On the Existence of Crepant Resolutions of Gorenstein Abelian Quotient Singularities in Dimensions ≥ 4

Dimitrios I. Dais, Martin Henk, and Günter M. Ziegler

ABSTRACT. For which finite subgroups G of $\mathrm{SL}(r,\mathbb{C})$, $r \geq 4$, are there *crepant desingularizations* of the quotient space $\mathbb{C}^r/G$? A complete answer to this question (also known as "Existence Problem" for such desingularizations) would classify all those groups for which the high-dimensional versions of *McKay correspondence* are valid. In the paper we consider this question in the case of *abelian* finite subgroups of $\mathrm{SL}(r,\mathbb{C})$ by using techniques from toric and discrete geometry. We give two necessary existence conditions, involving the Hilbert basis elements of the cone supporting the junior simplex, and an Upper Bound Theorem, respectively. Moreover, to the known series of Gorenstein abelian quotient singularities admitting projective, crepant resolutions (which are briefly recapitulated) we add a new series of non-c.i. cyclic quotient singularities having this property.

1. Introduction

The *McKay correspondence* can be understood as a "bridge" between the irreducible representations (or, dually, the conjugacy classes) of finite subgroups G of the special linear group $\mathrm{SL}(r,\mathbb{C})$ and the (co)homology of $\widehat{X}$'s, for any crepant desingularization $\widehat{X} \longrightarrow X$ of $X = \mathbb{C}^r/G$. (Here, *crepant* simply means that the canonical divisor $K_{\widehat{X}}$ of $\widehat{X}$ is trivial.) Before we are going to focus on the constant companion of this correspondence (i.e., the so-called "Existence Problem", whenever $r \geq 4$), let us briefly recall some basic facts about quotient singularities and summarize both classical and recent results concerning it.

• **Quotient singularities**. Let G be a finite subgroup of $\mathrm{GL}(r,\mathbb{C})$ which is *small*, i.e., with no pseudoreflections, acting linearly on $\mathbb{C}^r$, and let

$$\varpi : \mathbb{C}^r \longrightarrow \mathbb{C}^r/G = \mathrm{Spec}(\mathbb{C}\,[\mathfrak{x}_1, \ldots, \mathfrak{x}_r]^G)$$

be the quotient map. Denote by $(\mathbb{C}^r/G, [\mathbf{0}])$ the corresponding *quotient singularity* as germ at $[\mathbf{0}] := \varpi(\mathbf{0})$. Quotient singularities are known to be normal and Cohen-Macaulay (see [**80**, p. 129], [**61**, Proposition 13], and [**115**, Thm. 3.2]).

2000 *Mathematics Subject Classification.* 14E15 (Primary); 14M25, 52B20 (Secondary).

PROPOSITION 1.1 (Singular locus). *If G is a small finite subgroup of* $\mathrm{GL}(r,\mathbb{C})$*, then the singular locus of the entire geometric quotient space* $\mathbb{C}^r/G$ *equals*

$$\mathrm{Sing}\,(\mathbb{C}^r/G) = \varpi\,(\{\mathbf{z}\in\mathbb{C}^r \mid G_{\mathbf{z}} \neq \{\mathrm{Id}\}\})$$

where $G_{\mathbf{z}} := \{g\in G \mid g\cdot\mathbf{z} = \mathbf{z}\}$ *is the isotropy group of* $\mathbf{z} = (z_1,\ldots,z_r)\in\mathbb{C}^r$.

THEOREM 1.2 (Prill's isomorphism criterion, [**99**, Thm. 2]). *Let G_1, G_2 be two small finite subgroups of* $\mathrm{GL}(r,\mathbb{C})$, $r\geq 2$. *Then there exists an analytic isomorphism* $(\mathbb{C}^r/G_1,[\mathbf{0}])\cong(\mathbb{C}^r/G_2,[\mathbf{0}])$ *(i.e.,* $\mathcal{O}_{\mathbb{C}^r/G_i,[\mathbf{0}]} = \mathbb{C}\{\mathfrak{x}_1,\ldots,\mathfrak{x}_r\}^{G_i}$, $i=1,2$, *are isomorphic as local* $\mathbb{C}$*-algebras) if and only if G_1 and G_2 are conjugate to each other within* $\mathrm{GL}(r,\mathbb{C})$.

Hence, the classification of quotient singularities is reduced to the classification (up to conjugacy) of the small finite subgroups of $\mathrm{GL}(r,\mathbb{C})$, or equivalently, to those of $\mathrm{U}(r,\mathbb{C})$, as it follows from the next Lemma.

LEMMA 1.3 ([**114**, Lemma 4.2.15 (i), p. 82.]). *If G is a finite subgroup of* $\mathrm{GL}(r,\mathbb{C})$ *(resp., of* $\mathrm{SL}(r,\mathbb{C})$*), then G is conjugate in* $\mathrm{GL}(r,\mathbb{C})$ *(resp., in* $\mathrm{SL}(r,\mathbb{C})$*) to a finite subgroup of* $\mathrm{U}(r,\mathbb{C})$ *(resp., of* $\mathrm{SU}(r,\mathbb{C})$*).*

Subclasses of quotient singularities $(\mathbb{C}^r/G,[\mathbf{0}])$ of special theoretical value are those dictated by the hierachy of the local Noetherian rings $\mathcal{O}_{\mathbb{C}^r/G,[\mathbf{0}]}$ (cf. [**78**, §VI.3]):

$$\boxed{\text{(complete intersection (“c.i”))} \Longrightarrow \text{(Gorenstein)} \Longrightarrow \text{(Cohen-Macaulay)}}$$

• **Gorenstein quotient singularities**. These are characterized as follows:

THEOREM 1.4 (cf. [**60, 123**]). $(\mathbb{C}^r/G,[\mathbf{0}])$ *is a Gorenstein quotient singularity if and only if*[1] $G\subset\mathrm{SL}(r,\mathbb{C})$.

NOTE 1.5 (Classification for small r). The finite subgroups of $\mathrm{SL}(r,\mathbb{C})$ have been completely classified (up to conjugacy) only for r small.

(i) For $r=2$ the classification appears already in Klein's book “Vorlesungen über das Ikosaeder” [**75**] in 1884. (See also [**80**, Thm. on p. 35] and [**114**, §4.4]). The list contains the cyclic groups $\mathbf{Cyc}_k$ of order $k\geq 2$, the binary dihedral groups $\mathbf{Dih}_{n-2}$ of order $4(n-2)$, $n\geq 4$, and the binary tetrahedral, octahedral and icosahedral groups $\mathbf{T},\mathbf{O},\mathbf{I}$, having orders $24, 48$ and 120, respectively.

(ii) For $r=3$ the main part of the classification goes back to works of Jordan, Klein, Gordan, Maschke, Valentiner, Wiman, Gerbaldi, and Blichfeldt, written around the end of the 19th century. For the corresponding list of groups we refer to Blichfeldt's book [**8**], as well as to Yau & Yu [**125**] (containing some useful updates).

(iii) For $r=4$ see Hanany & He [**56**]. In the group theory literature there are lots of scattered results concerning this topic for $r\in\{5,\ldots,10\}$ though they are far from giving complete classifications. One of the main problems seems to be the appearance of “individual” groups. For instance, already for $r=6$ we meet the complex irreducible representation of the Hall-Janko sporadic simple group of order 604800 within $\mathrm{SL}(6,\mathbb{C})$. Bigger r's lead to more “exotic” groups. On the other hand, there exist many families of finite subgroups of $\mathrm{SL}(r,\mathbb{C})$ (e.g., the abelian ones, dihedral-like groups, imprimitive groups etc) being present in *all* dimensions.

[1] Note that every finite subgroup of $\mathrm{SL}(r,\mathbb{C})$ is small (cf. [**115**, p. 503]).

Remark 1.6. Let g be an element of a finite subgroup G of $\mathrm{SL}(r,\mathbb{C})$. By Lemma 1.3 there exists an $h \in \mathrm{SL}(r,\mathbb{C})$ such that $hGh^{-1} \subset \mathrm{SU}(r,\mathbb{C})$. Since hgh^{-1} is a unitary matrix, it is known (see, e.g., [**94**, Thm. 10.2, p. 208]) that
(i) hgh^{-1} is unitary similar to a diagonal matrix,
(ii) the diagonal entries of this matrix are the eigenvalues of hgh^{-1}, and
(iii) the eigenvalues of hgh^{-1} have absolute value 1.
Thus, there is a suitable matrix $k \in \mathrm{U}(r,\mathbb{C})$, so that

$$k\left(hgh^{-1}\right)k^{-1} = (kh)\,g\,(kh)^{-1} \in \mathrm{SU}\,(r,\mathbb{C})$$

is of the form

$$(kh)\,g\,(kh)^{-1} = \ \mathrm{diag}(e^{2\pi\sqrt{-1}\ \gamma_1},\ldots,e^{2\pi\sqrt{-1}\ \gamma_r}),$$

for some $\gamma_1,\ldots,\gamma_r \in \mathbb{Q} \cap \ [0,1)$.

Definition 1.7 ("Ages" and "heights"). (i) The *age*[2] of an element $g \in G$ is defined to be the sum

$$\boxed{\mathrm{age}\,(g) := \gamma_1 + \gamma_2 + \cdots + \gamma_r.} \tag{1.1}$$

(ii) The *height* $\mathrm{ht}(g)$ of an element $g \in G$ is defined to be the rank

$$\boxed{\mathrm{ht}\,(g) := \mathrm{rank}(g - \mathrm{Id}_G).} \tag{1.2}$$

Proposition 1.8 ([**3**, Prop. 5.2.]). *For every $g \in G$ we have*

$$\mathrm{ht}\,(g) = \mathrm{ht}\left(g^{-1}\right) = \mathrm{age}\,(g) + \mathrm{age}\left(g^{-1}\right). \tag{1.3}$$

Note 1.9. (i) Obviously, $0 \leq \mathrm{age}(g) \leq r-1$, with $\mathrm{age}(g) = 0 \Longleftrightarrow g = \mathrm{Id}_G$, and $\mathrm{age}(g_1) = \mathrm{age}(g_2)$ for all pairs $(g_1,g_2) \in G \times G$ of group elements belonging to the same conjugacy class. Moreover, $2 \leq \mathrm{ht}(g) \leq r$, for all $g \in G \smallsetminus \{\mathrm{Id}_G\}$.
(ii) The group elements having age 1 (resp., age $i \geq 2$) are usually called *junior elements* (resp., *senior elements*) of G. (Correspondingly, by (i), we may speak of *junior* (resp., *senior*) *conjugacy classes*, or in general of *conjugacy classes of age* $i \in \{0,1,\ldots r-1\}$).
(iii) As it was pointed out by Ito & Reid [**67**, Thm. 1.3], the "Tate twist"

$$G\,(-1) := \ \mathrm{Hom}(\widehat{\mathbb{Z}}\,(1)\,,G),\ (\text{with}\ \ \widehat{\mathbb{Z}}\,(1) := \ \varprojlim\,(\mathbb{Z}\,/\,d\,\mathbb{Z})),$$

(which is isomorphic to G as long as one makes a concrete choice of roots of unity) has a canonical grading inherited by the ages of its elements, invariant under conjugacy in $G\,(-1)$ (or G), and it is essentially used in Theorem 1.18.
(iv) If r is even and $\mathbb{C}^r/G$ *symplectic* (i.e., $G \subset \mathrm{Sp}(r,\mathbb{C})$), let

$$\mathcal{F}_\bullet(\mathbb{C}[G]) = \{\,F_k(\mathbb{C}[G])|\,k \in \mathbb{Z}_{\geq 0}\}$$

denote the increasing filtration of the group algebra $\mathbb{C}[G]$ defined by setting

$$F_k(\mathbb{C}[G]) := \mathbb{C}\text{-span of }\{\,g \in G \mid \ \mathrm{ht}(g) \leq k\},\quad \forall k \in \mathbb{Z}_{\geq 0}.$$

[2]In [**2**, §6] and [**3**, §5] the age of an element $g \in G$ is called the *weight* of g. Here, we shall adopt the terminology of [**67**]. To the word *weight* we ascribe a different meaning. (See below Definition 3.1.)

$\mathcal{F}_\bullet(\mathbb{C}[G])$ is compatible with the algebra structure on $\mathbb{C}[G]$. Using the induced filtration $\mathcal{F}_\bullet(\mathcal{Z}G)$ on the center $\mathcal{Z}G$ of $\mathbb{C}[G]$ (see [**1, 49**]) one determines the associated graded algebra

$$\mathbf{gr}^{\mathcal{F}_\bullet}(\mathcal{Z}G) := \{\, F_{k+1}(\mathcal{Z}G)/F_k(\mathcal{Z}G) \mid \; k \in \mathbb{Z}_{\geq 0}\} \tag{1.4}$$

whose singificance is revealed in Theorem 1.21.

• **Quotient c.i.-singularities**. In the mid 1980's Nakajima & Watanabe [**92**], and independently Gordeev [**54**], classified the quotient singularities which are complete intersections ("c.i's") in *all* dimensions. Even to write down without further ado their group lists would demand several pages. Instead, let us remind a previous result which constitutes the foundation stone for their classification.

Theorem 1.10 (Kac & Watanabe [**68**]). *If* $(\mathbb{C}^r/G, [\mathbf{0}])$ *is a quotient c.i.- singularity, then* G *is generated by the set* $\{\, g \in G \mid \operatorname{ht}(g) \leq 2\}$.

• **McKay correspondence in dimension** $r = 2$. In dimension 2, the *classical McKay correspondence* exploits the elegance of the invariant theory of finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ and the uniqueness (and simple description) of the *minimal* desingularization of the quotient spaces $\mathbb{C}^2/G$.

Theorem 1.11 (cf. [**75**, II. 9-13], [**80**, Ch. II, §8], [**114**, §4.5]). *The quotient spaces* $\mathbb{C}^2/G = \mathrm{Spec}(\mathbb{C}[\mathfrak{x}_1, \mathfrak{x}_2]^G)$, *for* G *a finite subgroup of* $\mathrm{SL}(2, \mathbb{C})$, *are minimally embedded as hypersurfaces* $\left\{(z_1, z_2, z_3) \in \mathbb{C}^3 \,|\varphi(z_1, z_2, z_3) = 0\right\}$, *i.e.,*

$$\mathbb{C}[\mathfrak{x}_1, \mathfrak{x}_2]^G \cong \mathbb{C}[z_1, z_2, z_3] \,/\, (\varphi(z_1, z_2, z_3))\,.$$

(The normal form of the ideal generator is given in the 4th column of Table 1.)

Nr.	Groups	Type	$\varphi(z_1, z_2, z_3)$
1.	$\mathbf{Cyc}_{n+1},\ n \geq 1$	A_n	$z_1^{n+1} + z_2^2 + z_3^2$
2.	$\mathbf{Dih}_{n-2},\ n \geq 4$	D_n	$z_1^{n-1} + z_1 z_2^2 + z_3^2$
3.	$\mathbf{T}$	E_6	$z_1^4 + z_2^3 + z_3^2$
4.	$\mathbf{O}$	E_7	$z_1^3 z_2 + z_2^3 + z_3^2$
5.	$\mathbf{I}$	E_8	$z_1^5 + z_2^3 + z_3^2$

Table 1.

Theorem 1.12 (cf. [**37**], [**38**, §5], [**80**, Thm. 2 on p. 152]). *Let* G *be finite subgroup of* $\mathrm{SL}(2, \mathbb{C})$ *and* $X = \mathbb{C}^2/G$. *The minimal (= crepant) resolution*

$$(\widehat{X}, \mathrm{Exc}(f)) \xrightarrow{f} (X, [\mathbf{0}])$$

of the Gorenstein quotient singularity $(X, [\mathbf{0}])$ *has exceptional set* $\mathrm{Exc}(f)$ *consisting of a configuration of rational smooth curves with self-intersection number* -2. *The intersection form* $(\ ,\ ) : H_2(\widehat{X}, \mathbb{Z}) \times H_2(\widehat{X}, \mathbb{Z}) \to \mathbb{Z}$ *of* $\mathrm{Exc}(f)$ *is negative definite, and therefore the dual graphs* $\mathrm{DG}(\mathrm{Exc}(f))$ *of the irreducible components of* $\mathrm{Exc}(f)$ *are exactly the Dynkin diagrams (of simply connected complex Lie groups) of type* A-D-E. (See Tables 1 and 2.)

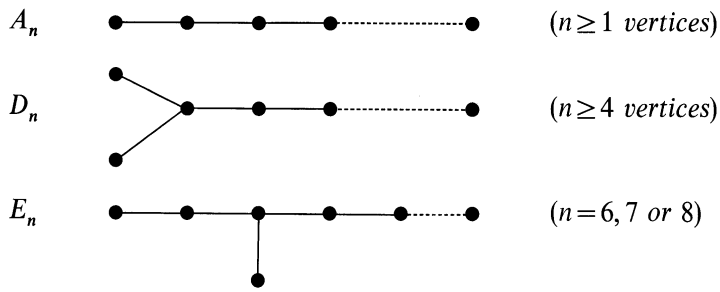

TABLE 2.

