delta$ and see how $\chi(\mathcal{O}_{X_\Delta}(D))$ is described by means of its combinatorial data (4.11) and (4.12).

THEOREM 7.2 (Riemann-Roch formula for compact toric surfaces). *If D is a Weil divisor on X_Δ with*

$$D\sim\sum_{i=1}^{\nu}\lambda_i C_i\in \mathrm{Div}_{\mathrm{W}}^{\mathbb{T}}(X_\Delta),\qquad (\lambda_1,\ldots,\lambda_\nu\in\mathbb{Z}),$$

then

$$\boxed{\begin{aligned}\chi(\mathcal{O}_{X_\Delta}(D))= \;& -\tfrac{1}{2}\sum_{i=1}^{\nu}\lambda_i(\lambda_i+1)\,r_i+\sum_{i\in I'_\Delta}\lambda_i(\lambda_i+1)\left(\tfrac{q_{i-1}-\widehat{p}_{i-1}}{2q_{i-1}}+\tfrac{q_i-p_i}{2q_i}\right)\\ &+\sum_{i\in I''_\Delta}\lambda_i(\lambda_i+1)\left(\tfrac{q_i-p_i}{2q_i}\right)+\sum_{i\in J'_\Delta}\lambda_i(\lambda_i+1)\left(\tfrac{q_{i-1}-\widehat{p}_{i-1}}{2q_{i-1}}\right)\\ &+\sum_{i=1}^{\nu}(\lambda_i+\lambda_{i+1}+2\lambda_i\lambda_{i+1})\,\tfrac{1}{2q_i}+\mathfrak{Y}_{X_\Delta}(D)+1,\end{aligned}}$$

where

$$\mathfrak{Y}_{X_\Delta}(D)=\sum_{i\in I_\Delta}\mathfrak{Y}_{X_\Delta,\,\mathrm{orb}(\sigma_i)}(D),$$

and

$$\boxed{\begin{aligned}\mathfrak{Y}_{X_\Delta,\,\mathrm{orb}(\sigma_i)}(D)= \;& -\tfrac{1}{2}(\lambda_i+1)\left\langle\tfrac{(q_i-p_i)\lambda_i+\lambda_{i+1}}{q_i}\right\rangle+(\lambda_{i+1}+1)\left\langle\tfrac{\lambda_i+(q_i-\widehat{p}_i)\lambda_{i+1}}{q_i}\right\rangle\\ &+\sum_{j=1}^{s_i}\left\langle\tfrac{\gamma_j^{(i)}\lambda_i+\delta_j^{(i)}\lambda_{i+1}}{q_i}\right\rangle\left(\left\lfloor\tfrac{\gamma_j^{(i)}\lambda_i+\delta_j^{(i)}\lambda_{i+1}}{q_i}\right\rfloor+1\right)\tfrac{(b_j^{(i)})^2}{2}\\ &-\sum_{1\le j<k\le s_i}\tfrac{1}{2}\left(\left\langle\tfrac{\gamma_j^{(i)}\lambda_i+\delta_j^{(i)}\lambda_{i+1}}{q_i}\right\rangle\left(\left\lfloor\tfrac{\gamma_k^{(i)}\lambda_i+\delta_k^{(i)}\lambda_{i+1}}{q_i}\right\rfloor+1\right)\right)\\ &-\sum_{1\le j<k\le s_i}\tfrac{1}{2}\left(\left\langle\tfrac{\gamma_k^{(i)}\lambda_i+\delta_k^{(i)}\lambda_{i+1}}{q_i}\right\rangle\left(\left\lfloor\tfrac{\gamma_j^{(i)}\lambda_i+\delta_j^{(i)}\lambda_{i+1}}{q_i}\right\rfloor+1\right)\right)\end{aligned}}$$

for all $i\in I_\Delta$.

PROOF. Since $\chi(\mathcal{O}_{X_\Delta})=1$, and

$$D\cdot(D-K_{X_\Delta})=\left(\sum_{i=1}^{\nu}\lambda_i C_i\right)\cdot\left(\sum_{i=1}^{\nu}(\lambda_i+1)\,C_i\right)$$

$$=\sum_{i=1}^{\nu}\lambda_i(\lambda_i+1)\,C_i^2+\sum_{1\le i<j\le\nu}(\lambda_i+\lambda_j+2\lambda_i\lambda_j)\,(C_i\cdot C_j)$$

$$=\sum_{i=1}^{\nu}\lambda_i(\lambda_i+1)\,C_i^2+\sum_{i=1}^{\nu}(\lambda_i+\lambda_{i+1}+2\lambda_i\lambda_{i+1})\,(C_i\cdot C_{i+1})$$

$$= -\sum_{i=1}^{\nu} \lambda_i (\lambda_i + 1) r_i + \sum_{i \in I'_\Delta} \lambda_i (\lambda_i + 1) \left(\tfrac{q_{i-1} - \widehat{p}_{i-1}}{q_{i-1}} + \tfrac{q_i - p_i}{q_i} \right)$$
$$+ \sum_{i \in I''_\Delta} \lambda_i (\lambda_i + 1) \left(\tfrac{q_i - p_i}{q_i} \right) + \sum_{i \in J'_\Delta} \lambda_i (\lambda_i + 1) \left(\tfrac{q_{i-1} - \widehat{p}_{i-1}}{q_{i-1}} \right)$$
$$+ \sum_{i=1}^{\nu} (\lambda_i + \lambda_{i+1} + 2\lambda_i \lambda_{i+1}) \tfrac{1}{q_i},$$