McKay [**87, 88**] established a remarkable connection between the representation theory of the finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ and the above Dynkin diagrams. This was the starting point for Gonzalez-Sprinberg, Verdier [**53**], and Knörrer [**76**], to construct a purely geometric, direct correspondence

$$\boxed{\mathbf{McK}(G; f) : \mathrm{Irr}^0\,(G) \longrightarrow \mathrm{DG}(\mathrm{Exc}\,(f))} \tag{1.5}$$

"of McKay-type" between the set $\mathrm{Irr}^0\,(G)$ of non-trivial irreducible representations of G and $\mathrm{DG}(\mathrm{Exc}(f))$ or, equivalently, between the irreducible representations of G and the members of the natural base of the cohomology ring $H^\bullet(\widehat{X}, \mathbb{Z})$ (cf. [**66**, §4]). The bijection $\mathbf{McK}(G; f)$ induces an isomorphism between $\mathrm{Irr}^0\,(G)$ and the graph $\mathrm{DG}(\mathrm{Exc}(f))$, i.e., the product of the images of two distinct elements of $\mathrm{Irr}^0\,(G)$ under $\mathbf{McK}(G; f)$ is mapped onto the exceptional prime divisor corresponding to the "right" graph vertex. Brylinski [**13**] constructed subsequently a canonical "dual" correspondence

$$\boxed{\mathbf{McK}(G; f)^{\mathrm{dual}} : \mathrm{DG}(\mathrm{Exc}\,(f)) \longrightarrow \left\{ \begin{array}{c} \text{Non-trivial conjugacy} \\ \text{classes of } G \end{array} \right\}} \tag{1.6}$$

relating the natural base of $H_\bullet(\widehat{X}, \mathbb{Z})$ to the set of conjugacy classes of G. Finally, Ito & Nakamura reinterpreted (1.5) in terms of[3] G-$\mathrm{Hilb}(\mathbb{C}^2)$:

THEOREM 1.13 (Hilbert scheme interpretation [**66**, Thm. 10.4, p. 190]). *$\widehat{X}$ is isomorphic to G-$\mathrm{Hilb}(\mathbb{C}^2)$ and (1.5) can be viewed as the bijection*

$$\mathrm{Irr}^0\,(G) \ni \rho \longmapsto D_\rho := \left\{ I \in G\text{-}\mathrm{Hilb}(\mathbb{C}^2) \,\middle|\, V(I) \supset V(\rho) \right\} \in \mathrm{DG}(\mathrm{Exc}\,(f)),$$

where $V(\rho)$ is the G-module corresponding to ρ, $\mathfrak{m} = \mathfrak{m}_{\mathbb{C}^2}$ (resp., $\mathfrak{m}_X$) the maximal ideal of $\mathbb{C}^2$ (resp., of X) at the origin, $\mathfrak{n} := \mathfrak{m}_X \mathcal{O}_{\mathbb{C}^2}$, and $V(I) := I/(\mathfrak{m}I + \mathfrak{n})$.

[3]In general, for $r \geq 2$, by a *cluster* in $\mathbb{C}^r$ is meant a zero-dimensional subscheme $Z \subset \mathbb{C}^r$, defined by an ideal $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{C}^r}$, so that $\mathcal{O}_Z = \mathcal{O}_{\mathbb{C}^r}/\mathcal{I}_Z$ is a finite dimensional $\mathbb{C}$-vector space. If G is a finite subgroup of $\mathrm{SL}(r, \mathbb{C})$ of order $l := |G|$, then every G-invariant cluster (G-*cluster*, for short) in $\mathbb{C}^r$ has global sections $H^0(Z, \mathcal{O}_Z)$ isomorphic (as $\mathbb{C}[G]$-module) to the regular representation of G. Consider the quasiprojective Hilbert scheme $\mathrm{Hilb}^l(\mathbb{C}^r)$ parametrizing all clusters Z of degree $\dim(\mathcal{O}_Z) = l$ (cf. [**106**, Lemma 5.1]), the Hilbert-Chow morphism $\mathrm{Hilb}^l(\mathbb{C}^r) \to \mathrm{Sym}^l(\mathbb{C}^r)$, as well as the unique irreducible component G-$\mathrm{Hilb}(\mathbb{C}^r)$ of the fixed locus $(\mathrm{Hilb}^l(\mathbb{C}^r))^G$ containing a general orbit of G on $\mathbb{C}^r$. Then the Hilbert-Chow morphism induces a proper birational morphism G-$\mathrm{Hilb}(\mathbb{C}^r) \to \mathbb{C}^r/G$, and G-$\mathrm{Hilb}(\mathbb{C}^r)$ parametrizes all G-clusters in $\mathbb{C}^r$. (That's why it is shortly called *the Hilbert scheme of G-clusters.*)

REMARK 1.14. From the above mentioned results, the following attributes of the quotient space $X = \mathbb{C}^2/G$ and of the desingularizing crepant morphism $f : \widehat{X} \longrightarrow X$ are worth recording:

(i) X is always minimally embedded as a *hypersurface* in $\mathbb{C}^3$.

(ii) The desingularizing crepant morphism f *exists* for all groups G of Table 1, and is uniquely determined over X up to *isomorphism*.

(iii) The singularity $[\mathbf{0}] \in X$ is *isolated* (and consequently all exceptional prime divisors w.r.t. f are compact).

(iv) f is a *projective* birational morphism (i.e., $\widehat{X}$ is always a quasiprojective complex variety), and can be decomposed into a finite sequence of blow-ups of *points.*

(v) $\widehat{X} \cong G\text{-Hilb}(\mathbb{C}^2)$.

None of (i)-(iv) "survives" *in general* in higher dimensions (and, as we explain below, G-Hilb$(\mathbb{C}^r)$ is *in general* a good choice for an $\widehat{X}$ *only for* $r = 3$). Nevertheless, passing to dimensions $r \geq 3$, it is useful to keep in mind under which additional conditions the one or the other property (or a reasonably *weakened version* thereof) is preserved.

• **McKay correspondence in dimension** $r = 3$. Based on the classification table [**125**] of the finite subgroups of $\mathrm{SL}(3, \mathbb{C})$, Ito [**63, 64**], Markushevich [**83, 84**], and Roan [**107**] provided (by a case-by-case thorough examination) a constructive proof of the following:

THEOREM 1.15 (Existence Theorem in Dimension 3). *All three-dimensional Gorenstein quotient singularities possess crepant resolutions.*

The resolution morphisms are unique only up to "isomorphism in codimension 1" (i.e., up to a finite number of canonical *flops*, cf. [**77**, §6.4]), and to win projectivity[4] one has to make particular choices (leading to *smooth* "minimal models"). Furthermore, for any such desingularization $f : \widehat{X} \longrightarrow X = \mathbb{C}^3/G$, $H^{\bullet}(\widehat{X}, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank equal to the number of conjugacy classes of G.

THEOREM 1.16 (McKay Correspondence over $\mathbb{Q}$ in dimension 3; [**3**, Prop. 5.6], [**67**, 1.5-1.6]). *For any crepant desingularization $f : \widehat{X} \longrightarrow X = \mathbb{C}^3/G$ there are canonical one-to-one correspondences*

$$\{\text{conjugacy classes of } G \text{ of age } i\} \longleftrightarrow \{\text{a basis of } H^{2i}(\widehat{X}, \mathbb{Q})\}. \tag{1.7}$$

Besides, in analogy to the two-dimensional case, it turns out that, among all possible projective crepant resolutions of X, the Hilbert scheme G-Hilb$(\mathbb{C}^3)$ of G-clusters is a *distinguished choice*. (See Nakamura [**93**], and Craw & Reid [**24**] for the abelian case, and [**10, 51, 52**] for the non-abelian case.) There are also several articles devoted to *intrinsic interpretations* of (1.7) for $\widehat{X} = G$-Hilb$(\mathbb{C}^3)$ in terms of the relevant ideals (see, e.g., [**22**] for a GIT- and [**65**] for a K-theoretic description). More recently, Craw [**21**, Thm. 1.1], working with a natural base of the cohomology ring of $\widehat{X}$ with *integer* coefficients, succeeded in establishing an explicit 3-*dimensional version of* (1.5) in the abelian case.

[4] According to a result of Kawamata & Matsuki [**73**], there is only a finite number of projective crepant desingularizations of $\mathbb{C}^3/G$.

• **McKay correspondence in higher dimensions**. In lack of space we recall only a few highlights and refer to the survey articles [**43, 82, 106**] for further reading.

There are many obstructions in generalizing McKay correspondence in dimensions $r \geq 4$, beginning with the Existence Problem (see comments below). But even if one *assumes* the existence of crepant desingularizations $f : \widehat{X} \longrightarrow X$ of a given $X = \mathbb{C}^r/G$, it is not -as yet- clear if there might be a *direct* analogue of (1.5) or (1.6) *over* $\mathbb{Z}$ (cf. [**105**, §1]). Most of the known results use homology and cohomology with coefficients taken from $\mathbb{Q}$ or $\mathbb{C}$.

Theorem 1.17 (Batyrev [**2**, Thm 8.4], Denef & Loeser [**34**, Corollary 5.3]). *If G is a finite subgroup of* $\mathrm{SL}(r,\mathbb{C})$, *then for any crepant desingularization $\widehat{X} \longrightarrow X$ of $X = \mathbb{C}^r/G$ we have*

$$\dim_{\mathbb{Q}} H^{2i}(\widehat{X},\mathbb{Q}) = \sharp \left\{ \text{conjugacy classes of } G \text{ having age } i \right\}, \tag{1.8}$$

whereas the odd dimensional cohomology groups of $\widehat{X}$ are trivial. In particular, the Euler number $\chi(\widehat{X})$ of $\widehat{X}$ equals the number of the conjugacy classes of G.

Theorem 1.18 (Ito-Reid Correspondence, [**67**, Thm. 1.4]). *If* $(\mathbb{C}^r/G, [\mathbf{0}])$ *is a Gorenstein quotient singularity, then there is a canonical one-to-one correspondence between the junior conjugacy classes in* $G\,(-1)$ (or G, cf. 1.9 (iii)) *and the crepant discrete valuations of* $\mathbb{C}^r/G$.

Passing to Borel-Moore homology $H_{\bullet}^{\mathrm{BM}}(\widehat{X},\mathbb{Q})$ (the dual of cohomology $H_c^{\bullet}(\widehat{X},\mathbb{Q})$ with compact supports) for which the notion of fundamental class of an algebraic cycle is well-defined, Theorem 1.18 indicates what the exact expectation for a *high-dimensional McKay correspondence over* $\mathbb{Q}$ ought to be.

Conjecture 1.19 ([**106**], [**70**, 2.8]). *If G is a finite subgroup of* $\mathrm{SL}(r,\mathbb{C})$, *and $\widehat{X} \longrightarrow X$ a crepant desingularization of $X = \mathbb{C}^r/G$, then there is a canonical one-to-one correspondence*

$$\boxed{\left\{ \begin{array}{c} \text{conjugacy classes of } G \\ \text{having age } i \end{array} \right\} \ni [g] \longleftrightarrow \mathrm{CL}\,(Z_g) \in H^{\mathrm{BM}}_{2(r-i)}(\widehat{X},\mathbb{Q}),}$$

mapping $[g]$ onto the fundamental class of the algebraic cycle Z_g, where Z_g denotes the Zariski closure of the center of the monomial valuation (*of the function field of* X) *corresponding to g.*

Theorem 1.20 (Kaledin [**70**, 2.9]). *Conjecture* 1.19 *is true for symplectic X's.*

In fact, in the symplectic case, working with coefficients from $\mathbb{C}$, it is also possible to confirm a *multiplicative version* of the high-dimensional McKay correspondence.

Theorem 1.21 (Ginzburg & Kaledin [**49**, Thm. 1.2], [**72**, Thm. 2.4]). *For symplectic X's, there is a canonical graded algebra isomorphism*

$$\boxed{H^{\bullet}(\widehat{X},\mathbb{C}) \cong \mathbf{gr}^{\mathcal{F}\bullet}(\mathcal{Z}G),} \tag{1.9}$$

with $\mathbf{gr}^{\mathcal{F}\bullet}(\mathcal{Z}G)$ *as defined in* (1.4).

For further information about symplectic quotient singularities and their relation to McKay correspondence, the reader is referred to [**6, 9, 41, 42, 43, 44, 45, 71**].

• **The failure of** G-Hilb$(\mathbb{C}^r)$ **in the role of an** $\widehat{X}$ **for** $r \geq 4$. Another serious problem occurring in dimensions $r \geq 4$ is that, in most of the cases, and though it has plenty of nice properties, the Hilbert scheme of G-clusters is no more suitable for our purposes. There are namely only a few exceptional examples in which G-Hilb$(\mathbb{C}^r)$ serves as a crepant desingularization of $X = \mathbb{C}^r/G$, like the four-dimensional cyclic quotient singularity of type $\frac{1}{15}(1,2,4,8)$ (cf. Note 8.13 (ii)), the symplectic singularities $(\mathbb{C}^r/\mathfrak{S}_{r/2}, [\mathbf{0}])$, $r \in 2\mathbb{Z}$, (with the symmetric group $\mathfrak{S}_{r/2}$ acting on $\mathbb{C}^r$ by permuting coordinates) or $(\mathbb{C}^r/\Gamma \wr \mathfrak{S}_{r/2}, [\mathbf{0}])$,where Γ denotes a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ and "$\wr$" the wreath product, and some others. (See Haiman [**55**, §5], Wang [**122**], and Kuznetsov [**79**].) In general, G-Hilb$(\mathbb{C}^r)$ *has not necessarily* trivial canonical divisor, can be *singular*, or even *non-normal* (see Note 5.7 and [**23**, Corollary 1.6], respectively).

• **On the existence of crepant resolutions in dimensions** $r \geq 4$. The presence of *terminal*[5] Gorenstein quotient singularities in dimensions $r \geq 4$ (for which there are no "crepant divisors" to pull out, cf. [**91**]) means automatically that, in contrast to what happens in dimension 2 and 3, *not all* Gorenstein quotient spaces $\mathbb{C}^r/G$ can be desingularized by crepant birational morphisms.

► **Existence Problem** (cf. [**67**, §4.5] and [**106**, §7]): *For which* $G \subset \mathrm{SL}(r,\mathbb{C})$, $r \geq 4$, *do there exist crepant (preferably projective) desingularizations of* $\mathbb{C}^r/G$?

Our first guess is that if we do not move too far away from the hypersurface case (cf. Theorem 1.11 and Remark 1.14 (i)), then the existence of birational morphisms of this sort is indeed guaranteed.

CONJECTURE 1.22 ([**29**]). *All quotient c.i.-singularities admit projective, crepant resolutions in all dimensions.*

By (1.3) and Theorem 1.10 we see that for every quotient c.i.-singularity $(\mathbb{C}^r/G, [\mathbf{0}])$ the group G *is generated by its junior elements.* In view of Theorem 1.18, we believe that this property is sufficient for the existence of the desired desingularizations of $\mathbb{C}^r/G$. Conjecture 1.22 is true whenever G is abelian (see below Theorem 5.1). Moreover, the same assertion for all *toric* (not necessarily quotient) c.i.-singularities has been proven to be true in [**26**].

Now going *beyond* the "c.i.'s", and putting the terminal ones aside, the remaining Gorenstein quotient singularities have *rarely* resolutions of this kind. Nonetheless, to our surprise, the singularity series which do so, are *not* negligible as one would expect at first sight. (See below §7 and §8.)

Henceforth, we consider exclusively the Existence Problem for Gorenstein ***abelian*** quotient singularities. There are several reasons to give priority to the abelian ones:

(i) Abelian finite subgroups G of $\mathrm{SL}(r,\mathbb{C})$ exist in all dimensions r, their conjugacy classes are singletons, and their character groups are isomorphic to themselves. Hence, letting them act linearly on $\mathbb{C}^r$, the age and the height of any element $g \in G$ are determined by its *weights* appearing in the *type* of $(\mathbb{C}^r/G, [\mathbf{0}])$ (as long as we fix eigencoordinates and generators; cf. 3.1).

[5] For the definition of terms like *canonical* (resp., *terminal*) *singularities* (of *index* $i \geq 1$), *crepant divisor* etc., see [**85, 102, 103**].

(ii) For abelian G's the Gorenstein quotient spaces $\mathbb{C}^r/G$ can be treated by the toric machinery and, particularly, by studying the properties of the so-called *junior simplices* $\mathfrak{s}_G$. Note that there is no loss of generality when one works in the toric category because the existence of an *arbitrary* projective crepant desingularization of $\mathbb{C}^r/G$ implies the existence of a $\mathbb{T}_{N_G}$-*equivariant* projective crepant desingularization, where $\mathbb{T}_{N_G} := (\mathbb{C}^*)^r/G$ (cf. [**86**, proof of Lemma 1]). Moreover, the *secondary polytope* of $\mathfrak{s}_G$ describes conveniently the corresponding flops.

(iii) It is expected (cf. [**67**, §4.6]) that abelian quotient singularities will be good candidates for proving both Conjecture 1.19 and an analogue of Theorem 1.20, and for removing the restrictive hypothesis on $\mathbb{C}^r/G$ (i.e., to be symplectic).

(iv) Given a *non-abelian* group G, it is also conjectured (see [**104**, §3]) that the existence of crepant desingularizations of $\mathbb{C}^r/G$ may be related to the existence of such desingularizations for the quotients $\mathbb{C}^r/H$, for all *maximal* cyclic (or abelian) subgroups H contained in G.

The present paper has been written trying to be self-contained and partially expository. In particular, it includes more background material than the average research paper has. The new results are essentially in sections 3, 8, 9 and 10, together with some parts of §4 and of Appendices A and D. (In §5, §6 , and §7 we summarize results from [**27, 29**] and from the unpublished manuscript [**28**].)