(by Lemma 4.7), it suffices to consider the minimal desingularization (4.8) of X_Δ, and to determine $\mathfrak{Y}_{X_\Delta}(D)$ by (7.4) and (7.5). Let $\overline{D}$ be the strict transform of D by this f, and

$$f^* D - \overline{D} = \sum_{i \in I_\Delta} \sum_{j=1}^{s_i} \mu_j^{(i)} E_j^{(i)}$$

the $\mathbb{Q}$-Cartier divisor on $X_{\widetilde{\Delta}}$ supported in $\bigcup_{i \in I_\Delta} \bigcup_{j=1}^{s_i} E_j^{(i)}$ with coefficients satisfying the linear system

$$\overline{D} \cdot E_{j'}^{(i)} = -\left(\sum_{i \in I_\Delta} \sum_{j=1}^{s_i} \mu_j^{(i)} E_j^{(i)} \right) \cdot E_{j'}^{(i)},$$

for all $i \in I_\Delta$ and all $j' \in \{1, \ldots, s_i\}$. For every fixed $i \in I_\Delta$, this system is equivalent to the following:

$$\left(\mathbf{L}_{s_i}(b_1^{(i)}, \ldots, b_{s_i}^{(i)}) \right) (\mu_1^{(i)}, \ldots, \mu_{s_i}^{(i)})^T = (\lambda_i, 0, 0, \ldots, 0, 0, \lambda_{i+1})^T.$$

By Lemma 3.13,

$$\mu_j^{(i)} = \tfrac{1}{q_i} \left(\gamma_j^{(i)} \delta_1^{(i)} \lambda_i + \gamma_{s_i}^{(i)} \delta_j^{(i)} \lambda_{i+1} \right) = \tfrac{1}{q_i} \left(\gamma_j^{(i)} \lambda_i + \delta_j^{(i)} \lambda_{i+1} \right). \tag{7.6}$$

Now we write

$$-2\, \mathfrak{Y}_{X_\Delta,\ \mathrm{orb}(\sigma_i)}(D) =$$
$$= \sum_{j=1}^{s_i} \left\langle \mu_j^{(i)} \right\rangle E_j^{(i)} \cdot \left((\lambda_i \overline{C}_i + \lambda_{i+1} \overline{C}_{i+1} + \sum_{j=1}^{s_i} \left\lfloor \mu_j^{(i)} \right\rfloor E_j^{(i)}) + (\overline{C}_i + \overline{C}_{i+1} + \sum_{j=1}^{s_i} E_j^{(i)}) \right)$$
$$= \sum_{j=1}^{s_i} \left\langle \mu_j^{(i)} \right\rangle E_j^{(i)} \cdot \left((\lambda_i + 1) \overline{C}_i + (\lambda_{i+1} + 1) \overline{C}_{i+1} + \sum_{j=1}^{s_i} \left(\left\lfloor \mu_j^{(i)} \right\rfloor + 1 \right) E_j^{(i)} \right)$$
$$= (\lambda_i + 1) \left\langle \mu_1^{(i)} \right\rangle + (\lambda_{i+1} + 1) \left\langle \mu_{s_i}^{(i)} \right\rangle - \sum_{j=1}^{s_i} \left\langle \mu_j^{(i)} \right\rangle \left(\left\lfloor \mu_j^{(i)} \right\rfloor + 1 \right) (b_j^{(i)})^2$$
$$+ \sum_{1 \le j < k \le s_i} \left(\left\langle \mu_j^{(i)} \right\rangle \left(\left\lfloor \mu_k^{(i)} \right\rfloor + 1 \right) + \left\langle \mu_k^{(i)} \right\rangle \left(\left\lfloor \mu_j^{(i)} \right\rfloor + 1 \right) \right),$$

and use formulae (7.6) for $\mu_j^{(i)}$'s. This completes the proof. □

8. Stringy invariants of compact toric surfaces

Stringy Hodge numbers $h_{\mathrm{str}}^{p,q}(X)$ of normal, projective complex varieties X with at worst Gorenstein quotient or toroidal singularities were introduced in [**8**] in an attempt to determine a suitable mathematical formulation (and generalization) for the numbers which are encoded into the Poincaré polynomial of the chiral and antichiral rings of the physical "integer charge orbifold theory". Batyrev generalized further this definition in [**6**] and made it work also for the case in which one allows X to have at worst log-terminal singularities. In this framework, ones has to introduce appropriate $\mathcal{E}_{\mathrm{str}}$-*functions* $\mathcal{E}_{\mathrm{str}}(X;u,v)$ instead which may be not even rational. As we shall see below, it is possible to express the stringy invariants of any compact toric surface X_Δ in terms of its combinatorial data (4.11) and (4.12), and to verify that the so-called *stringy Euler number* $e_{\mathrm{str}}(X_\Delta) := \lim_{u,v\to 1}\mathcal{E}_{\mathrm{str}}(X_\Delta;u,v)$ is an integer ≥ 3.

• **$\mathcal{E}$-polynomials.** As it was shown by Deligne in [**17**, §8], the cohomology groups $H^i(X,\mathbb{Q})$ of any complex variety X are equipped with a functorial *mixed Hodge stucture*. The same remains true if one works with cohomologies $H^i_c(X,\mathbb{Q})$ with compact supports. There exist namely an increasing *weight-filtration*

$$\mathcal{W}_\bullet:\quad 0=W_{-1}\subset W_0\subset W_1\subset\cdots\subset W_{2i-1}\subset W_{2i}=H^i_c(X,\mathbb{Q})$$

and a decreasing *Hodge-filtration*

$$\mathcal{F}^\bullet:\quad H^i_c(X,\mathbb{C})=F^0\supset F^1\supset\cdots\supset F^i\supset F^{i+1}=0,$$

such that $\mathcal{F}^\bullet$ induces a natural filtration

$$F^\alpha\left(Gr_k^{\mathcal{W}_\bullet}(H^i_c(X,\mathbb{C}))\right)=$$
$$\left(W_k\left(H^i_c(X,\mathbb{C})\right)\cap F^\alpha\left(H^i_c(X,\mathbb{C})\right)+W_{k-1}\left(H^i_c(X,\mathbb{C})\right)\right)\,/\,W_{k-1}\left(H^i_c(X,\mathbb{C})\right)$$

(denoted again $\mathcal{F}^\bullet$) on the complexification of the graded pieces

$$Gr_k^{\mathcal{W}_\bullet}(H^i_c(X,\mathbb{Q}))=W_k/W_{k-1}.$$

Let now

$$h^{\alpha,\beta}\left(H^i_c(X,\mathbb{C})\right):=\ \dim_{\mathbb{C}} Gr^\alpha_{\mathcal{F}^\bullet}Gr^{\mathcal{W}_\bullet}_{\alpha+\beta}\left(H^i_c(X,\mathbb{C})\right)$$

denote the corresponding *Hodge-Deligne numbers.* The so-called $\mathcal{E}$*-polynomial* of X is defined as follows:

$$\mathcal{E}(X;u,v):=\sum_{\alpha,\beta}\left(\sum_{i\geq 0}\ (-1)^i\ \ h^{\alpha,\beta}\left(H^i_c(X,\mathbb{C})\right)\right)u^\alpha v^\beta\in\mathbb{Z}[u,v].$$

In fact, the $\mathcal{E}$-polynomial is to be viewed as "generating function" of these numbers. In particular, if X happens to be projective, equipped with a *pure Hodge structure*, then

$$\mathcal{E}(X;u,v)=\sum_{\alpha,\beta}(-1)^{\alpha+\beta}\,h^{\alpha,\beta}(X)\,u^\alpha v^\beta,\tag{8.1}$$

where $h^{\alpha,\beta}(X)$ denote the *usual* Hodge numbers w.r.t. this structure.

• **$\mathcal{E}_{\mathrm{str}}$-functions.** Allowing the existence of log-terminal singularities to pass to stringy invariants, one takes essentially into account the discrepancy coefficients.

Definition 8.1. Let $\varphi:\widetilde{X}\longrightarrow X$ denote an *snc*-desingularization of a $\mathbb{Q}$-Gorenstein normal complex variety X, that is, a desingularization of X whose exceptional locus $\mathrm{Exc}(\varphi)=\cup_{i=1}^l D_i$ consists of **s**mooth prime divisors $D_1,\ldots,D_l$ with only **n**ormal **c**rossings. Setting $L:=\{1,2,\ldots,l\}$, assume that X has at worst log-terminal singularities, i.e., discrepancy divisor

$$K_{\widetilde{X}}-\varphi^*(K_X)=\sum_{j=1}^{l}\ \eta_j\ D_j,$$

with $\eta_j > -1$ for all $j \in L$. For every subset $J \subseteq L$ we introduce the following notation:

$$D_J := \begin{cases} \widetilde{X}, & \text{if } J = \varnothing \\ \bigcap_{j \in J} D_j, & \text{if } J \neq \varnothing \end{cases} \quad \text{and} \quad D_J^\circ := D_J \smallsetminus \bigcup_{j \in L \smallsetminus J} D_j \;.$$

The algebraic function

$$\boxed{\mathcal{E}_{\text{str}}(X; u, v) := \sum_{J \subseteq L} \mathcal{E}(D_J^\circ; u, v) \prod_{j \in J} \tfrac{uv-1}{(uv)^{\eta_j+1}-1}} \tag{8.2}$$

(under the convention for $\prod_{j \in J}$ to be 1, if $J = \varnothing$, and $\mathcal{E}(\varnothing; u, v) := 0$) is called the *stringy $\mathcal{E}$-function* of X.