More precisely, the paper is organized as follows: In §2 we recall fundamental notions from toric geometry and introduce our notation. A detailed study of the abelian quotient singularities as toric singularities (including various properties of the junior and senior simplices of those which are Gorenstein) is presented in §3. In section 4 we explain: (a) why the Existence Problem (in the abelian case) is equivalent to the problem of finding junior simplices possessing *basic* (preferably *coherent*) *lattice triangulations*, (b) how one can compute (1.8) by means of the Ehrhart polynomials of these simplices, and (c) why two different maximal (partial or full) projective crepant desingularizations of a Gorenstein abelian quotient space $\mathbb{C}^r/G$ can be obtained from each other by a finite succession of flops.

Wide classes of Gorenstein abelian quotient singularities admitting projective, crepant resolutions are given in §5, §7, and §8. (In §8 we prove a long-standing conjecture concerning the so-called *GP-singularity series*, cf. [**28**, §10].)

On the other hand, to *exclude* candidates for having crepant resolutions (whenever the available lattice points in the junior simplex are either "strangely located" or "not enough" to triangulate suitably), we apply two *necessary* existence conditions. The first of them (see (6.1) in §6) informs us that, provided such a resolution is present, each of the Hilbert basis elements of the cone supporting $\mathfrak{s}_G$ has to be either a junior element or a vertex of $\mathfrak{s}_G$. The second one (see (9.2)-(9.3) in §9) states that the existence of a crepant resolution implies the boundedness of the acting group order from above by a number which depends on the number of lattice points of $\mathfrak{s}_G$ and of $\partial\mathfrak{s}_G$.

Next, combining our results, we outline an algorithm by means of which it is possible to handle the Existence Problem (in the abelian case); see §10 (and especially Figure 7). Only in the last two steps of this algorithm the computational complexity grows rapidly. In particular, Step 5 (involving the determination of maximal coherent triangulations of $\mathfrak{s}_G$) is added just for excluding some "sporadic" counterexamples which happen to "survive" after having used the above mentioned existence criteria. (Computer programs like `Puntos` [**31**] or `TOPCOM` [**100**] offer practical assistance in this situation.)

Useful technical notions and results from the theory of subdivisions, triangulations, lattice triangulations, and lattice point enumerators, are presented separately in four appendices at the end of the paper. An extensive part of Appendix A is devoted to Upper Bound Theorems (UBT's). Apart from the UBT *for the facets of simplicial balls* (Theorem A.5), we conjecture the validity of a more effective UBT *for the facets of geometric simplex triangulations*, and give a proof of it in dimension 3 by means of PL-topological methods (see Theorem A.22). The *coherence* of triangulations and certain combinatorial properties of *bistellar flips* belong to the topics covered in Appendix B. In Appendix C we explain how one passes from the coordinates of the $\mathbf{h}^*$-vector of a lattice polytope P (or, equivalently, from the coordinates of the $\mathbf{h}$-vector of any *basic* triangulation $\mathcal{T}$ of P) to the coefficients of its Ehrhart polynomial. Finally, in Appendix D we compute the coefficients of the Ehrhart polynomial of *any junior simplex* by making use of Mordell-Pommersheim and Diaz-Robins formulae.

2. Toric Glossary

At first we recall some basic facts from the theory of toric varieties. We mostly use the same notation as in [**27, 28, 29**]. Our standard references on toric geometry are the books of Oda [**96**] and Fulton [**47**].

• **General notation**. The *linear hull,* the *affine hull*, the *integral affine hull*, the *positive hull* and *the convex hull* of a set B of vectors of $\mathbb{R}^r$, $r \geq 1$, will be denoted by $\mathrm{lin}(B)$, $\mathrm{aff}(B)$, $\mathrm{aff}_{\mathbb{Z}}(B)$, $\mathrm{pos}(B)$ (or $\mathbb{R}_{\geq 0} B$) and $\mathrm{conv}(B)$, respectively. The *dimension* $\dim(B)$ of a $B \subset \mathbb{R}^r$ is defined to be the dimension of its affine hull.

• **Lattice determinants**. Let $N \cong \mathbb{Z}^r$ be a free $\mathbb{Z}$-module of rank $r \geq 1$. N can be regarded as a *lattice* in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r$. (For fixed identification, we shall represent the elements of $N_{\mathbb{R}}$ by column-vectors in $\mathbb{R}^r$). If $\{n_1, \ldots, n_r\}$ is a $\mathbb{Z}$-basis of N, then

$$\det(N) := |\det(n_1, \ldots, n_r)|$$

is the *lattice determinant.* An $n \in N$ is called *primitive* if $\mathrm{conv}(\{\mathbf{0}, n\}) \cap N$ contains no other points except $\mathbf{0}$ and n.

• **Cones**. Let $N \cong \mathbb{Z}^r$ be as above, $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual lattice, $N_{\mathbb{R}}, M_{\mathbb{R}}$ their real scalar extensions, and $\langle .,. \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ the natural $\mathbb{R}$-bilinear pairing. A subset σ of $N_{\mathbb{R}}$ is called *strongly convex polyhedral cone* (*s.c.p. cone*, for short), if there exist $n_1, \ldots, n_k \in N_{\mathbb{R}}$, such that $\sigma = \mathrm{pos}(\{n_1, \ldots, n_k\})$ and $\sigma \cap (-\sigma) = \{\mathbf{0}\}$. The *dual cone* of such a σ is $\sigma^{\vee} := \{\mathbf{x} \in M_{\mathbb{R}} \mid \langle \mathbf{x}, \mathbf{y} \rangle \geq 0, \ \forall \mathbf{y}, \ \mathbf{y} \in \sigma\}$. A subset τ of a s.c.p. cone σ is called a *face* of σ (notation: $\tau \prec \sigma$), if $\tau = \{\mathbf{y} \in \sigma \mid \langle m_0, \mathbf{y} \rangle = 0\}$, for some $m_0 \in \sigma^{\vee}$. A s.c.p. cone $\sigma = \mathrm{pos}(\{n_1, \ldots, n_k\})$ is called *simplicial* (resp., *rational*) if $n_1, \ldots, n_k$ are $\mathbb{R}$-linearly independent (resp., if $n_1, \ldots, n_k \in N_{\mathbb{Q}}$, where $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$).

• **Hilbert bases**. If $\sigma \subset N_{\mathbb{R}} \cong \mathbb{R}^r$ is a rational s.c.p. cone, then σ has $\mathbf{0}$ as its apex and the subsemigroup $\sigma \cap N$ of N is a monoid.

Proposition 2.1 (Gordan's lemma). *$\sigma \cap N$ is finitely generated as an additive semigroup, i.e. there exist $n_1, \ldots, n_\nu \in \sigma \cap N$ such that*

$$\sigma \cap N = \mathbb{Z}_{\geq 0}\, n_1 + \cdots + \mathbb{Z}_{\geq 0}\, n_\nu \ .$$

PROPOSITION 2.2 (Minimal generating system, [**111**, p. 233]). *Among all the systems of generators of* $\sigma \cap N$, *there is a unique system* $\mathbf{Hlb}_N(\sigma)$ *of* minimal cardinality, *namely*:

$$\mathbf{Hlb}_N(\sigma) = \left\{ n \in \sigma \cap (N \smallsetminus \{\mathbf{0}\}) \;\middle|\; \begin{array}{c} n \text{ cannot be expressed as} \\ \text{the sum of two other vectors} \\ \text{belonging to } \sigma \cap (N \smallsetminus \{\mathbf{0}\}) \end{array} \right\}. \tag{2.1}$$

DEFINITION 2.3. $\mathbf{Hlb}_N(\sigma)$ is called *the Hilbert basis of* σ w.r.t. N.

THEOREM 2.4 (Sebö [**113**]). *Given a rational s.c.p. cone* $\sigma \subset N_{\mathbb{R}}$ *and an element* $n \in \sigma \cap N$, *it is* co$\mathcal{NP}$*-complete to decide whether* n *is contained in* $\mathbf{Hlb}_N(\sigma)$.

REMARK 2.5. Sebö's Theorem shows the difficulty of deciding whether an integral vector is additively reducible. In general, at least $r + \left\lfloor \frac{r}{6} \right\rfloor$ elements of $\mathbf{Hlb}_N(\sigma)$ are needed to write an $n \in \sigma \cap N$ as non-negative integer linear combination of elements of $\mathbf{Hlb}_N(\sigma)$ (see [**11**]). For an algorithm computing $\mathbf{Hlb}_N(\sigma)$ by the determination of the Graver basis of a suitable integer matrix we refer to Sturmfels [**121**, Algorithm 13.2, p. 128]. Another efficient algorithm (which relies on a project-and-lift approach, without making use of additional variables, and is implemented in the computer program `MLP`) is due to Hemmecke [**57**].

• **Affine toric varieties**. For a lattice $N \cong \mathbb{Z}^r$ having M as its dual, we define an r-dimensional *algebraic torus* $\mathbb{T}_N \cong (\mathbb{C}^*)^r$ by setting

$$\mathbb{T}_N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^*.$$

We identify M with the character group of $\mathbb{T}_N$ and N with the group of 1-parameter subgroups of $\mathbb{T}_N$. Let σ be a rational s.c.p. cone with

$$M \cap \sigma^{\vee} = \mathbb{Z}_{\geq 0}\, m_1 + \mathbb{Z}_{\geq 0}\, m_2 + \cdots + \mathbb{Z}_{\geq 0}\, m_k.$$

To the finitely generated, normal, monoidal $\mathbb{C}$-subalgebra $\mathbb{C}[M \cap \sigma^{\vee}]$ of $\mathbb{C}[M]$ we associate an affine complex variety

$$U_\sigma := \mathrm{Spec}\,(\mathbb{C}[M \cap \sigma^{\vee}])$$

endowed with a canonical $\mathbb{T}_N$-action. The analytic structure induced on U_σ is independent of the semigroup generators $\{m_1, \ldots, m_k\}$. Moreover, $\sharp\,(\mathbf{Hlb}_M(\sigma^{\vee}))\ (\leq k)$ is nothing but the *embedding dimension* of U_σ, i.e. the minimal number of generators of the maximal ideal of the local $\mathbb{C}$-algebra $\mathcal{O}_{U_\sigma,\mathbf{0}}$ (cf. [**96**, 1.2-1.3]).

• **Fans**. A *fan* w.r.t. $N \cong \mathbb{Z}^r$ is a finite collection Δ of rational s.c.p. cones in $N_{\mathbb{R}}$, such that the faces of any member belongs to Δ and such that the intersection of any two members is a face of each of them. We denote by $|\Delta|$ the support, and by $\Delta(i)$ the set of i-dimensional cones of Δ. If ϱ is a *ray* of Δ, i.e, if $\varrho \in \Delta(1)$, then there exists a unique primitive vector $n(\varrho) \in N \cap \varrho$ with $\varrho = \mathbb{R}_{\geq 0}\, n(\varrho)$ and each cone $\sigma \in \Delta$ can be therefore written as $\sigma = \sum_{\varrho \in \Delta(1),\, \varrho \prec \sigma} \mathbb{R}_{\geq 0}\, n(\varrho)$. The set $\mathrm{Gen}(\sigma) := \{n(\varrho) \mid \varrho \in \Delta(1), \varrho \prec \sigma\}$ is called the *set of minimal generators* of σ. For Δ itself one defines analogously

$$\mathrm{Gen}(\Delta) := \bigcup_{\sigma \in \Delta} \mathrm{Gen}(\sigma).$$

• **General toric varieties**. The *toric variety* $X(N, \Delta)$ associated to a fan Δ w.r.t. the lattice N is by definition the identification space $X(N, \Delta) := \bigcup_{\sigma \in \Delta} U_\sigma \,/\, \sim,$

with $U_{\sigma_1} \ni u_1 \sim u_2 \in U_{\sigma_2}$ if and only if there is a $\tau \in \Delta$ such that $\tau \prec \sigma_1 \cap \sigma_2$ and $u_1 = u_2$ within U_τ. $X(N, \Delta)$ is called *simplicial* if all cones of Δ are simplicial. If $X(N, \Delta)$ is r-dimensional, then its topological Euler number $\chi(X(N, \Delta))$ equals

$$\chi(X(N, \Delta)) = \sharp\Delta(r)\,. \text{ (See [\mathbf{47}, p. 59].)} \tag{2.2}$$

$X(N, \Delta)$ is also equipped with a canonical $\mathbb{T}_N$-action which is compatible with the above mentioned $\mathbb{T}_N$-actions on U_σ's. The orbits w.r.t. this action are parametrized by the set of all cones belonging to Δ. For a $\tau \in \Delta$, we denote by $\mathrm{orb}(\tau)$ (resp., by $\mathbf{V}(\tau)$) the orbit (resp., the closure of the orbit) which is associated to τ. If $\tau \in \Delta$, then $\mathbf{V}(\tau) = X(N(\tau), \mathrm{Star}(\tau; \Delta))$ is itself a toric variety w.r.t.

$$N(\tau) := N/N_\tau\ , \quad \mathrm{Star}(\tau; \Delta) := \{\overline{\sigma} \mid \sigma \in \Delta,\ \tau \prec \sigma\}\,,$$

where N_τ denotes the sublattice of N generated (as subgroup) by the intersection $N \cap \mathrm{lin}(\tau)$, and $\overline{\sigma} = (\sigma + (N_\tau)_\mathbb{R}) / (N_\tau)_\mathbb{R}$ is the image of σ in $N(\tau)_\mathbb{R} = N_\mathbb{R}/(N_\tau)_\mathbb{R}$.

• **Equivariant maps**. A *map of fans* $\varpi : (N', \Delta') \to (N, \Delta)$ is a $\mathbb{Z}$-linear homomorphism $\varpi : N' \to N$ whose scalar extension $\varpi = \varpi_\mathbb{R} : N'_\mathbb{R} \to N_\mathbb{R}$ satisfies the property: $\forall \sigma' \in \Delta'\ \exists\ \sigma \in \Delta$ with $\varpi(\sigma') \subset \sigma$. Note that the dual of the homomorphism $\varpi \otimes_\mathbb{Z} \mathrm{id}_{\mathbb{C}^*} : \mathbb{T}_{N'} = N' \otimes_\mathbb{Z} \mathbb{C}^* \to \mathbb{T}_N = N \otimes_\mathbb{Z} \mathbb{C}^*$ induces an *equivariant holomorphic map* $\varpi_* : X(N', \Delta') \to X(N, \Delta)$. This map is *proper* if and only if $\varpi^{-1}(|\Delta|) = |\Delta'|$. In particular, if $N = N'$ and Δ' is a refinement of Δ, then $\mathrm{id}_* : X(N, \Delta') \to X(N, \Delta)$ is *proper* and *birational* (cf. [**96**, 1.15 and 1.18]).

• **Basic cones and desingularization**. Let $N \cong \mathbb{Z}^r$ be a lattice of rank r and $\sigma \subset N_\mathbb{R}$ a simplicial, rational s.c.p. cone of dimension $k \le r$. σ can be obviously written as $\sigma = \varrho_1 + \cdots + \varrho_k$, for distinct 1-dimensional cones $\varrho_1, \ldots, \varrho_k$. Let

$$\mathbf{Par}(\sigma) := \left\{ \mathbf{y} \in (N_\sigma)_\mathbb{R} \;\middle|\; \mathbf{y} = \sum_{j=1}^{k} \varepsilon_j\ n(\varrho_j)\,,\ \text{with}\ \ 0 \le \varepsilon_j < 1,\ \forall j,\ 1 \le j \le k \right\}$$

be the *fundamental* (*half-open*) *parallelotope* associated to σ. The *multiplicity* $\mathrm{mult}(\sigma; N)$ of σ with respect to N is defined to be $\mathrm{mult}(\sigma; N) := \sharp(\mathbf{Par}(\sigma) \cap N_\sigma)$. As it turns out,

$$\mathrm{mult}(\sigma; N) = \mathrm{Vol}_{N_\sigma}(\mathbf{Par}(\sigma))\,, \tag{2.3}$$

where $\mathrm{Vol}(\mathbf{Par}(\sigma))$ denotes the usual volume (Lebesgue measure) of $\mathbf{Par}(\sigma)$, and

$$\mathrm{Vol}_{N_\sigma}(\mathbf{Par}(\sigma)) := \tfrac{\mathrm{Vol}(\mathbf{Par}(\sigma))}{\det(N_\sigma)} = \tfrac{\det(\mathbb{Z}\,n(\varrho_1) \oplus \cdots \oplus \mathbb{Z}\,n(\varrho_k))}{\det(N_\sigma)}$$

its relative volume. If $\mathrm{mult}(\sigma; N) = 1$, then σ is called a *basic cone w.r.t.* N.

Proposition 2.6 ([**96**, Thm. 1.10 and Prop. 1.25]). *The affine toric variety U_σ is $\mathbb{Q}$-factorial* (*resp., smooth*) *if and only if σ is simplicial* (*resp., basic w.r.t. N*). *Correspondingly, a toric variety $X(N, \Delta)$ is $\mathbb{Q}$-factorial* (*resp., smooth*) *if and only if it is simplicial (resp., simplicial and each s.c.p. cone $\sigma \in \Delta$ is basic*).

By Carathéodory's theorem concerning convex polyhedral cones one can choose a refinement Δ' of any given fan Δ, so that Δ' becomes simplicial. Since further subdivisions of Δ' reduce the multiplicities of its cones, we may arrive (after finitely many subdivisions) at a fan $\widetilde{\Delta}$ having only basic cones.