The main result of Batyrev in [**6**] says that:

THEOREM 8.2. *The stringy $\mathcal{E}$-function $\mathcal{E}_{\text{str}}(X; u, v)$ is independent of the choice of the snc-desingularization $\varphi : \widetilde{X} \longrightarrow X$.*

REMARK 8.3. (a) To define (8.2) it is sufficient for $\varphi : \widetilde{X} \longrightarrow X$ to fulfil the snc-condition only for those D_j's for which $\eta_j \neq 0$.

(b) If X admits a *crepant* desingularization $\varphi : \widetilde{X} \longrightarrow X$, i.e., $K_{\widetilde{X}} = \varphi^* K_X$ with $\widetilde{X}$ nonsingular, then $\mathcal{E}_{\text{str}}(X; u, v) = \mathcal{E}(\widetilde{X}; u, v)$.

(c) In general, $\mathcal{E}_{\text{str}}(X; u, v)$ may be not a rational function in the two variables u, v. Nevertheless, if X has at worst Gorenstein singularities, then

$$\mathcal{E}_{\text{str}}(X; u, v) \in \mathbb{Z}[\![u, v]\!] \cap \mathbb{Q}(u, v).$$

(Of course, for X projective, *stringy Hodge numbers* $h_{\text{str}}^{\alpha,\beta}(X)$ can be defined only if $\mathcal{E}_{\text{str}}(X; u, v) \in \mathbb{Z}[u, v]$).

(d) Since all $\mathbb{Q}$-Gorenstein toric varieties have at worst log-terminal singularities (see Note 2.6 (b)), their stringy $\mathcal{E}$-function is defined by (8.2).

DEFINITION 8.4. One defines the rational number

$$\boxed{e_{\text{str}}(X) := \lim_{u,v \to 1} \mathcal{E}_{\text{str}}(X; u, v) = \sum_{J \subseteq L} e(D_J^\circ) \prod_{j \in J} \frac{1}{\eta_j + 1}} \tag{8.3}$$

as the *stringy Euler number* of X.

• **Back to compact toric surfaces**. Let X_Δ be a compact toric surface constructed by a two-dimensional complete fan Δ (as in § 4). We intend to compute the stringy invariants of X_Δ in terms of its combinatorial data. At first, it should be mentioned that X_Δ, as an orbifold, is endowed with a canonical *pure Hodge structure*, with Hodge numbers

$$h^{\alpha,\beta}(X_\Delta) = \dim_{\mathbb{C}} H^\beta(X_\Delta, \widehat{\Omega}^\alpha_{X_\Delta}), \quad \alpha, \beta \in \{0, 1, 2\},$$

where $\widehat{\Omega}^\alpha_{X_\Delta} := \iota_* \Omega^\alpha_{X_\Delta \smallsetminus \text{Sing}(X_\Delta)}$ denotes *the Zariski sheaf of germs of α-forms* (see [**42**, Thm. 3.6, pp. 121-122]), and $\iota : X_\Delta \smallsetminus \text{Sing}(X_\Delta) \hookrightarrow X_\Delta$ is the open embedding of the regular locus of X_Δ into itself. This is due to the fact that the so-called *Danilov's spectral sequence*

$$\mathbf{E}_1^{\alpha,\beta} = H^\beta(X_\Delta, \widehat{\Omega}^\alpha_{X_\Delta}) \Longrightarrow H^{\alpha+\beta}(X_\Delta, \mathbb{C}),$$

(cf. [**15**, § 12] and [**42**, p. 133]), degenerates at the $\mathbf{E}_1$-term, consituting a direct analogue of the (usual) Hodge spectral sequence for the case in which one works with projective complex *manifolds*. Moreover, by [**42**, Thm. 3.11], we have

$$h^{\alpha,\beta}\left(X_{\Delta}\right)=\begin{cases}0, & \text{if } \alpha\neq\beta,\\ \\ \sum_{j=0}^{2}\left(-1\right)^{\alpha}\binom{2-j}{\alpha-j}\sharp(\Delta(j)), & \text{if } \alpha=\beta.\end{cases}\tag{8.4}$$

THEOREM 8.5 (Stringy invariants of X_Δ). *The stringy $\mathcal{E}$-function of X_Δ equals*

$$\boxed{\begin{array}{l}\mathcal{E}_{\rm str}\left(X_{\Delta};u,v\right)=1+\left(\nu-2\right)uv+\left(uv\right)^{2}\\ +\sum\limits_{i\in I_{\Delta}}\left(\left(uv\right)^{2}\sum\limits_{j=0}^{q_{i}-1}\left(uv\right)^{-\frac{\left[j\left(q_{i}-p_{i}+1\right)\right]_{q_{i}}}{q_{i}}}-1\right)\end{array}}\tag{8.5}$$

In particular, the stringy Euler number of X_Δ is always a positive integer ≥ 3, because

$$\boxed{e_{\rm str}\left(X_{\Delta}\right)=\nu+\sum_{i\in I_{\Delta}}\left(q_{i}-1\right)}\tag{8.6}$$

PROOF. Since $\sharp(\Delta(0))=1$, $\sharp(\Delta(1))=\sharp(\Delta(2))=\nu$, by formulae (8.4) we get

$$h^{\alpha,\alpha}\left(X_{\Delta}\right)=\begin{cases}1, & \text{if } \alpha\in\{0,2\},\\ \nu-2, & \text{if } \alpha=1.\end{cases}$$

Hence, by (8.2) and (8.1),

$$\mathcal{E}_{\rm str}\left(X_{\Delta};u,v\right)=\mathcal{E}\left(X_{\Delta};u,v\right)+\sum_{i\in I_{\Delta}}\left(\mathcal{E}_{\rm str}\left(\left(X_{\Delta},\mathrm{orb}\left(\sigma_{i}\right)\right);u,v\right)-1\right)$$

$$=\sum_{0\leq\alpha,\beta\leq 2}\left(-1\right)^{\alpha+\beta}h^{\alpha,\beta}\left(X_{\Delta}\right)u^{\alpha}v^{\beta}+\sum_{i\in I_{\Delta}}\left(\mathcal{E}_{\rm str}\left(U_{i};u,v\right)-1\right)$$

$$=1+\left(\nu-2\right)uv+\left(uv\right)^{2}+\sum_{i\in I_{\Delta}}\left(\left(uv\right)^{2}\sum_{j=0}^{q_{i}-1}\left(uv\right)^{-\frac{\left[j\left(q_{i}-p_{i}+1\right)\right]_{q_{i}}}{q_{i}}}-1\right)$$

where for the last equality one applies [**7**, Lemma 7.4, pp. 28-29], i.e., that the stringy function of each $U_i\cong\mathbb{C}^2/G_i$, $i\in I_\Delta$, is nothing but the so-called *orbifold $\mathcal{E}$-function* of the quotient space $\mathbb{C}^2/G_i$, under the consideration of the element $g_i:=\mathrm{diag}(\zeta_{q_i}^{(q_i-p_i)},\zeta_{q_i})$ as the distinguished generator of the cyclic subgroup G_i of $\mathrm{GL}(2,\mathbb{C})$ which acts on $\mathrm{T}^{\rm hol}_{X_\Delta,\mathrm{orb}(\sigma_i)}\cong\mathbb{C}^2$ as follows:

$$G\times\mathbb{C}^{2}\ni\left(g_{i}^{j},(z_{1},z_{2})\right)\longmapsto\left(\zeta_{q_{i}}^{j(q_{i}-p_{i})}z_{1},\zeta_{q_{i}}^{j}z_{2}\right)\in\mathbb{C}^{2},$$

$\forall j\in\{0,1,\ldots,q_i-1\}$. To compute $e_{\rm str}\,(X_\Delta)$ one can take the limit of $\mathcal{E}_{\rm str}\,(X_\Delta;u,v)$ whenever $u,v\to 1$, or, alternatively, make use of [**7**, Corollary 7.6, p. 30]:

$$\begin{aligned} e_{\rm str}\,(X_\Delta) &= e(X_\Delta \smallsetminus {\rm Sing}(X_\Delta)) + \sum_{i\in I_\Delta} |G_i| \\ &= e(X_\Delta) - \sharp(I_\Delta) + \sum_{i\in I_\Delta} q_i \\ &= \nu + \sum_{i\in I_\Delta} (q_i-1)\,. \end{aligned}$$

Thus, $e_{\rm str}\,(X_\Delta)$ is always a positive integer $\geq \nu \geq 3$. □

REMARK 8.6. (a) Since

$$e_{\rm str}\,(X_\Delta) = e(X_\Delta) + \sum_{i\in I_\Delta} \left(e_{\rm str}\,((X_\Delta, {\rm orb}\,(\sigma_i))) - 1\right),$$

working directly with the initial definition (8.3) and with the (good) minimal desingularization (4.8) of X_Δ, we obtain

$$e_{\rm str}\,((X_\Delta, {\rm orb}\,(\sigma_i))) = e_{\rm str}\,(U_i)$$

$$= e\,(U_i) + \sum_{j=1}^{s_i} e((E_j^{(i)})^\circ)\tfrac{q_i}{\gamma_j^{(i)}+\delta_j^{(i)}} + \sum_{j=1}^{s_i-1} e(E_j^{(i)}\cap E_{j+1}^{(i)})\tfrac{q_i^2}{(\gamma_j^{(i)}+\delta_j^{(i)})(\gamma_{j+1}^{(i)}+\delta_{j+1}^{(i)})} - 1,$$