Theorem 2.7 (Existence of Desingularizations). *For every toric variety $X(N, \Delta)$ there exists a refinement $\widetilde{\Delta}$ of Δ consisting of exclusively basic cones w.r.t. N, i.e., such that $f = \mathrm{id}_* : X(N, \widetilde{\Delta}) \longrightarrow X(N, \Delta)$ is a desingularization.*

3. AQS as Toric Singularities

Abelian quotient singularities (*AQS*, for short) can be directly investigated by means of the theory of toric varieties. If G is a finite abelian subgroup of $\mathrm{GL}(r,\mathbb{C})$, then $(\mathbb{C}^*)^r/G$ is automatically an algebraic torus embedded in $\mathbb{C}^r/G$.

• **General notation**. For $n \in \mathbb{N}$, $m \in \mathbb{Z}$, we denote by $[m]_n$ the (uniquely determined) integer for which $0 \le [m]_n < n$, $m \equiv [m]_n \,(\mathrm{mod}\; n)$. If $x \in \mathbb{Q}$, we define $\lceil x \rceil$ (resp., $\lfloor x \rfloor$) to be the least integer number $\ge x$ (resp., the greatest integer number $\le x$). "gcd" and "lcm" will be abbreviations for greatest common divisor and least common multiple. Furthermore, for $n \in \mathbb{Z}_{\ge 2}$, we denote by $\zeta_n := e^{\frac{2\pi\sqrt{-1}}{n}}$ the "first" n-th primitive root of unity.

• **The equivalence relation "$\backsim$"**. For $(q,r) \in (\mathbb{Z}_{\ge 2})^2$ we define[6]

$$\Lambda(q;r) := \left\{ (\alpha_1,..,\alpha_r) \in \{0,1,2,..,q-1\}^r \;\middle|\; \begin{array}{c} \gcd(q,\alpha_1,..,\widehat{\alpha_i},..,\alpha_r) = 1, \\ \text{for all } i, \;\; 1 \le i \le r \end{array} \right\}$$

and for $((\alpha_1,\dots,\alpha_r),(\alpha'_1,\dots,\alpha'_r)) \in \Lambda(q;r) \times \Lambda(q;r)$ the relation

$$(\alpha_1,\dots,\alpha_r) \backsim (\alpha'_1,\dots,\alpha'_r) :\Longleftrightarrow \left\{ \begin{array}{l} \text{there exists a permutation} \\ \phi : \{1,\dots,r\} \to \{1,\dots,r\} \\ \text{and an integer } \lambda,\ 1 \le \lambda \le l-1, \\ \text{with } \gcd(\lambda,l) = 1, \text{ such that} \\ \alpha'_{\phi(i)} = [\lambda \cdot \alpha_i]_l\,,\ \forall i,\ 1 \le i \le r \end{array} \right\}. \tag{3.1}$$

It is easy to see that $\backsim$ is an equivalence relation. We denote by

$$\overline{\Lambda}(q;r) := \Lambda(q;r) / \backsim$$

the corresponding set of equivalent classes determined by "$\backsim$".

• **The "type" of an AQS**. Let G be a finite, small, abelian subgroup of $\mathrm{GL}(r,\mathbb{C})$, $r \ge 2$, with order $l = |G| \ge 2$. Consider as "starting point" a maximal decomposition of G (viewed as an abstract group) into a direct product of cyclic groups of orders, say, $q_1,\dots,q_\kappa$:

$$\boxed{G \cong (\mathbb{Z}/q_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/q_\kappa\mathbb{Z})} \tag{3.2}$$

and let

$$\boxed{\exp(G) := \ \mathrm{lcm}(q_1,\dots,q_\kappa)}$$

be the *exponent* of G and $\xi_\mu := \exp(G) \cdot q_\mu^{-1}$, for all $\mu \in \{1,\dots,\kappa\}$. Since G is small, it is easy to prove that G possesses at most $r-1$ generators. Therefore we may assume, from now on, that $\kappa \le \min(r-1, \lfloor \frac{l}{2} \rfloor)$. Choose after this fixing in advance of isomorphism (3.2) suitable coordinates on $\mathbb{C}^r$ to diagonalize the action of each factor $(\mathbb{Z}/q_\mu\mathbb{Z})$ on $\mathbb{C}^r$. According to Theorem 1.2, and since every representation of a finite cyclic group in a $\mathbb{C}$-vector space is the direct sum of the one-dimensional representations, the action

$$(\mathbb{Z}/q_\mu\mathbb{Z}) \times \mathbb{C}^r \ni (g_\mu,(z_1,\dots,z_r)) \mapsto \ \mathrm{diag}(\zeta_{q_\mu}^{\alpha_{\mu,1}}\ z_1,\dots,\zeta_{q_\mu}^{\alpha_{\mu,r}}\ z_r) \in \mathbb{C}^r \tag{3.3}$$

can be uniquely determined by the choice of a generator $g_\mu := \mathrm{diag}\big(\zeta_{q_\mu}^{\alpha_{\mu,1}},\dots,\zeta_{q_\mu}^{\alpha_{\mu,r}}\big)$ of each cyclic factor, i.e., by the choice of an r-tuple $(\alpha_{\mu,1},\dots,\alpha_{\mu,r}) \in \Lambda(q_\mu;r)$ as a representative of an equivalence class within $\overline{\Lambda}(q_\mu;r)$ w.r.t. $\backsim$ which is defined by (3.1). (Any two r-tuples from $\Lambda(q_\mu;r)$ belong to the same equivalence class w.r.t.

[6]The symbol $\widehat{\alpha_i}$ means here that α_i is omitted.

$\backsim$ if and only if the two associated representations of $\mathbb{Z}/q_\mu\mathbb{Z}$ within $\mathrm{GL}(r,\mathbb{C})$, which correspond to these two (probably different) generators of $\mathbb{Z}/q_\mu\mathbb{Z}$, are conjugate to each other, i.e., if and only if the corresponding quotient spaces are isomorphic; cf. [**46**, Lemma 2, p. 296]). Each element $g \in G$, identified with the diagonalization of its image under (3.2), is of the form

$$\boxed{g = \mathrm{diag}\left(\zeta_{\exp(G)}^{\delta_1(j_1,\ldots,j_\kappa)}, \ldots, \zeta_{\exp(G)}^{\delta_r(j_1,\ldots,j_\kappa)}\right)} \tag{3.4}$$

induced by a unique κ-tuple $(j_1,\ldots,j_\kappa) \in \{0,1,\ldots,q_1-1\}\times\cdots\times\{0,1,\ldots,q_\kappa-1\}$, where

$$\delta_i\left(j_1,\ldots,j_\kappa\right) := \left[\sum_{\mu=1}^{\kappa} j_\mu \cdot \xi_\mu \cdot \alpha_{\mu,i}\right]_{\exp(G)}, \ \forall i \in \{1,\ldots,r\}. \tag{3.5}$$

Definition 3.1. For a G having decomposition (3.2) and for a given, predeterminated choice of the representation of the generators of its factors within $\mathrm{GL}(r,\mathbb{C})$, as above in (3.3), (3.4), (3.5), we say that the AQS $(\mathbb{C}^r/G,[\mathbf{0}])$ *is of type*

$$\boxed{\tfrac{1}{q_1}\left(\alpha_{1,1},\alpha_{1,2},\ldots,\alpha_{1,r}\right) \times \ \cdots \ \cdots \ \times \tfrac{1}{q_\kappa}\left(\alpha_{\kappa,1},\alpha_{\kappa,2},\ldots,\alpha_{\kappa,r}\right)} \tag{3.6}$$

(and view all the entries $\alpha_{i,j}$ as its *weights*.) If G happens to be cyclic, then we fix a choice of a generator of $G \cong \mathbb{Z}/l\mathbb{Z}$ by making use of suitable exponents of ζ_l (i.e., by shortening of (3.2), in order to have just one factor, and by suitable diagonalization), and we omit the first subscript index of each of these regarded weights. In this case, we simply say that $(\mathbb{C}^r/G,[\mathbf{0}])$ is a cyclic quotient singularity (*CQS*, for short) *of type*

$$\boxed{\frac{1}{l}\left(\alpha_1,\alpha_2,\ldots,\alpha_r\right).} \tag{3.7}$$

• **Abelian quotient spaces as toric varieties**. Let G be a finite, small, *abelian* subgroup of $\mathrm{GL}(r,\mathbb{C})$, $r \geq 2$, having order $l = |G| \geq 2$, and let

$$\{e_1 = (1,0,\ldots,0,0)^\intercal, \ldots, e_r = (0,0,\ldots,0,1)^\intercal\}$$

denote the standard basis of $\mathbb{R}^r$, $N_0 := \mathbb{Z}^r = \sum_{i=1}^r \mathbb{Z}e_i$ the standard rectangular lattice, M_0 its dual, and $\mathbb{T}_{N_0} := \mathrm{Spec}\big(\mathbb{C}\left[\mathfrak{x}_1^{\pm1},\ldots,\mathfrak{x}_r^{\pm1}\right]\big) = (\mathbb{C}^*)^r$. Clearly,

$$\mathbb{T}_{N_G} := \mathrm{Spec}(\mathbb{C}\left[\mathfrak{x}_1^{\pm1},\ldots,\mathfrak{x}_r^{\pm1}\right]^G) = (\mathbb{C}^*)^r/G$$

is an r-dimensional algebraic torus with N_G as its 1-parameter group and with M_G as its group of characters. Using the map

$$(N_0)_\mathbb{R} \ni (y_1,\ldots,y_r)^\intercal = \mathbf{y} \longmapsto \theta\left(\mathbf{y}\right) := (e^{(2\pi\sqrt{-1})y_1},\ldots,e^{(2\pi\sqrt{-1})y_r})^\intercal \in \mathbb{T}_{N_0}$$

and the injection $\iota : \mathbb{T}_{N_0} \hookrightarrow \mathrm{GL}(r,\mathbb{C})$ defined by

$$\mathbb{T}_{N_0} \ni (t_1,\ldots,t_r)^\intercal \longmapsto \mathrm{diag}\left(t_1,\ldots,t_r\right) \in \ \mathrm{GL}\left(r,\mathbb{C}\right),$$

we have obviously $N_G = (\iota\circ\,\theta)^{-1}(G)$ with $\det(N_G) = \frac{1}{l}$. In fact, using (3.4) and (3.5), we get

$$N_G = N_0 + \sum_{\mu=1}^{\kappa} \mathbb{Z}\,(\frac{1}{q_\mu}\left(\alpha_{\mu,1},\ldots,\alpha_{\mu,r}\right)^\intercal)$$

$$= N_0 + \sum_{(j_1,\ldots,j_\kappa)\in\{0,1,\ldots,q_1-1\}\times\cdots\times\{0,1,\ldots,q_\kappa-1\}} \mathbb{Z}\left(\tfrac{\delta_1(j_1,\ldots,j_\kappa)}{\exp(G)},\ldots,\tfrac{\delta_r(j_1,\ldots,j_\kappa)}{\exp(G)}\right)^{\intercal} \quad (3.8)$$

and

$$M_G = \left\{ m \in M_0 \;\middle|\; \begin{array}{c} \mathfrak{x}^m = \mathfrak{x}_1^{\mu_1}\cdots\mathfrak{x}_r^{\mu_r} \quad \text{is a } G\text{-invariant} \\ \text{Laurent monomial } (m=(\mu_1,\ldots,\mu_r)) \end{array} \right\} \text{ (with } \det(M_G)=l).$$

If we define

$$\boxed{\sigma_0 := \operatorname{pos}(\{e_1,..,e_r\})}$$

to be the r-dimensional positive orthant, and Δ_G to be the fan

$$\Delta_G := \{\sigma_0 \text{ together with its faces}\},$$

then by the exact sequence $0 \to G \cong N_G/N_0 \to \mathbb{T}_{N_0} \to \mathbb{T}_{N_G} \to 0$ induced by the canonical duality pairing

$$M_0/M_G \times N_G/N_0 \to \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}^*$$

(cf. [**47**, p. 34] and [**96**, pp. 22-23]), we get $\mathbb{C}^r = X(N_0,\Delta_G) \to X(N_G,\Delta_G)$ as projection map, where

$$\boxed{X(N_G,\Delta_G) = U_{\sigma_0} = \mathbb{C}^r/G = \operatorname{Spec}\left(\mathbb{C}[\mathfrak{x}_1,\ldots,\mathfrak{x}_r]^G\right) \hookleftarrow \mathbb{T}_{N_G}}$$

Formally, we identify $[\mathbf{0}]$ with $\operatorname{orb}(\sigma_0)$. Moreover, the singular locus of $X(N_G,\Delta_G)$ can be written (by Propositions 1.1 and 2.6) as the union

$$\operatorname{Sing}(X(N_G,\Delta_G)) = \{\operatorname{orb}(\sigma_0)\} \bigcup \left\{ \operatorname{Spec}(\mathbb{C}[\overline{\sigma}_0^\vee \cap M_G(\tau)]) \;\middle|\; \begin{array}{c} \tau \precneqq \sigma_0, \\ \dim(\tau)\geq 2, \text{and} \\ \operatorname{mult}(\tau;N_G)\geq 2 \end{array} \right\}.$$

Proposition 3.2. *For an AQS* $(\mathbb{C}^r/G,[\mathbf{0}])$ *the following are equivalent*:

(i) $\operatorname{Sing}(X(N_G,\Delta_G)) = \{\operatorname{orb}(\sigma_0)\}$, *i.e.,* $\operatorname{orb}(\sigma_0) = [\mathbf{0}] \in X(N_G,\Delta_G)$ *is isolated.*

(ii) *For all* $\tau, \tau \precneqq \sigma_0$, *with* $\dim(\tau)\geq 2$, *we have* $\operatorname{mult}(\tau)=1$.

Definition 3.3. The *splitting codimension* of $U_{\sigma_0} = \mathbb{C}^r/G$ is defined to be the number

$$\operatorname{splcod}(U_{\sigma_0}) := \min\left\{ \mathfrak{k} \in \{2,\ldots,r\} \;\middle|\; \begin{array}{c} U_{\sigma_0} \cong U_\tau \times \mathbb{C}^{r-\mathfrak{k}}, \text{ s.t.} \\ \tau \preceq \sigma_0, \dim(\tau) = \mathfrak{k} \\ \text{and } \operatorname{Sing}(U_\tau) \neq \varnothing \end{array} \right\}$$

If $\operatorname{splcod}(U_{\sigma_0}) = r$, then $\operatorname{orb}(\sigma_0)$ is called an *msc-singularity*, i.e., a singularity having the maximum splitting codimension.

Proposition 3.4. *For an AQS* $(\mathbb{C}^r/G,[\mathbf{0}])$ *of type* (3.6) *the following conditions are equivalent*:

(i) $\operatorname{orb}(\sigma_0)$ *is a non-msc-singularity.*

(ii) $\operatorname{splcod}(\operatorname{orb}(\sigma_0); U_{\sigma_0}) = \mathfrak{k}_0$ *with* $2 \leq \mathfrak{k}_0 \leq r-1$.

(iii) *There exists a subfamily* $\{y_1,y_2,\ldots,y_{r-\mathfrak{k}_0}\} \subset \{1,\ldots,r\}$, *such that*

$$\delta_{y_1}(j_1,\ldots,j_\kappa) = \cdots = \delta_{y_{r-\mathfrak{k}_0}}(j_1,\ldots,j_\kappa) = 0,$$

for all $(j_1,\ldots,j_\kappa) \in \{0,1,2,\ldots,q_1-1\}\times\cdots\times\{0,1,2,\ldots,q_\kappa-1\}$;

$\{y_1, y_2, \ldots, y_{r-\mathfrak{k}_0}\}$ is, in addition, the largest subfamily of $\{1, \ldots, r\}$ having this property.

THEOREM 3.5 (Gorenstein condition). *Let $(\mathbb{C}^r/G, [\mathbf{0}])$ be an AQS of type* (3.6). *Then the following conditions are equivalent:*

(i) $X(N_G, \Delta_G) = \mathbb{C}^r/G$ *is Gorenstein.*

(ii) $\sum_{i=1}^{r} \alpha_{\mu,i} \equiv (0 \bmod q_\mu)$, *for all μ, $1 \leq \mu \leq \kappa$.*

(iii) $\langle (1, 1, \ldots .1, 1), n\rangle \geq 1$, *for all n, $n \in \sigma_0 \cap (N_G \smallsetminus \{\mathbf{0}\})$.*

(iv) $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ *is a canonical singularity of index* 1.

PROOF. It follows from Theorem 1.4 and [**102**, Thm. 3.1, p. 292]. □

• **Junior and senior elements**. Let $(\mathbb{C}^r/G, [\mathbf{0}])$ be a Gorenstein AQS of type (3.6). For all $i \in \{1, \ldots, r-1\}$, we denote by

$$\boxed{\mathcal{H}_i := \{(x_1, \ldots, x_r)^\intercal \in \mathbb{R}^r \mid x_1 + x_2 + \cdots + x_r = i\}}$$

the affine hyperplane *of level i*, and by

$$\boxed{\mathfrak{s}_G^{[i]} := \sigma_0 \cap \mathcal{H}_i = \mathrm{conv}(\{ie_1, \ldots, ie_r\})}$$

the $(r-1)$-dimensional lattice simplex *of age i* which lies on $\mathcal{H}_i$. (We adopt here the terminology of [**67**, §1-2]). In particular, *the junior simplex* is defined to be

$$\boxed{\mathfrak{s}_G := \mathfrak{s}_G^{[1]} = \sigma_0 \cap \mathcal{H}_1 = \mathrm{conv}(\{e_1, \ldots, e_r\})\,.}$$

An element $g \in G$ as in (3.4) is a *junior element* (resp., a *senior element of age i*, in the sense of 1.9 (ii)) whenever

$$n_g \text{ belongs to } \mathfrak{s}_G \cap N_G \quad (\text{resp., to } \mathfrak{s}_G^{[i]} \cap N_G, \text{ for } i \in \{2, \ldots, r-1\})\,,$$

where

$$\boxed{n_g := \left(\frac{\delta_1(j_1, \ldots, j_\kappa)}{\exp(G)}, \ldots, \frac{\delta_r(j_1, \ldots, j_\kappa)}{\exp(G)}\right)^\intercal.} \tag{3.9}$$

$\mathbf{Par}(\sigma_0)$ is nothing but the unit half-open cube in $\mathbb{R}^r$, with

$$\begin{aligned}&\{\text{lattice points } n_g \text{ representing the junior elements } g \text{ of } G\}\\ &= (\mathfrak{s}_G \cap \mathbf{Par}(\sigma_0) \cap N_G) = (\mathfrak{s}_G \cap (N_G \smallsetminus \{e_1, \ldots, e_r\}))\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}&\{\text{lattice points } n_g \text{ representing the seniors } g \in G \text{ whose age is } i\}\\ &= (\mathfrak{s}_G^{[i]} \cap \mathbf{Par}(\sigma_0) \cap N_G) \subset (\mathfrak{s}_G^{[i]} \cap (N_G \smallsetminus \{ie_1, \ldots, ie_r\}))\end{aligned} \tag{3.11}$$

for all $i \in \{2, \ldots, r-1\}$ (cf. (3.8)). Obviously,

$$\sum_{i=1}^{r-1} (\sharp(\mathfrak{s}_G^{[i]} \cap \mathbf{Par}(\sigma_0) \cap N_G)) = l - 1. \tag{3.12}$$

NOTE 3.6 (Geometric interpetation via hypersimplices). If $i \in \{1, \ldots, r-1\}$, then the $(r-1)$-dimensional polytope

$$\begin{aligned}\mathrm{HypS}(i, r) &:= \mathrm{conv}(\{e_{\nu_1} + \cdots + e_{\nu_i} \mid 1 \leq \nu_1 < \nu_2 < \cdots < \nu_i \leq r\}) \\ &= \left\{ (x_1, \ldots, x_r)^\intercal \in \mathbb{R}^r \;\middle|\; 0 \leq x_j \leq 1 \text{ for } 1 \leq j \leq r, \ \sum_{j=1}^{r} x_j = i \right\}\end{aligned} \tag{3.13}$$

has $\binom{r}{i}$ vertices, $2r$ facets for $i \in \{2, \ldots, r-2\}$, and only r facets for $i \in \{1, r-1\}$. It is the so-called (i, r)-*hypersimplex* and can be viewed as the convex hull of the barycenters of the $(i-1)$-dimensional faces of the standard $(r-1)$-dimensional simplex (see [**126**, pp. 19-20]). Figure 1 shows Hyp$(2,4)$ (which is a regular octahedron). Note, in particular, that HypS$(1,r)$ and HypS$(r-1,r)$ are simplices. The sets $\mathfrak{s}_G^{[i]} \cap \mathbf{Par}(\sigma_0)$ can be expressed in terms of hypersimplices as follows:

$$\mathfrak{s}_G \cap \mathbf{Par}(\sigma_0) = \mathrm{HypS}(1,r) \smallsetminus \{e_1, \ldots, e_r\} = \mathfrak{s}_G \smallsetminus \{e_1, \ldots, e_r\}, \tag{3.14}$$

and for $i \in \{2, \ldots, r-1\}$, respectively,

$$\mathfrak{s}_G^{[i]} \cap \mathbf{Par}(\sigma_0) = \mathrm{HypS}(i,r) \smallsetminus \left\{ \mathbf{x} \in \mathrm{HypS}(i,r) \;\middle|\; \begin{array}{c} \text{with } x_j = 1 \\ \text{for all least one} \\ j \in \{1, \ldots, r\} \end{array} \right\}. \tag{3.15}$$

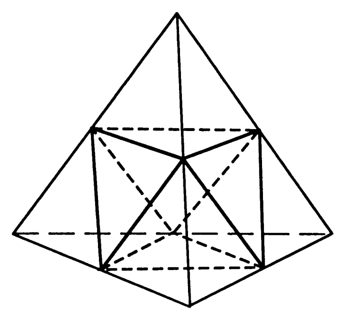

FIGURE 1.

REMARK 3.7. The height (1.2) of an element g of G (as in (3.4)) equals

$$\boxed{\mathrm{ht}(g) = \mathrm{rank}(g - \mathrm{Id}_G) = \sharp\{i \mid 1 \le i \le r \text{ with } \delta_i(j_1, \ldots, j_\kappa) \neq 0\}} \tag{3.16}$$

and specifies the dimension of the face of σ_0 on which $n_g \in N_G$ lies.

DEFINITION 3.8. For $i \in \{1, \ldots, r-1\}$ and $k \in \{2, \ldots, r\}$ we define

$$\boxed{\mathfrak{B}_G(i,k) := \{g \in G \mid \mathrm{age}(g) = i, \ \mathrm{ht}(g) = k\}.}$$

LEMMA 3.9. $\mathfrak{B}_G(i,k) = \varnothing$ *for* $k \le i$.

PROOF. By Proposition 1.8, for each $g \in G \smallsetminus \{\mathrm{Id}_G\}$ we have $\mathrm{ht}(g) > \mathrm{age}(g)$. □

$G \smallsetminus \{\mathrm{Id}_G\}$ can be decomposed as follows:

$$G \smallsetminus \{\mathrm{Id}_G\} = \mathrm{JAI}(G) \stackrel{\bullet}{\bigcup} \mathrm{SAI}(G),$$

where

$$\mathrm{JAI}(G) := \stackrel{\bullet}{\bigcup} \{\mathfrak{B}_G(i,k) \mid 1 \le i < k \le r, \ k \neq 2i\},$$

and

$$\mathrm{SAI}(G) := \stackrel{\bullet}{\bigcup} \left\{ \mathfrak{B}_G(i,2i) \;\middle|\; 1 \le i \le \left\lfloor \frac{r}{2} \right\rfloor \right\}.$$

We call JAI(G) (resp., SAI(G)) the set of elements of $G \smallsetminus \{\mathrm{Id}_G\}$ with inverses *of jumping age* (resp., *of stationary age*), as we have:

LEMMA 3.10 ("Ping-Pong Lemma"). (i) *For* $1 \leq i < k \leq r$, $k \neq 2i$, *there is an one-to-one correspondence*

$$\mathrm{JAI}\,(G) \ni \mathfrak{B}_G\,(i,k) \ni g \longmapsto g^{-1} \in \mathfrak{B}_G\,(k-i,k) \in \mathrm{JAI}\,(G)$$

(ii) *If* $k = 2i$ *and* $g \in \mathrm{SAI}(G)$, *then* $g^{-1} \in \mathrm{SAI}(G)$.

Note that JAI(G) (resp. SAI(G)) is expressed as the disjoint union of exactly $\left(\binom{r}{2} - \left\lfloor \frac{r}{2} \right\rfloor\right)$ $(\in 2\,\mathbb{Z})$ sets (resp. of exactly $\left\lfloor \frac{r}{2} \right\rfloor$ sets). In addition, JAI(G) consists of group elements the cardinality of which is always an *even* number.

DEFINITION 3.11. To treat the group elements on each face of $\mathfrak{s}_G^{[i]}$'s separately, for $1 \leq i \leq r-1$, $2 \leq k \leq r$, and indices $1 \leq \nu_1 < \nu_2 < \cdots < \nu_k \leq r$, we define

$$\boxed{\begin{array}{c} \mathfrak{s}_G^{[i]}\,(\nu_1, \nu_2, \ldots, \nu_k) := \mathrm{conv}\,(\{ie_{\nu_1}, ie_{\nu_2}, \ldots, ie_{\nu_k}\}) \\[2ex] (\mathfrak{s}_G^{[i]} = \mathfrak{s}_G^{[i]}\,(1,2,\ldots,r)\ \ ) \end{array}}$$

and

$$\boxed{\begin{array}{c} \mathfrak{B}_G\,(i,k;\nu_1,\nu_2,\ldots,\nu_k) := \\[2ex] \left\{ g \text{ as in (3.4)},\ g \in \mathfrak{B}_G\,(i,k) \;\middle|\; n_g \in \mathfrak{s}_G^{[i]}\,(\nu_1,\nu_2,\ldots,\nu_k) \cap \mathbf{Par}\,(\sigma_0) \cap N_G \right\}. \end{array}}$$

LEMMA 3.12. *For any* $g \in G$ *the following conditions are equivalent*:

(i) $g \in \mathfrak{B}_G\,(i,k;\nu_1,\nu_2,\ldots,\nu_k)$.

(ii) $n_g \in \mathrm{int}(\mathfrak{s}_G^{[i]}\,(\nu_1,\nu_2,\ldots,\nu_k)) \cap \mathbf{Par}\,(\sigma_0) \cap N_G$.

PROOF. The lattice points which belong to $\partial(\mathfrak{s}_G^{[i]}\,(\nu_1,\nu_2,\ldots,\nu_k)) \cap \mathbf{Par}\,(\sigma_0) \cap N_G$ represent group elements with height $< k$. $\square$

LEMMA 3.13 ("Refined Ping-Pong Lemma"). *For* $1 \leq i < k \leq r$, *and indices* $1 \leq \nu_1 < \nu_2 < \cdots < \nu_k \leq r$, *there is an one-to-one correspondence*

$$\mathfrak{B}_G\,(i,k;\nu_1,\nu_2,\ldots,\nu_k) \ni g \longmapsto g^{-1} \in \mathfrak{B}_G\,(k-i,k;\nu_1,\nu_2,\ldots,\nu_k)$$

PROOF. As both g and g^{-1} (as diagonal matrices) must have the entries which are equal to 1 at the same positions, the assertion can be varified by Lemma 3.10 (i).$\square$

LEMMA 3.14. *For* $1 \leq i \leq r-1$ *and* $k = 2i$ *we have*:

(i) *if* $g \in \mathfrak{B}_G\,(i,2i;\nu_1,\nu_2,\ldots,\nu_{2i})$, *then* $g^{-1} \in \mathfrak{B}_G\,(i,2i;\nu_1,\nu_2,\ldots,\nu_{2i})$, *and*

(ii) *if* $l = |G| \equiv 1\,(\mathrm{mod}\ 2)$, *then* $\mathfrak{B}_G\,(i,2i;\nu_1,\nu_2,\ldots,\nu_{2i})$ *consists of an even number of group elements. Among them there are no elements* g *with* $g = g^{-1}$.

PROOF. For (i) we use the same argument as in the proof of Lemma 3.13. Assertion (ii) can be shown easily as well. If $\mathfrak{B}_G\,(i,2i;\nu_1,\nu_2,\ldots,\nu_{2i})$ would contain an element g with $g = g^{-1}$, then this g would have order 2 and therefore $l \equiv 0\,(\mathrm{mod}\ 2)$ by Lagrange Theorem. This would contradict to our assumption. $\square$

PROPOSITION 3.15. *For a Gorenstein AQS* $(\mathbb{C}^r/G, [\mathbf{0}])$ *the following conditions are equivalent*:

(i) $\mathrm{orb}\,(\sigma_0) \in X\,(N_G, \Delta_G)$ *is isolated.*

(ii) $\bigcup_{i=1}^{r-1}(\partial\mathfrak{s}_G^{[i]} \cap N_G) = \varnothing$ *and* $\bigcup_{i=1}^{r-1}(\mathrm{int}(\mathfrak{s}_G^{[i]}) \cap N_G) \neq \varnothing$.

(iii) $\mathfrak{B}_G\,(i,k;\nu_1,\nu_2,\ldots,\nu_k) = \varnothing$, *for all* $i \in \{1,\ldots,r-1\}, k \in \{2,\ldots,r-1\}$, *and* $1 \leq \nu_1 < \cdots < \nu_k \leq r$, *while* $\mathfrak{B}_G\,(i,r) \neq \varnothing$ *for at least one* $i \in \{1,\ldots,r-1\}$.

PROOF. It follows from Proposition 3.2 and Theorem 3.5. $\square$

LEMMA 3.16. *If there exists an element $g \in G$, such that* $\mathrm{ht}(g) = r$, *then* $\mathrm{orb}(\sigma_0)$ *is an msc-singularity. For G cyclic, the converse is also true.*

PROOF. It follows from (3.16) and the definition of splitting codimension. $\square$

COROLLARY 3.17. *If* $\mathrm{orb}(\sigma_0) \in X(N_G, \Delta_G)$ *is isolated, then* $\mathrm{orb}(\sigma_0)$ *is an msc-singularity.*

EXAMPLE 3.18. Let $(\mathbb{C}^4/G, [\mathbf{0}])$ be the Gorenstein CQS of type $\frac{1}{12}(1,2,3,6)$. This is a non-isolated msc-singularity (by Propositions 3.2, 3.15 and Lemma 3.16). If $G = \{g_0 = \mathrm{Id}_G, g_1, g_2, \ldots, g_{11}\}$ and $n_i := n_{g_i}$ denotes the lattice point of N_G representing g_i, for $1 \leq i \leq 11$, then, up to reenumeration of indices, we find

i	n_i	n_{i+6}
0	$(0,0,0,0)^\intercal$	$\frac{1}{12}(6,0,6,0)^\intercal$
1	$\frac{1}{12}(1,2,3,6)^\intercal$	$\frac{1}{12}(7,2,9,6)^\intercal$
2	$\frac{1}{12}(2,4,6,0)^\intercal$	$\frac{1}{12}(9,6,3,6)^\intercal$
3	$\frac{1}{12}(4,8,0,0)^\intercal$	$\frac{1}{12}(8,4,0,0)^\intercal$
4	$\frac{1}{12}(3,6,9,6)^\intercal$	$\frac{1}{12}(10,8,6,0)^\intercal$
5	$\frac{1}{12}(5,10,3,6)^\intercal$	$\frac{1}{12}(11,10,9,6)^\intercal$

Obviously, g_1, g_2, g_3, g_6, g_9 are juniors, $g_4, g_5, g_7, g_8, g_{10}$ are seniors of age 2, and g_{11} is the only senior of age 3. Furthermore,

$$\begin{cases} \mathrm{JAI}(G) = \mathfrak{B}_G(1,3) \overset{\bullet}{\bigcup} \mathfrak{B}_G(1,4) \overset{\bullet}{\bigcup} \mathfrak{B}_G(2,3) \overset{\bullet}{\bigcup} \mathfrak{B}_G(3,4), \\ \mathrm{SAI}(G) = \mathfrak{B}_G(1,2) \overset{\bullet}{\bigcup} \mathfrak{B}_G(2,4), \end{cases}$$

where

$$\mathfrak{B}_G(1,3) = \mathfrak{B}_G(1,3;1,2,3) = \{g_2\}, \ \mathfrak{B}_G(1,4) = \{g_1\},$$
$$\mathfrak{B}_G(2,3) = \mathfrak{B}_G(2,3;1,2,3) = \{g_{10}\}, \mathfrak{B}_G(3,4) = \{g_{11}\},$$

$$\mathfrak{B}_G(1,2) = \mathfrak{B}_G(1,2;1,2) \overset{\bullet}{\bigcup} \mathfrak{B}_G(1,2;1,3), \ \mathfrak{B}_G(1,2;1,2) = \{g_3, g_9\},$$

and

$$\mathfrak{B}_G(1,2;1,3) = \{g_6\}, \ \mathfrak{B}_G(2,4) = \{g_4, g_5, g_7, g_8\}.$$

Figure 2 shows the location of the lattice points $n_1, \ldots, n_{11}$ on the junior tetrahedron $\mathfrak{s}_G = \mathfrak{s}_G^{[1]}$ and on the two tetrahedra $\mathfrak{s}_G^{[2]}$ and $\mathfrak{s}_G^{[3]}$, containing the representatives of the junior and of the senior elements of ages 2 and 3, respectively. (For aesthetic reasons, $\mathfrak{s}_G^{[2]}$ and $\mathfrak{s}_G^{[3]}$ are scaled by $\frac{1}{2}$ and $\frac{1}{3}$, respectively.) The dotted lines (with arrows at their ends) indicate how the refined Ping-Pong Lemma 3.13 and Lemma 3.14 (i) are applied in practice. Note that $g_{12-i} = g_i^{-1}$, for all i, $1 \leq i \leq 6$.