by (4.9) and (4.13). Since

$$e\,(U_i)=1,\quad e(E_j^{(i)}) = e(\mathbb{P}^1_{\mathbb{C}}) = 2,\quad e(E_j^{(i)}\cap E_{j+1}^{(i)}) = 1,$$

and

$$e((E_j^{(i)})^\circ) = \begin{cases} 1, & \text{if } j=1 \text{ and } s_i>1, \\ 0, & \text{if } j\in\{2,\ldots,s_i-1\} \text{ and } s_i>2, \\ 1, & \text{if } j=s_i \text{ and } s_i>1, \\ 2, & \text{if } j=s_i=1, \end{cases}$$

we deduce finally that

$$e_{\rm str}\,(U_i) = \begin{cases} q_i, & \text{if } s_i=1, \\ q_i\left(\tfrac{1}{q_i-p_i+1} + \tfrac{1}{q_i-\widehat{p}_i+1} + \sum_{j=1}^{s_i-1} \tfrac{q_i}{(\gamma_j^{(i)}+\delta_j^{(i)})(\gamma_{j+1}^{(i)}+\delta_{j+1}^{(i)})}\right), & \text{if } s_i>1, \end{cases}$$

and

$$\begin{aligned} e_{\rm str}\,(X_\Delta) &= \nu + \sum_{i\in I_\Delta \text{ with } s_i=1} (q_i-1) \\ &+ \sum_{i\in I_\Delta \text{ with } s_i>1} (q_i(\tfrac{1}{q_i-p_i+1} + \tfrac{1}{q_i-\widehat{p}_i+1} + \sum_{j=1}^{s_i-1} \tfrac{q_i}{(\gamma_j^{(i)}+\delta_j^{(i)})(\gamma_{j+1}^{(i)}+\delta_{j+1}^{(i)})}) - 1). \end{aligned} \tag{8.7}$$

Comparing formulae (8.6) and (8.7), and taking into account that the two computational methods are of local nature (i.e., focused on each of the involved singularities severally) we get the (nontrivial) identity

$$\tfrac{1}{q_i-p_i+1} + \tfrac{1}{q_i-\widehat{p}_i+1} + \sum_{j=1}^{s_i-1} \tfrac{q_i}{(\gamma_j^{(i)}+\delta_j^{(i)})(\gamma_{j+1}^{(i)}+\delta_{j+1}^{(i)})} = 1,$$

for all $i\in I_\Delta$ with $s_i>1$. This identity is exactly what one needs for the understanding of the "intrinsic" role played by the orders of the inertia subgroups of G_i's (corresponding to the irreducible components of codimension 1 in the ramification locus of the covering $\mathbb{C}^2 \longrightarrow \mathbb{C}^2/G_i$) during the direct computation which has been performed in Theorem 8.5 (cf. [**7**, Proof of Lemma 7.4, pp. 28-29]).

(b) The reader should not confuse the *stringy Euler number* $e_{\rm str}(X_\Delta)$ with $c_2(\widehat{\Omega}^1_{X_\Delta})$, which has been named *orbifold Euler number* by some authors (see, e.g., [**10**, Thm. 7.3, p. 332] and [**54**, Remark 0.6, p. 117]). In our case,

$$c_2(\widehat{\Omega}^1_{X_\Delta}) = e(X_\Delta) - \sum_{i\in I_\Delta}\left(1-\frac{1}{|G_i|}\right) = \nu - \sum_{i\in I_\Delta}\left(1-\frac{1}{q_i}\right).$$

It is worthwhile mentioning that the stringy Euler number of any projective surface with quotient singularities is a topological invariant, whereas the orbifold Euler number is not (in general).

Corollary 8.7. *If X_Δ is Gorenstein, then $\mathcal{E}_{\rm str}(X_\Delta; u, v)$ is a polynomial, and the stringy Hodge numbers of X_Δ are the following non-negative integers*:

$$\boxed{h^{\alpha,\beta}_{\rm str}(X_\Delta) = \begin{cases} 0, & \text{if } \alpha \neq \beta, \\ 1, & \text{if } \alpha=\beta=0, \\ \nu-2, & \text{if } \alpha=\beta=1, \\ 1+\sum\limits_{i\in I_\Delta} s_i, & \text{if } \alpha=\beta=2. \end{cases}} \tag{8.8}$$

Proof. If X_Δ is Gorenstein, then $p_i = 1$ and $s_i = q_i - 1$ for all $i \in I_\Delta$, and the function (8.5) is a polynomial whose coefficients are those given in (8.8). □

Note 8.8. (a) For the study of local contibution to the stringy $\mathcal{E}$-function of surface singularities which are log-terminal but not cyclic quotient singularities, as well as for a natural generalization of the definition (8.2) of stringy $\mathcal{E}$-function for wider classes of surface singularities, the reader is referred to Veys' article [**54**].

(b) For *every* toric variety X_Δ (of arbitrary dimension) with at worst Gorenstein singularities, $\mathcal{E}_{\rm str}(X_\Delta; u, v)$ is always a polynomial, as it was shown in [**6**, Proposition 4.4, pp. 12-13].

Acknowledgments. The author would like to thank V.V. Batyrev and M. Henk for valuable discussions on various questions concerning the combinatorics involved in the study of log-terminal cones (during the week in which the Conference on "Algebraic and Geometric Combinatorics" was held in the academic village of Anogia), as well as the referee for useful suggestions.

References

[1] Alexeev V.A. & Nikulin V.V: *Classification of log Del Pezzo surfaces of index* ≤ 2, archiv: math.AG/0406536v5, December 29, 2005.

[2] Bădescu L.: *Algebraic Surfaces*, Universitext, Springer-Verlag, 2001.

[3] Barth W.P., Hulek C., Peters C.A.M. & Van de Ven A.: *Compact Complex Surfaces*, Erg. der Math. und ihrer Grenzgebiete, 3 Folge, Bd. **4**, Second Ed., Springer-Verlag, 2004.

[4] Batyrev V.V.: *Toroidal Fano threefolds*, Math. USSR Izvestija **19** (1982), 13-25.

[5] ———, *Higher dimensional toric varieties with ample anticanonical class*, PhD Thesis (in Russian), Moscow State University, 1985.

[6] ———, *Stringy Hodge numbers of varieties with Gorenstein canonical singularities*. In: "Integrable Systems and Algebraic Geometry", Proc. of Taniguchi Symposium, (edited by M.H. Saito, Y. Shimizu, and K. Ueno), World Scientific Pub. Co., 1998, pp. 1-32.

[7] ———, *Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs,* Journal of European Math. Soc. **1** (1999), 5-33.

[8] Batyrev V.V. & Dais D.I.: *Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry*, Topology **35** (1996), 901-929.

[9] Beauville A.: *Complex Algebraic Surfaces*, Second Edition, L.M.S., Student Texts, Vol. **34**, Cambridge University Press, 1996.

[10] Blache R.: *Riemann-Roch theorem for normal surfaces and applications*, Abh. Math. Sem. Univ. Hamburg **65** (1995), 307-340.

[11] Borisov A. & Borisov L.: *Singular toric Fano varieties*, Izvestija Acad. Sci. USSR Sb. Math. **75** (1993), 227-283.

[12] Brenton L.: *On the Riemann-Roch equation for singular complex surfaces*, Pacific Jour. of Math. **71** (1977), 299-312.

[13] Brieskorn E.: *Über die Auflösung gewisser Singularitäten von holomorphen Abbildungen*, Math. Ann. **166** (1966), 76-102.

[14] Dais D.I., Haus U.-U. & Henk M.: *On crepant resolutions of 2-parameter series of Gorenstein cyclic quotient singularities*, Results in Math. **33** (1998), 208-265.

[15] Danilov V.I.: *The geometry of toric varieties*, Russian Math. Surveys **33** (1978), 97-154.

[16] Dedekind R.: *Erläuterungen zu zwei Fragmenten von Riemann.* In: Riemann's "Gesammelte Mathematische Werke", 2. Aufl., (1892), [reprinted by Dover Pub., 1953], pp. 466-478. (See also: Dedekind's "Gesammelte Mathematische Werke", Band II, Vieweg, 1930, pp. 159-173.)

[17] Deligne P.: *Théorie de Hodge* **III**, Publ. Math. I.H.E.S. **44** (1975), 5-77.

[18] Fulton W.: *Introduction to Toric Varieties*, Annals of Mathematics Studies, Vol. **131**, Princeton University Press, 1993.

[19] ———, *Intersection Theory*, Springer-Verlag, Second Edition, 1998.

[20] Grauert H.: *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **124** (1962), 331-368. [See also: "Collected Papers", Volume **I**, Springer-Verlag, 1994, pp. 271-308.]

[21] Griffiths Ph. & Harris J.: *Principles of Algebraic Geometry*, J. Wiley & Sons, Inc., 1978.

[22] Hartshorne R.: *Ample Subvarieties of Algebraic Varieties*, Lecture Notes in Mathematics, Vol. **156**, Springer-Verlag, 1970.

[23] ———, *Algebraic Geometry*, Graduate Texts in Mathematics, Vol. **52**, Springer-Verlag, 1977.

[24] ———, *Stable reflexive sheaves*, Math. Ann. **254** (1980), 121-176.

[25] Hensley D.: *Lattice vertex polytopes with interior lattice points*, Pacific Jour. of Math. **105** (1983), 183-191.

[26] Hidaka F. & Watanabe K.: *Normal Gorenstein surfaces with ample anticanonical divisor*, Tokyo Jour. of Math. **4** (1981), 319-330.

[27] Hirzebruch F.: *Über eine Klasse von einfach-zusammenhängenden komplexen Mannigfaltigkeiten*, Math. Ann. **124**, (1951), 1-22. [See also: "Gesammelte Abhandlungen", Band **I**, Springer-Verlag, 1987, pp. 1-11.]