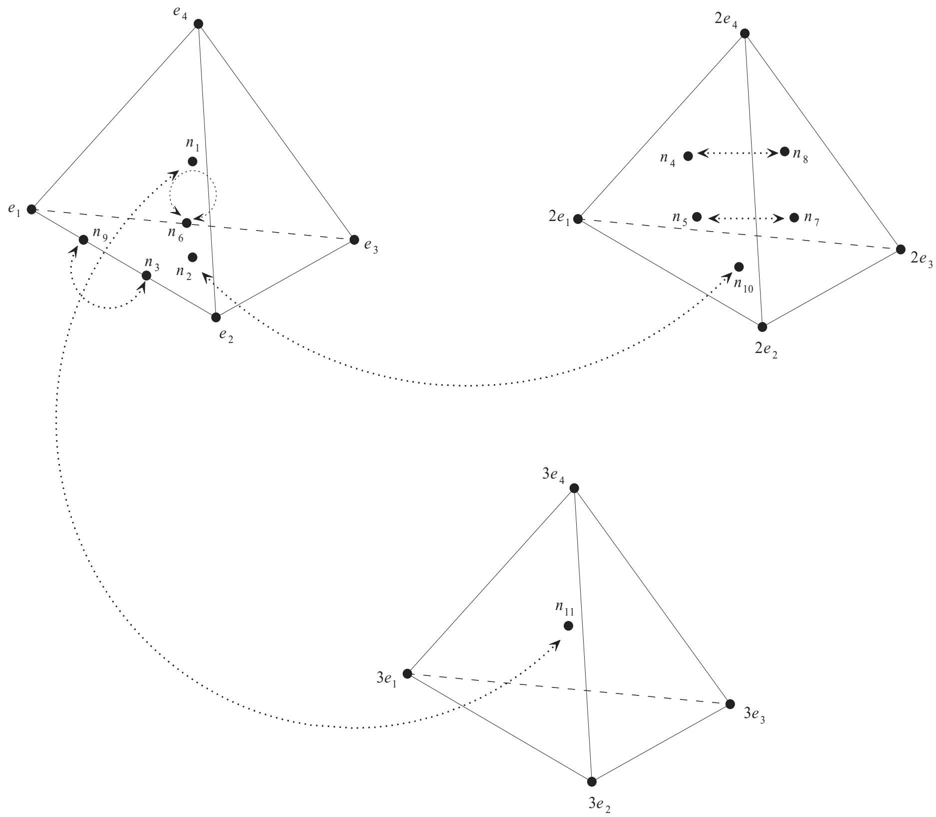

FIGURE 2.

• **On the cardinality of** $\mathfrak{B}_G\,(1,k)$. Let $(\mathbb{C}^r/G,[\mathbf{0}])$ be a Gorenstein AQS of type (3.6). The *counting* of the lattice points of N_G representing all the elements of $\mathfrak{B}_G\,(i,k;\nu_1,\nu_2,\ldots,\nu_k)$'s and $\mathfrak{B}_G\,(i,k)$'s (with emphasis on $\mathfrak{B}_G\,(1,k)$) is of particular interest (cf. §9). By Lemma 3.12 we get

$$\sharp\,(\mathfrak{B}_G\,(i,k;\nu_1,\nu_2,\ldots,\nu_k)) = \sharp(\mathrm{int}(\mathfrak{s}_G^{[i]}\,(\nu_1,\nu_2,\ldots,\nu_k)\cap \mathbf{Par}\,(\sigma_0)\cap N_G)), \quad (3.17)$$

and for $i=1$,

$$\sharp\,(\mathfrak{B}_G\,(1,k;\nu_1,\nu_2,\ldots,\nu_k)) \overset{(3.10)}{=} \sharp(\mathrm{int}(\mathfrak{s}_G\,(\nu_1,\nu_2,\ldots,\nu_k)\cap N_G)). \quad (3.18)$$

Moreover,

$$\sharp\,(\mathfrak{B}_G\,(i,k)) = \sum_{1\leq\nu_1<\cdots<\nu_k\leq r} \sharp\,(\mathfrak{B}_G\,(i,k;\nu_1,\nu_2,\ldots,\nu_k)) \quad (3.19)$$

$$= \sum_{1\leq\nu_1<\cdots<\nu_k\leq r} \sharp(\mathrm{int}(\mathfrak{s}_G^{[i]}\,(\nu_1,\nu_2,\ldots,\nu_k)\cap \mathbf{Par}\,(\sigma_0)\cap N_G))$$

and for $i = 1$,

$$\boxed{\begin{aligned} \sharp\left(\mathfrak{B}_G\left(1,k\right)\right) &= \sum_{1\leq\nu_1<\cdots<\nu_k\leq r} \sharp\left(\mathfrak{B}_G\left(1,k;\nu_1,\nu_2,\ldots,\nu_k\right)\right) \\ &= \sum_{1\leq\nu_1<\cdots<\nu_k\leq r} \sharp(\mathrm{int}(\mathfrak{s}_G\left(\nu_1,\nu_2,\ldots,\nu_k\right)\cap N_G)) \end{aligned}} \tag{3.20}$$

The cardinality (3.17) can be written as follows:

$$\sharp\left(\mathfrak{B}_G\left(i,k;\nu_1,\nu_2,\ldots,\nu_k\right)\right) =$$

$$= \sharp\left\{ \lambda\in[0,l)\cap\mathbb{Z} \;\middle|\; \begin{array}{c} \sum\limits_{\rho=1}^{r} \delta_\rho\left(j_1,\ldots,j_\kappa\right) = i\cdot\exp\left(G\right) \text{ and} \\ \delta_\rho\left(j_1,\ldots,j_\kappa\right) \left\{ \begin{array}{ll} \neq 0, & \text{if } \rho\in\{\nu_1,\ldots,\nu_k\} \\ = 0, & \text{if } \rho\notin\{\nu_1,\ldots,\nu_k\} \end{array}\right. \\ \text{for all} \quad \rho\in\{1,2,\ldots,r\} \end{array} \right\}. \tag{3.21}$$

Proposition 3.19. *Let $r\geq 3$, $2\leq k\leq r-1$, $\{\nu_1,\nu_2,\ldots,\nu_k\}\subset\{1,2,\ldots,r\}$ be a family of indices, such that $1\leq\nu_1<\cdots<\nu_k\leq r$, and $\left\{\nu_1',\nu_2',\ldots,\nu_{r-k}'\right\} = \{1,2,\ldots,r\}\smallsetminus\{\nu_1,\nu_2,\ldots,\nu_k\}$ be its complement. If $(\mathbb{C}^r/G,[\mathbf{0}])$ is Gorenstein msc-CQS of type* (3.7), *then the number of group elements having fixed height k and arbitrary age equals*

$$\begin{aligned} \sum_{i=1}^{r-1}\sharp\left(\mathfrak{B}_G\left(i,k;\nu_1,\nu_2,\ldots,\nu_k\right)\right) &= \sum_{i=1}^{k-1}\sharp\left(\mathfrak{B}_G\left(i,k;\nu_1,\nu_2,\ldots,\nu_k\right)\right) \\ &= \left[\gcd\left(\alpha_{\nu_1'},\ldots,\alpha_{\nu_{r-k}'},l\right)-1\right] \end{aligned}$$

$$-\sum_{\substack{1\leq\rho_1<\cdots<\rho_y\leq k \\ \text{with } 1\leq y\leq k-2 \text{ and} \\ \{\rho_1,\ldots,\rho_y\}\subset\{\nu_1,\nu_2,\ldots,\nu_k\}}} \left[\gcd\left(\alpha_{\nu_1'},..,\alpha_{\nu_{r-k}'},\alpha_{\rho_1},..,\alpha_{\rho_y},l\right)-1\right] \tag{3.22}$$

(For $k = 2$ we simply omit this last sum.)

Proof. The first equality is clear by Lemma 3.9. On the other hand,

$$\begin{aligned} &\sum_{i=1}^{r-1}\sharp(\mathfrak{s}_G^{[i]}\left(\nu_1,\nu_2,\ldots,\nu_k\right)\cap\mathbf{Par}\left(\sigma_0\right)\cap N_G) \\ &= \sum_{i=1}^{k-1}\sharp(\mathfrak{s}_G^{[i]}\left(\nu_1,\nu_2,\ldots,\nu_k\right)\cap\mathbf{Par}\left(\sigma_0\right)\cap N_G) \\ &= \sharp\left\{\lambda\in\mathbb{Z}\;\middle|\;1\leq\lambda\leq l-1:\lambda\alpha_{\nu_i'}\equiv 0\,(\mathrm{mod}\ l),\ \forall i,\ 1\leq i\leq r-k\right\} \\ &= \gcd(\alpha_{\nu_1'},\ldots,\alpha_{\nu_{r-k}'},l)-1. \end{aligned}$$

The sum we subtract in (3.22) is nothing but the evaluation of the number

$$\sum_{i=1}^{k-1}\sharp\left(\partial\mathfrak{s}_G^{[i]}\left(\nu_1,\nu_2,\ldots,\nu_k\right)\cap\mathbf{Par}\left(\sigma_0\right)\cap N_G\right)$$

of lattice points lying on the corresponding relative boundaries. □

Remark 3.20. For arbitrary $r\geq 3$, the numbers $\sharp\ (\mathfrak{B}_G\left(1,2;\nu_1,\nu_2\right))$ and

$$\sharp\ \left(\mathfrak{B}_G\left(1,3;\nu_1,\nu_2,\nu_3\right)\right) = \sharp\ \left(\mathfrak{B}_G\left(2,3;\nu_1,\nu_2,\nu_3\right)\right)$$

can be determined by the formula (3.22); the first of them directly (because in this case the left-hand side of (3.22), consists of only one summand), and the latter just as the half of what we get from the right-hand side of (3.22). For $r \in \{3,4\}$, as a byproduct of this formula and of the refined Ping-Pong Lemma 3.13, one gets a simple method to count the number of the lattice points lying in the relative interior of each proper face of the junior simplex separately!

NOTE 3.21. One can analogously compute $\sum_{i=1}^{r-1} \sharp(\mathfrak{B}_G(i,k;\nu_1,\nu_2,\ldots,\nu_k))$ whenever the acting group G is abelian but not cyclic. In this case, the formula generalizing (3.22) contains denumerants of restricted weighted vectorial partitions instead of greatest common divisors.

4. Crepant Resolutions of Gorenstein AQS

Let $(\mathbb{C}^r/G,[\mathbf{0}])$, $r \geq 3$, be a Gorenstein AQS. In this section we explain how the crepant (partial or full) $\mathbb{T}_{N_G}$-equivariant desingularizations of its underlying space are to be studied by means of *lattice triangulations* of the junior simplex $\mathfrak{s}_G$.

DEFINITION 4.1. We denote the set of all lattice triangulations $\mathcal{T}$ of $\mathfrak{s}_G$ w.r.t. N_G (with $\mathrm{vert}(\mathcal{T}) \subseteq \mathfrak{s}_G \cap N_G$, cf. C.6) by $\mathbf{LTR}_{N_G}(\mathfrak{s}_G)$, and define

$$\mathbf{LTR}^{\max}_{N_G}(\mathfrak{s}_G) := \left\{ \mathcal{T} \in \mathbf{LTR}_{N_G}(\mathfrak{s}_G) \;\middle|\; \begin{array}{l} \mathcal{T} \text{ is a } \textit{maximal triangulation} \\ \text{of } \mathfrak{s}_G, \text{ i.e., } \mathrm{vert}(\mathcal{T}) = \mathfrak{s}_G \cap N_G \end{array} \right\},$$

$$\mathbf{LTR}^{\mathrm{basic}}_{N_G}(\mathfrak{s}_G) := \left\{ \mathcal{T} \in \mathbf{LTR}^{\max}_{N_G}(\mathfrak{s}_G) \;\middle|\; \begin{array}{l} \mathcal{T} \text{ is a } \textit{basic} \text{ triangulation} \\ \text{of } \mathfrak{s}_G \text{ (see Definition C.7)} \end{array} \right\}.$$

Adding the prefix **Coh-** to any one of the above sets, we mean the subset consisting of coherent triangulations (in the sense of B.1). The *hierarchy of lattice triangulations* of $\mathfrak{s}_G$ is given by the following inclusion diagram:

$$\begin{array}{ccccc} \mathbf{LTR}^{\mathrm{basic}}_{N_G}(\mathfrak{s}_G) & \subset & \mathbf{LTR}^{\max}_{N_G}(\mathfrak{s}_G) & \subset & \mathbf{LTR}_{N_G}(\mathfrak{s}_G) \\ \bigcup & & \bigcup & & \bigcup \\ \mathbf{Coh\text{-}LTR}^{\mathrm{basic}}_{N_G}(\mathfrak{s}_G) & \subset & \mathbf{Coh\text{-}LTR}^{\max}_{N_G}(\mathfrak{s}_G) & \subset & \mathbf{Coh\text{-}LTR}_{N_G}(\mathfrak{s}_G) \end{array}$$

NOTE 4.2. (i) There is a bijection between the triangulations belonging to $\mathbf{LTR}_{N_G}(\mathfrak{s}_G)$ (resp., to $\mathbf{Coh\text{-}LTR}_{N_G}(\mathfrak{s}_G)$) and the vertex set of the *universal polytope* $\mathrm{Un}(\mathcal{V})$ (resp., of the *secondary polytope* $\mathrm{Sec}(\mathcal{V})$) of $\mathfrak{s}_G$ w.r.t. the point configuration $\mathcal{V} = \mathfrak{s}_G \cap N_G$; see Appendix B.

(ii) There exist always coherent maximal triangulations of $\mathfrak{s}_G$'s (see B.2).

(iii) For $r \geq 3$ there are $\mathfrak{s}_G$'s admitting maximal non-coherent triangulations.

(iv) For $r \geq 4$ there are lots of $\mathfrak{s}_G$'s admitting maximal non-basic triangulations.

(v) It is not known, as yet, if there are $\mathfrak{s}_G$'s for which $\mathbf{LTR}^{\mathrm{basic}}_{N_G}(\mathfrak{s}_G) \neq \varnothing$, whereas $\mathbf{Coh\text{-}LTR}^{\mathrm{basic}}_{N_G}(\mathfrak{s}_G) = \varnothing$ (see C.8 (iii)).

(v) An immediate consequence of Theorem 4.4 is that the "Existence Problem", as stated in §1, is in the abelian case *equivalent* to the following:

► **Existence Problem for Gorenstein AQS**: For *which abelian finite subgroups* $G \subset \mathrm{SL}(r,\mathbb{C})$, $r \geq 4$, *do there exist triangulations* $\mathcal{T} \in \mathbf{LTR}^{\mathrm{basic}}_{N_G}(\mathfrak{s}_G)$ (*and preferably* $\mathcal{T} \in \mathbf{Coh\text{-}LTR}^{\mathrm{basic}}_{N_G}(\mathfrak{s}_G)$)?

This demonstrates one more example of the interplay between algebraic and discrete geometry. The initial problem is fairly difficult, yet once translated into discrete geometry, it can be treated by using familiar tools.

DEFINITION 4.3. Identifying $\mathbb{C}^r/G$ with $X(N_G,\Delta_G)$ as in §3, let

$$\sigma_{\mathbf{s}} := \mathrm{pos}(\mathbf{s}) \subset (N_G)_{\mathbb{R}} \cong \mathbb{R}^r$$

denote the cone supporting a simplex $\mathbf{s}$ of a $\mathcal{T} \in \mathbf{LTR}_{N_G}(\mathfrak{s}_G)$. We define the fan

$$\widehat{\Delta_G}(\mathcal{T}) := \{\sigma_{\mathbf{s}} \mid \mathbf{s} \in \mathcal{T}\}$$

and

$$\mathbf{PCDES}(X(N_G,\Delta_G)) := \left\{ \begin{array}{c} \textit{partial crepant } \mathbb{T}_{N_G}\text{-equivariant} \\ \text{desingularizations of } X(N_G,\Delta_G) \\ \text{with overlying spaces having} \\ \text{at worst } \mathbb{Q}\text{-factorial } \textit{canonical} \\ \text{singularities of index 1} \end{array} \right\},$$

$$\mathbf{PCDES}^{\max}(X(N_G,\Delta_G)) := \left\{ \begin{array}{c} \textit{partial crepant } \mathbb{T}_{N_G}\text{-equivariant} \\ \text{desingularizations of } X(N_G,\Delta_G) \\ \text{with overlying spaces having} \\ \text{at worst } \mathbb{Q}\text{-factorial } \textit{terminal} \\ \text{singularities of index 1} \end{array} \right\},$$

$$\mathbf{CDES}(X(N_G,\Delta_G)) := \left\{ \begin{array}{c} \textit{crepant } \mathbb{T}_{N_G}\text{-equivariant (full)} \\ \text{desingularizations of } X(N_G,\Delta_G) \end{array} \right\}.$$

(Whenever we put the prefix **QP-** in the front of any one of them, we mean the corresponding subset of it consisting of those desingularizations whose overlying spaces are *quasiprojective*.)

THEOREM 4.4 (Desingularizing by triangulations). *Let $(\mathbb{C}^r/G,[\mathbf{0}])$ be a Gorenstein AQS $(r \geq 3)$. Then there exist one-to-one correspondences*:

$$\boxed{\begin{array}{ccc} (\mathbf{Coh}\text{-})\,\mathbf{LTR}^{\mathrm{basic}}_{N_G}(\mathfrak{s}_G) & \overset{1:1}{\longleftrightarrow} & (\mathbf{QP}\text{-})\,\mathbf{CDES}(X(N_G,\Delta_G)) \\ \cap & & \cap \\ (\mathbf{Coh}\text{-})\,\mathbf{LTR}^{\max}_{N_G}(\mathfrak{s}_G) & \overset{1:1}{\longleftrightarrow} & (\mathbf{QP}\text{-})\,\mathbf{PCDES}^{\max}(X(N_G,\Delta_G)) \\ \cap & & \cap \\ (\mathbf{Coh}\text{-})\,\mathbf{LTR}_{N_G}(\mathfrak{s}_G) & \overset{1:1}{\longleftrightarrow} & (\mathbf{QP}\text{-})\,\mathbf{PCDES}(X(N_G,\Delta_G)) \end{array}}$$

which are realized by crepant $\mathbb{T}_{N_G}$-equivariant birational morphisms of the form

$$\boxed{f_{\mathcal{T}} = \mathrm{id}_* : X(N_G,\ \widehat{\Delta_G}(\mathcal{T})) \longrightarrow X(N_G,\Delta_G)} \tag{4.1}$$

induced by mapping

$$\mathcal{T} \longmapsto \widehat{\Delta_G}(\mathcal{T}), \qquad \widehat{\Delta_G}(\mathcal{T}) \longmapsto X(N_G,\ \widehat{\Delta_G}(\mathcal{T})).$$

Sketch of proof. $X(N_G, \Delta_G)$ is Gorenstein and has at most rational singularities, i.e., canonical singularities of index 1. Moreover, its dualizing sheaf is trivial. Let

$$f = \mathrm{id}_* : X(N_G,\ \widetilde{\Delta_G}) \longrightarrow X(N_G, \Delta_G)$$

denote an arbitrary partial desingularization. Studying the behaviour of the highest rational differentials on $X(N_G,\ \widetilde{\Delta_G})$ (see [**102**, §3] or [**29**, Proposition 4.1]), one proves

$$K_{X(N_G,\ \widetilde{\Delta_G})} = f^*\left(K_{X(N_G,\Delta_G)}\right) - \sum_{\varrho\in\widetilde{\Delta_G}(1)} \left(\langle(1,\ldots,1), n(\varrho)\rangle - 1\right)\ D_{n(\varrho)}\ ,$$

where

$$D_{n(\varrho)} := \mathbf{V}(\varrho) = \mathbf{V}(\mathrm{pos}(\{n(\varrho)\}))\,.$$

Obviously, f is crepant if and only if $\mathrm{Gen}(\widetilde{\Delta_G}) \subset \mathcal{H}_1$, and since the number of crepant exceptional prime divisors is independent of the specific choice of f, the first and second 1-1 correspondences (from below) are obvious by the adjunction-theoretic definition of terminal (resp., canonical) singularities. In particular, all $\mathbb{T}_{N_G}$-equivariant partial crepant desingularizations of $X(N_G, \Delta_G)$ of the form (4.1) have overlying spaces with at most $\mathbb{Q}$-factorial singularities; and conversely, each partial $\mathbb{T}_{N_G}$-equivariant crepant desingularization with overlying space with at worst $\mathbb{Q}$-factorial singularities, has to be of this form. ($\mathbb{Q}$-factoriality is here equivalent to the consideration only of triangulations instead of more general polytopal subdivisions; cf. Proposition 2.6. Furthermore, by maximal triangulations we exhaust all crepant prime divisors). The top 1-1 correspondence follows from the equivalence

$$\mathcal{T} \ni \mathbf{s} \text{ is a basic simplex w.r.t. } N_G \Longleftrightarrow \sigma_{\mathbf{s}} \in \widehat{\Delta_G}(\mathcal{T}) \text{ is a basic cone w.r.t. } N_G.$$

To prove the 1-1 correspondences after omitting the brackets, it suffices to use the fact that the coherence of a triangulation $\mathcal{T} \in \mathbf{LTR}_{N_G}(\mathfrak{s}_G)$ implies the existence of a strictly upper convex function defined on the support of the entire fan $\widehat{\Delta_G}(\mathcal{T})$, and then to apply ampleness criterion; cf. [**29**, Proposition 4.5, p. 211]. □

Remark 4.5. Concerning the existence or non-existence of lattice triangulations $\mathcal{T} \in \mathbf{LTR}^{\mathrm{basic}}_{N_G}(\mathfrak{s}_G)$ for Gorenstein AQS $(\mathbb{C}^r/G, [\mathbf{0}])$ of type (3.6) we can w.l.o.g. restrict ourselves to the class of *msc*-AQS (as defined in 3.3), because the existence question for a non-msc-singularity is obviously reduced to the same question for an msc-singularity of *strictly smaller* dimension. (The recognition of the those which are msc-singularities follows from Proposition 3.4.)

Note 4.6 (Exceptional prime divisors). If $\mathcal{T} \in \mathbf{LTR}^{\mathrm{basic}}_{N_G}(\mathfrak{s}_G)$, then the exceptional prime divisors $D_{n(\varrho)}$ ($\varrho \in \widehat{\Delta_G}(\mathcal{T})(1)\smallsetminus\Delta_G(1)$) w.r.t. $f_{\mathcal{T}}$ are $(r-1)$-dimensional toric varieties whose topological Euler number (2.2) equals

$$\chi(D_{n(\varrho)}) = \sharp\{(r-1)\text{-dimensional simplices of } \mathrm{star}_{n(\varrho)}(\mathcal{T})\}.$$

Moreover, $D_{n(\varrho)}$ is compact if and only if $n(\varrho) \in \mathrm{int}(\mathfrak{s}_G)$. On the other hand, if $\varrho \in \widehat{\Delta_G}(\mathcal{T})(1)$ and $n(\varrho) \in \mathrm{int}(\mathfrak{s}_G(\nu_1,\ldots,\nu_k)) \cap N_G$, for some k, $2 \le k \le r-1$, and certain $1 \le \nu_1 < \nu_2 < \cdots < \nu_k \le r$, then the non-compact $D_{n(\varrho)}$ can be viewed as the total space of a fibration $D_{n(\varrho)} \longrightarrow \mathbb{C}^{r-k}$. The generic fibers are isomorphic to the $(k-1)$-dimensional compact toric variety associated to the star of ϱ within

$$\{\sigma \in \widehat{\Delta_G}(\mathcal{T}) \mid \sigma \prec \sigma_{\mathbf{s}},\ \mathbf{s} \in \mathfrak{s}_G(\nu_1,\ldots,\nu_k)\cap\mathcal{T}\}$$

In many cases, looking at the $\mathrm{star}_{n(\varrho)}(\mathcal{T})$, one can say more about the structure of $D_{n(\varrho)}$. (For concrete classes of examples, see below Remark 5.6 (i), Theorem 7.2, and Remark 7.5 (ii).)

• **Cohomology dimensions**. Using (3.10) and (3.11) we deduce from Theorem 1.17 the following:

Theorem 4.7. *If $(\mathbb{C}^r/G,[\mathbf{0}])$ is a Gorenstein AQS, then for any crepant desingularization $\widehat{X} \longrightarrow X$ of $X = \mathbb{C}^r/G$ we have* $\dim_{\mathbb{Q}} H^0(\widehat{X},\mathbb{Q}) = 1$,

$$\dim_{\mathbb{Q}} H^{2i}(\widehat{X},\mathbb{Q}) = \sharp(\mathfrak{s}_G^{[i]} \cap \mathbf{Par}(\sigma_0)\cap N_G), \quad \forall i \in \{1,\ldots,r-1\}, \tag{4.2}$$

and the other cohomology groups of $\widehat{X}$ are trivial. In particular, $\chi(\widehat{X}) = |G|$.

To determine the cohomology dimensions (4.2) you may exploit the description (3.14)-(3.15) of $\mathfrak{s}_G^{[i]} \cap \mathbf{Par}(\sigma_0)$ in terms of hypersimplices $\mathrm{HypS}(i,r)$. But if you don't like to work directly with hypersimplices, here is an alternative: Compute the coefficients of the Ehrhart polynomial of the junior simplex $\mathfrak{s}_G$ by the formulae given in Appendix D, and then apply (4.3) instead of (4.2).

Theorem 4.8. *Maintaining the notation and the assumptions of* 4.7, *we have*

$$\boxed{\dim_{\mathbb{Q}} H^{2i}(\widehat{X},\mathbb{Q}) = \mathfrak{h}_i^*(\mathfrak{s}_G) = \sum_{j=0}^{r-1}\left(\sum_{\kappa=0}^{i}(-1)^{\kappa}\binom{r}{\kappa}(i-\kappa)^j\right)\mathbf{a}_j(\mathfrak{s}_G),} \tag{4.3}$$

for all $i \in \{0,1,\ldots,r-1\}$, where by $\mathfrak{h}_i^*(\mathfrak{s}_G)$ is denoted the i-th component of the $\mathbf{h}^*$-vector and by $\mathbf{a}_j(\mathfrak{s}_G)$ the j-th coefficient of the Ehrhart polynomial of $\mathfrak{s}_G$, respectively. (See C.1 and C.3.)

Proof. If there exists a crepant desingularization $\widehat{X} \longrightarrow X = \mathbb{C}^r/G$, then there exists also a $\mathbb{T}_{N_G}$-equivariant crepant desingularization (4.1) induced by a triangulation $\mathcal{T} \in \mathbf{LTR}_{N_G}^{\text{basic}}(\mathfrak{s}_G)$. Using Theorem 1.17, [**3**, Thm. 4.4] and Theorem C.9, we get for all $i \in \{0,1,\ldots,r-1\}$,

$$\dim_{\mathbb{Q}} H^{2i}(\widehat{X},\mathbb{Q}) = \dim_{\mathbb{Q}} H^{2i}(X(N_G,\widehat{\Delta_G}(\mathcal{T})),\mathbb{Q}) = \mathfrak{h}_i(\mathcal{T}) = \mathfrak{h}_i^*(\mathfrak{s}_G),$$

and it suffices to apply formula (C.5) (for $P = \mathfrak{s}_G$, $d = r-1$) to obtain (4.3). □

Remark 4.9 (A simple basicness criterion). If $(\mathbb{C}^r/G,[\mathbf{0}])$ is a Gorenstein AQS, and $\mathcal{T} \in \mathbf{LTR}_{N_G}(\mathfrak{s}_G)$, then by (A.2), (C.4) and (C.6) we get

$$\mathfrak{f}_{r-1}(\mathcal{T}) = \sharp\left\{\begin{array}{c}(r-1)\text{-dim.}\\ \text{simplices of } \mathcal{T}\end{array}\right\} = \sum_{i=0}^{r-1}\mathfrak{h}_i(\mathcal{T}) \le \sum_{i=0}^{r-1}\mathfrak{h}_i^*(\mathfrak{s}_G) = (r-1)!\,\mathrm{Vol}(\mathfrak{s}_G),$$

which implies

$$\mathfrak{f}_{r-1}(\mathcal{T}) = \chi(X(N_G,\widehat{\Delta}_G(\mathcal{T}))) \le (r-1)!\,\mathrm{Vol}(\mathfrak{s}_G) = |G| = \tfrac{1}{\det(N_G)},$$

cf. (2.2). This holds as equality:

$$\mathfrak{f}_{r-1}(\mathcal{T}) = \chi(X(N_G,\widehat{\Delta}_G(\mathcal{T}))) = (r-1)!\,\mathrm{Vol}(\mathfrak{s}_G) = |G| \tag{4.4}$$

if and only if $\mathcal{T} \in \mathbf{LTR}^{\text{basic}}_{N_G}(\mathfrak{s}_G)$ (by Theorem C.9). In practice, having a concrete maximal triangulation $\mathcal{T}$ of $\mathfrak{s}_G$ in hand, it suffices to compare $\mathfrak{f}_{r-1}(\mathcal{T})$ with $|G|$. If these two numbers coincide, then $\mathcal{T}$ has to be basic.

• **Flops**. If $\mathcal{T}, \mathcal{T}'$ are two *coherent* lattice triangulations of $\mathfrak{s}_G$, are there "elementary operations" whose repetitive use would geometrically describe how one can obtain $\mathcal{T}'$ from $\mathcal{T}$? On the level of triangulations a satisfactory answer is given by the *bistellar flips*[7] (as defined in combinatorial topology). If, in addition, $\mathcal{T}, \mathcal{T}'$ are assumed to be maximal, this answer on the level of birational maps connecting $X(N_G, \widehat{\Delta}_G(\mathcal{T}))$ with $X(N_G, \widehat{\Delta}_G(\mathcal{T}'))$ leads to algebro-geometric *flops*.

Theorem 4.10 (Bistellar flips, and flops). (i) *If* $\mathcal{T}, \mathcal{T}' \in \mathbf{Coh\text{-}LTR}_{N_G}(\mathfrak{s}_G)$, *then there exist finitely many circuits*

$$\mathcal{C}_1, \ldots, \mathcal{C}_\kappa \subset \mathfrak{s}_G \cap N_G, \quad \textit{and} \quad \mathcal{T}_1, \ldots, \mathcal{T}_\kappa \in \mathbf{Coh\text{-}LTR}_{N_G}(\mathfrak{s}_G)$$

such that $\mathcal{T}_{i+1} = \mathrm{FL}_{\mathcal{C}_i}(\mathcal{T}_i)$ (*i.e., such that* $\mathcal{T}_{i+1}$ *is the* bistellar flip *of* $\mathcal{T}_i$ *along* $\mathcal{C}_i$, cf. B.9) *for all* $i \in \{1, \ldots, \kappa - 1\}$, *with* $\mathcal{T}_1 = \mathcal{T}$ *and* $\mathcal{T}_\kappa = \mathcal{T}'$.

(ii) *In particular, if* $\mathcal{T}, \mathcal{T}' \in \mathbf{Coh\text{-}LTR}^{max}_{N_G}(\mathfrak{s}_G)$, *then the circuits* $\mathcal{C}_1, \ldots, \mathcal{C}_\kappa$ *can be chosen in such a way that* $\sharp(\mathcal{C}_i) = r + 1$ *and* $\dim(\mathrm{conv}(\mathcal{C}_i)) = r - 1$ *for all* $i \in \{1, \ldots, \kappa - 1\}$. *Setting* $X_i := X(N_G, \widehat{\Delta}_G(\mathcal{T}_i))$, $X := X_1$, *and* $X' := X_\kappa$, *we conclude that* X *and* X' *can be obtained from each other by a finite succession of flops*[8] *which fit together into the following diagram*:

$$\begin{tikzcd}
 & Y_1 & & Y_2 & & & & Y_{\kappa-1} & \\
X_1 \arrow[ur, "\varphi_1"] \arrow[ddrrrr, bend right, "f_{\mathcal{T}_1}"'] & & X_2 \arrow[ll, dashed] \arrow[ul, "\vartheta_1"'] \arrow[ur, "\varphi_2"] \arrow[ddrr, bend right, "f_{\mathcal{T}_2}"] & & X_3 \arrow[ll, dashed] \arrow[ul, "\vartheta_2"'] \arrow[dd, "f_{\mathcal{T}_3}"'] & \cdots \arrow[l, dashed] & X_{\kappa-1} \arrow[l, dashed] \arrow[ur, "\varphi_{\kappa-1}"] \arrow[ddll, bend left, "f_{\mathcal{T}_{\kappa-1}}"'] & & X_\kappa \arrow[ll, dashed] \arrow[ul, "\vartheta_{\kappa-1}"'] \arrow[ddllll, bend left, "f_{\mathcal{T}_\kappa}"] \\
 & & & & & & & & \\
 & & & & X(N_G, \Delta_G) & & & &
\end{tikzcd}$$

Here, by "flops" *we mean the upper triangles of the diagram, where both* φ_i *and* ϑ_i *are* small *birational morphisms* (*i.e., their exceptional loci have codimension* ≥ 2) *and* $X_{i+1} \dashrightarrow X_i$ *birational maps which are isomorphisms in codimension* 1.

Proof. We shall use the notation and the terminology introduced in Appendix B. (i) Consider an edge path $\overline{\mathbf{v}_1\mathbf{v}_2}, \overline{\mathbf{v}_2\mathbf{v}_3}, \ldots, \overline{\mathbf{v}_{\kappa-1}\mathbf{v}_\kappa}$ on the polytope $\mathrm{Sec}(\mathfrak{s}_G \cap N_G)$ connecting $\mathbf{v}_1 := \mathbf{v}_{\mathcal{T}_1}$ with $\mathbf{v}_\kappa := \mathbf{v}_{\mathcal{T}_\kappa}$. By Theorem B.10 one determines circuits $\mathcal{C}_1, \ldots, \mathcal{C}_\kappa \subset \mathfrak{s}_G \cap N_G$ such that $\mathbf{v}_i = \mathbf{v}_{\mathcal{T}_i}$ with $\mathcal{T}_{i+1} = \mathrm{FL}_{\mathcal{C}_i}(\mathcal{T}_i)$, $\forall i \in \{1, \ldots, \kappa - 1\}$.

---

[7] One of the main reasons for adding to the triangulations involved in the above formulation of Existence Problem the phrase "preferably coherent" is their connection by a sequence of bistellar flips. This does not hold in general for non-coherent triangulations. For instance, Santos provided in [**109**] a point set whose space of (all) triangulations is bistellarly *disconnected*.

[8] In the MMP-language (and as long as one may work in the category of *quasi*projective complex varieties) we say that the "minimal models" X and X' are connected by a sequence of flops whose "termination" is due to our specific setting; cf. [**85**, Thm. 3.4.6, p. 158]. In fact, these flops can be conceived of as high-dimensional analogues of the original "Atiyah's flop" (see [**85**, Example 3.4.3, p. 157]).