[28] ———, *Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen*, Math. Ann. **126**, (1953), 1-22. [See also: "Gesammelte Abhandlungen", Band **I**, Springer-Verlag, 1987, pp. 11-32.]

[29] Hirzebruch F.& Zagier D.: *The Atiyah-Singer Theorem and Elementary Number Theory.* Math. Lecture Series, Vol. **3**, Publish or Perish Inc., 1974.

[30] Ishida M.-N.: *Torus embeddings and dualizing complexes*, Tôhoku Math. Jour. **32** (1980), 111-146.

[31] Knöller F.W.: *Zweidimensionale Singularitäten und Differentialformen*, Math. Ann. **206** (1973), 205-213.

[32] Koelman R.J.: *The number of moduli of families of curves on toric varieties*, Katholieke Universiteit te Nijmegen, PhD Thesis, 1991, `ISBN 90-9004155-9`.

[33] Kollár J. & Kovács S.: *Birational geometry of log surfaces.* Notes from the "Seminar on Moduli of Surfaces", University of Utah, Summer 1994; available as e-print at the address: `http://www.math.princeton.edu/~kollar/`

[34] Lagarias J. & Ziegler G.M.: *Bounds for lattice polytopes containing a fixed number of interior points in a sublattice*, Canadian J. Math. **43** (1991), 1022-1035.

[35] Laufer H.: *Normal Two-Dimensional Singularities*, Annals of Math. Studies, Vol. **71**, Princeton University Press, 1971.

[36] Matsuki K: *Introduction to Mori's Program*, Universitext, Springer-Verlag, 2002.

[37] Miyanishi M: *Open Algebraic Surfaces*, CRM Monograph Series, Vol. **12**, A.M.S., 2001.

[38] Mumford D.: *The topology of normal singularities of an algebraic surface and a criterion of simplicity*, Publ. Math. I.H.E.S. **9** (1961), 5-22.

[39] Myerson G.: *On semi-regular finite continued fractions,* Archiv der Mathematik (Basel) **48** (1987), 420-425.

[40] Nill B.: *Gorenstein toric Fano varieties*, PhD Thesis, Universität Tübingen, 2005.

[41] Oda T.: *Lectures on Torus Embeddings and Applications.* (Based on joint work of the author with K. Miyake), Tata Institute of Fundamental Research, Springer-Verlag, 1978.

[42] ______, *Convex Bodies and Algebraic Geometry. An Introduction to the Theory of Toric Varieties*. Erg. der Math. und ihrer Grenzgebiete, 3 Folge, Bd. **15**, Springer-Verlag, 1988.

[43] Perron O.: *Die Lehre von den Kettenbrüchen*, Bd. **I**, Dritte Auflage, Teubner, 1954.

[44] Pommersheim J.E.: *Toric varieties, lattice points and Dedekind sums*, Math. Ann. **295** (1993), 1-24.

[45] Poonen B. & Rodriguez-Villegas F.: *Lattice polygons and the number* 12, Am. Math. Monthly **107** (2000), 238-250.

[46] Rademacher H. & Grosswald E.: *Dedekind Sums.* Carus Math. Monographs **16**, Math. Assoc. Amer., 1972.

[47] Reid M.: *Young person's guide to canonical singularities.* In : "Algebraic Geometry, Bowdoin 1985", (edited by S.J.Bloch), Proc. of Symp. in Pure Math., A.M.S., Vol. **46**, Part I, 1987, pp. 345-416.

[48] Riemenschneider O.: *Deformationen von Quotientensingularitäten (nach zyklischen Gruppen)*, Math. Ann. **209** (1974), 211-248.

[49] Sakai F.: *Anticanonical models of rational surfaces*, Math. Ann. **269** (1984), 389-410.

[50] ______, *Weil divisors on normal surfaces*, Duke Math. Jour. **51** (1984), 877-887.

[51] ______, *The structure of normal surfaces*, Duke Math. Jour. **52** (1985), 627-648.

[52] ______, *Classification of normal surfaces.* In : "Algebraic Geometry, Bowdoin 1985", (edited by S.J.Bloch), Proc. of Symp. in Pure Math., A.M.S., Vol. **46**, Part I, 1987, pp. 451-465.

[53] Uspensky J.V. & Heaslet M.A.: *Elementary Number Theory*, McGraw-Hill, 1939.

[54] Veys W.: *Stringy invariants of normal surfaces*, Journal of Alg. Geom. **13** (2004), 115-141.

[55] Watanabe K. & Watanabe M.: *The classification of Fano threefolds with torus embeddings*, Tokyo J. Math. **5** (1982), 37-48.

[56] Ye Q.: *On Gorenstein log Del Pezzo surfaces*, Japanese J. of Math. **28** (2002), 87-136.

University of Crete, Department of Mathematics, Division Algebra and Geometry, Knossos Avenue, P.O. Box 2208, GR-71409, Heraklion, Crete, Greece

E-mail address: `ddais@math.uoc.gr`