(ii) If $\mathcal{T}$ is a maximal triangulation of $\mathfrak{s}_G$, and $\mathbf{s}_1, \mathbf{s}_2$ two $(r-1)$-dimensional simplices of $\mathcal{T}$ having $\mathbf{s}_1 \cap \mathbf{s}_2$ as $(r-2)$-dimensional common face, then $\mathcal{T}|_{\mathbf{s}_1 \cup \mathbf{s}_2}$ is either the triangulation $\mathcal{Y}_+(\mathcal{C})$ or the triangulation $\mathcal{Y}_-(\mathcal{C})$ of conv(vert($\mathbf{s}_1$)∪vert($\mathbf{s}_2$)) w.r.t. the circuit $\mathcal{C} = \text{vert}(\mathbf{s}_1) \cup \text{vert}(\mathbf{s}_2)$ with $\sharp(\mathcal{C}) = r+1$ (cf. Lemma B.7). To pass from $\mathcal{T}$ to another maximal triangulation $\mathcal{T}'$ it suffices to apply (i) for circuits $\mathcal{C}_1, \ldots, \mathcal{C}_\kappa$ *only of this kind*. (This follows from results of Oda & Park [**97**, Corollary 3.9, Proposition 3.10, and Theorem 3.12, pp. 395-398].) After having determined such $\mathcal{T}_i$'s (with $\mathcal{T}_{i+1} = \text{FL}_{\mathcal{C}_i}(\mathcal{T}_i)$ for all $i \in \{1, \ldots, \kappa - 1\}$), it is enough to define Y_i to be the Gorenstein toric variety associated to the fan which consists of the cones supporting the lattice polytopes of the polytopal subdivision $\mathcal{T}_i \smallsetminus (\mathcal{Y}_\odot(\mathcal{C}_i) * \mathcal{T}^{[\mathcal{C}_i]}$ of $\mathfrak{s}_G$. By the birational morphism φ_i we contract $\mathbf{V}(\sigma_{\mathbf{s}_i})$, where $\mathbf{s}_i$ denotes the unique $(r-2)$-dimensional simplex of $\mathcal{Y}_\odot(\mathcal{C}_i)$ with $\text{int}(\mathbf{s}_i) \subset \text{int}(\mathcal{Y}_\odot(\mathcal{C}_i))$, and by ϑ_i we extract $\mathbf{V}(\sigma_{\mathbf{t}_i})$, where $\mathbf{t}_i$ denotes the unique $(r-2)$-dimensional simplex of $\mathcal{Y}_\boxtimes(\mathcal{C}_i)$ with $\text{int}(\mathbf{t}_i) \subset \text{int}(\mathcal{Y}_\boxtimes(\mathcal{C}_i))$. The ϑ_i's are non-divisorial extractions, because we do not introduce any new vertices in the triangulation $\mathcal{T}_{i+1}$. □

EXERCISE 4.11. Take again the example of CQS of type $\frac{1}{12}(1,2,3,6)$ as in 3.18. Working with `Puntos` (cf. Note B.5) we find all $\mathcal{T} \in \mathbf{Coh\text{-}LTR}^{\max}_{N_G}(\mathfrak{s}_G)$. These are altogether 12 triangulations: One of them has 9 simplices, two have 10 simplices, four have 11 simplices, and the remaining five have 12 simplices. The latter ones are necessarily the elements of the set $\mathbf{Coh\text{-}LTR}^{\text{basic}}_{N_G}(\mathfrak{s}_G)$. The vertex sets of their simplices (in the notation used in 3.18) are recorded in the following list.

$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$	$\mathcal{T}_5$
$\{e_2,e_4,n_1,n_3\}$	$\{e_2,e_4,n_1,n_3\}$	$\{e_2,e_4,n_1,n_3\}$	$\{e_2,e_4,n_1,n_3\}$	$\{e_2,e_4,n_1,n_3\}$
$\{e_3,n_1,n_2,n_6\}$	$\{e_3,n_1,n_2,n_6\}$	$\{e_3,n_1,n_2,n_9\}$	$\{e_3,n_1,n_2,n_6\}$	$\{e_3,n_1,n_2,n_6\}$
$\{e_3,e_4,n_1,n_6\}$	$\{e_3,e_4,n_1,n_6\}$	$\{e_3,e_4,n_1,n_6\}$	$\{e_3,e_4,n_1,n_6\}$	$\{e_3,e_4,n_1,n_6\}$
$\{e_2,e_3,e_4,n_1\}$	$\{e_2,e_3,e_4,n_1\}$	$\{e_2,e_3,e_4,n_1\}$	$\{e_2,e_3,e_4,n_1\}$	$\{e_2,e_3,e_4,n_1\}$
$\{e_2,e_3,n_1,n_2\}$	$\{e_2,e_3,n_1,n_2\}$	$\{e_2,e_3,n_1,n_2\}$	$\{e_2,e_3,n_1,n_2\}$	$\{e_2,e_3,n_1,n_2\}$
$\{e_2,n_1,n_2,n_6\}$	$\{e_2,n_1,n_2,n_6\}$	$\{e_3,n_1,n_6,n_9\}$	$\{n_1,n_2,n_6,n_9\}$	$\{n_1,n_3,n_6,n_9\}$
$\{e_1,e_4,n_1,n_9\}$	$\{e_1,e_4,n_1,n_9\}$	$\{e_1,e_4,n_1,n_9\}$	$\{e_1,e_4,n_1,n_9\}$	$\{e_1,e_4,n_1,n_9\}$
$\{e_4,n_1,n_3,n_9\}$	$\{e_4,n_1,n_3,n_9\}$	$\{e_4,n_1,n_3,n_9\}$	$\{e_4,n_1,n_3,n_9\}$	$\{e_4,n_1,n_3,n_9\}$
$\{e_1,e_4,n_1,n_6\}$	$\{e_1,e_4,n_1,n_6\}$	$\{e_1,e_4,n_1,n_6\}$	$\{e_1,e_4,n_1,n_6\}$	$\{e_1,e_4,n_1,n_6\}$
$\{e_2,n_1,n_3,n_6\}$	$\{e_2,n_1,n_2,n_3\}$	$\{e_2,n_1,n_2,n_3\}$	$\{e_2,n_1,n_2,n_3\}$	$\{e_2,n_1,n_2,n_3\}$
$\{n_1,n_3,n_6,n_9\}$	$\{n_1,n_2,n_3,n_9\}$	$\{n_1,n_2,n_3,n_9\}$	$\{n_1,n_2,n_3,n_9\}$	$\{n_1,n_2,n_3,n_6\}$
$\{e_1,n_1,n_6,n_9\}$	$\{e_1,n_1,n_2,n_9\}$	$\{e_1,n_1,n_6,n_9\}$	$\{e_1,n_1,n_6,n_9\}$	$\{e_1,n_1,n_6,n_9\}$

For $1 \leq i \leq j \leq 5$, $i \neq j$, find sequences of flops connecting $X(N_G, \widehat{\Delta}_G(\mathcal{T}_i))$ with $X(N_G, \widehat{\Delta}_G(\mathcal{T}_j))$, and distinguish those possessing the *smallest* number of flops.

5. The C.I.-Case

An evidence in support of Conjecture 1.22 is given by the following:

THEOREM 5.1 ([**29**]). *All abelian quotient c.i.-singularities admit projective, crepant resolutions in all dimensions.*

An extensive technical part of its proof is devoted to the rendering of the original (purely algebraic) group classification of Watanabe [**124**] into graph-theoretic terms and to a subsequent *convenient* description of the corresponding junior simplices. As it turns out, an AQS is a c.i.-singularity if and only if the junior simplex $\mathfrak{s}_G$ is (what we call) a *Watanabe simplex w.r.t.* N_G. (In addition, notice that every abelian quotient c.i.-msc-singularity of dimension ≥ 3 has to be *non-cyclic*!)

Definition 5.2. Let d be an integer ≥ 0 and N a free $\mathbb{Z}$-module of rank d, regarded as a lattice within $N_{\mathbb{R}} \cong \mathbb{R}^d$. The *Watanabe simplices* w.r.t. N are the lattice simplices $\mathbf{s}$ (w.r.t. N, of dimension $\leq d$) satisfying

$$\operatorname{aff}_{\mathbb{Z}}(\mathbf{s} \cap N) = \operatorname{aff}(\mathbf{s}) \cap N$$

which are defined inductively (starting in dimension 0) in the following manner:

(i) Every 0-dimensional lattice simplex $\mathbf{s} = \{n\}$, $n \in N$, is a Watanabe simplex.

(ii) A lattice simplex $\mathbf{s} \subset N_{\mathbb{R}}$ of dimension d', $1 \leq d' \leq d$, is a Watanabe simplex if

• *either* $\mathbf{s} = \mathbf{s}_1 * \mathbf{s}_2$ (the *join* of $\mathbf{s}_1$ and $\mathbf{s}_2$), where $\mathbf{s}_1$, $\mathbf{s}_2$ are Watanabe simplices of dimensions d_1, $d_2 \geq 0$, $d_1 + d_2 = d' - 1$, with respect to sublattices $N_1 \subset \operatorname{aff}(\mathbf{s}_1)$, $N_2 \subset \operatorname{aff}(\mathbf{s}_2)$ of N, such that $\operatorname{aff}_{\mathbb{Z}}(\mathbf{s} \cap N) = \operatorname{aff}_{\mathbb{Z}}(N_1 \cup N_2)$,

• *or* $\mathbf{s}$ is a lattice translate of some dilation $\lambda\,\mathbf{s}'$, where $\lambda \in \mathbb{Z}$, $\lambda \geq 2$, and $\mathbf{s}'$ is an d'-dimensional Watanabe simplex with respect to N.

(These conditions are mutually exclusive; with this definition every affine integral transformation that preserves N also preserves the Watanabe simplices w.r.t. N).

Theorem 5.1 results from the following:

Theorem 5.3. *All Watanabe simplices w.r.t. a lattice N possess basic, coherent triangulations.*

To prove 5.3 it suffices to show that: (i) joins and dilations of coherent triangulations of lattice polytopes remain coherent, (ii) the join of two basic simplices is basic, and (iii) the dilation of a basic simplex by a factor $k \in \mathbb{Z}$, $k \geq 2$, possesses a basic triangulation (see [**29**, Theorem 3.5, Lemma 3.6 and Proposition 6.1]).

Example 5.4. Let $\mathbb{H}_d$ denote the affine hyperplane arrangement of type $\widetilde{\mathcal{A}}_d$ in $\mathbb{R}^d$ consisting of the union of hyperplanes

$$\left\{\left\{\mathbf{x} \in \mathbb{R}^d \,|\, x_i = \kappa\right\}, 1 \leq i \leq d, \kappa \in \mathbb{Z}\right\} \cup \left\{\left\{\mathbf{x} \in \mathbb{R}^d \,|\, x_i - x_j = \lambda\right\}, 1 \leq i < j \leq d, \lambda \in \mathbb{Z}\right\}.$$

$\mathbb{H}_d$ induces a basic triangulation $\mathbf{T}_{\mathbb{H}_d}$ (w.r.t. $\mathbb{Z}^d$) on the *entire* space $\mathbb{R}^d$. Let Hvs: $\mathbb{R} \longrightarrow \mathbb{R}$ denote the Heaviside function

$$\operatorname{Hvs}(x) := \begin{cases} x, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The $\mathbf{T}_{\mathbb{H}_d}$-support function

$$\psi^{(d)}_{\mathrm{Hvs}}(\mathbf{x}) := -\sum_{0 \leq i < j \leq d} \left\{ \sum_{0 \leq \kappa \leq x_j - x_i} \operatorname{Hvs}(x_i - x_j - \kappa) + \sum_{x_j - x_i \leq \kappa \leq 0} \operatorname{Hvs}(\kappa - x_j + x_i) \right\},$$

$\forall \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, with $x_0 := 0$, is strictly upper convex. Thus, $\mathbf{T}_{\mathbb{H}_d}$ is also coherent. Next, define

$$\begin{aligned} \mathbf{s}_d &:= \operatorname{conv}(\{\mathbf{0}, e_1, e_1 + e_2, \ldots, e_1 + \cdots + e_d\}) \\ &= \left\{\mathbf{x} \in \mathbb{R}^d \mid 0 \leq x_d \leq x_{d-1} \leq \cdots \leq x_1 \leq 1\right\}, \end{aligned}$$

and let k be an integer ≥ 2. Since the affine hulls of the facets of $\mathbf{s}_d$ belong to $\mathbb{H}_d$, the restriction $\mathbf{T}(d;k) := \mathbf{T}_{\mathbb{H}_d}|_{k\mathbf{s}_d}$ of $\mathbf{T}_{\mathbb{H}_d}$ on $k\,\mathbf{s}_d$ is a basic, coherent triangulation of $k\,\mathbf{s}_d$ w.r.t. $\mathbb{Z}^d$. The triangulation $\mathbf{T}(2;4)$ of $4\,\mathbf{s}_2$ is depicted in Figure 3.

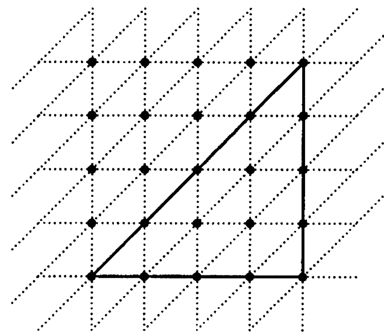

FIGURE 3.

APPLICATION 5.5 (Hypersurface case). The underlying space of a Gorenstein AQS $(\mathbb{C}^r/G,[\mathbf{0}])$, $r\geq 3$, is embeddable as a hypersurface in $\mathbb{C}^{r+1}$ if and only if G is conjugate (within $\mathrm{SL}(r,\mathbb{C})$) to a group of the form

$$G(r;k):=\left\langle \{\mathrm{diag}(1,...,1,\underbrace{\zeta_k}_{i\text{-th pos.}},\underbrace{\zeta_k^{-1}}_{(i+1)\text{-pos.}},1,...,1)\mid 1\leq i\leq r-1\}\right\rangle,$$

with $k\in\mathbb{Z}$, $k\geq 2$, i.e., if and only if it is of type

$$\frac{1}{k}(1,1,0,\ldots,0)\times\frac{1}{k}(0,1,1,0,\ldots,0)\times\ \cdots\ \cdots\ \times\frac{1}{k}(0,0,\ldots,0,1,1).$$

In this case, we may identify $\mathbb{C}^r/G(r;k)$ (or $\mathbb{C}^r/G$, cf. Thm. 1.2) with

$$\left\{(z_0,z_1,\ldots,z_r)\in\mathbb{C}^{r+1}\;\middle|\; z_0^k=\prod_{i=1}^{r}z_i\right\},$$

and write the junior simplex $\mathfrak{s}_G$ as the dilation of a basic simplex (w.r.t. N_G) by the factor k:

$$\mathfrak{s}_G=k\ \mathrm{conv}(\{\tfrac{1}{k}\ e_1,\ldots,\tfrac{1}{k}\ e_r\}).$$

There is an affine transformation $\mathbb{R}^r\longrightarrow\mathbb{R}^{r-1}\times\{0\}\subset\mathbb{R}^r$ whose restriction on the affine hull $\mathrm{aff}(\mathfrak{s}_G)$ of $\mathfrak{s}_G$ is a bijection, say Ξ, mapping the lattice $\mathrm{aff}(\mathfrak{s}_G)\cap N_G$ onto the standard rectangular lattice $\mathbb{Z}^{r-1}=\mathbb{Z}^{r-1}\times\{0\}\subset\mathbb{R}^{r-1}\times\{0\}$, and $\mathfrak{s}_G$ onto $\Xi(\mathfrak{s}_G)=k\,\mathbf{s}_{r-1}$. Since $\mathbf{T}(r-1;k)$ (as defined in 5.4, with $d=r-1$) is a basic coherent triangulation of $k\,\mathbf{s}_{r-1}$ w.r.t. $\mathbb{Z}^{r-1}$,

$$f_{\Xi^{-1}(\mathbf{T}(r-1;k))}:X(N_G,\widehat{\Delta}_G(\Xi^{-1}(\mathbf{T}(r-1;k)))\longrightarrow X(N_G,\Delta_G)=\mathbb{C}^r/G$$

is a projective crepant desingularization of the quotient space $\mathbb{C}^r/G\cong\mathbb{C}^r/G(r;k)$.

REMARK 5.6. If $G\subset\mathrm{SL}(r,\mathbb{C})$ is conjugate to $G(r;k)$, then
(i) the star of any vertex of $\mathbf{T}(r-1;k)$, belonging to the interior of $k\,\mathbf{s}_{r-1}$, is constructed as the image under an appropriate integral affine transformation of the *join* of the origin $\mathbf{0}\in\mathbb{R}^{r-1}$ with the facets of the zonotope which is defined as the convex hull of the union of the $[-1,0]$-cube and the $[0,1]$-cube; cf. [**26**]. Hence, every compactly supported exceptional prime divisor on $X(N_G,\widehat{\Delta}_G(\Xi^{-1}(\mathbf{T}(r-1;k)))$ is a Fano manifold obtained by a $\mathbb{T}_{N_G}$-equivariant projective crepant desingularization of a toric Fano variety (with at worst Gorenstein singularities). This Fano variety turns out to be a projective variety of degree $\binom{2r}{r}$ embedded in $\mathbb{P}_{\mathbb{C}}^{r(r+1)}$.

(ii) Besides $\Xi^{-1}(\mathbf{T}\,(r-1;k)$ there are lots of other basic, coherent triangulations of $\mathfrak{s}_G$ corresponding to different vertices of its secondary polytope. For instance, if G is conjugate to $G(4;2)$, `Puntos` [**31**] gives us 196 maximal coherent triangulations of $\mathfrak{s}_G$. 192 out of them are basic.

(iii) The non-trivial cohomology dimensions (4.3) of any crepant desingularization $\widehat{X}$ of $X=\mathbb{C}^r/G$ are equal to

$$\dim_{\mathbb{Q}} H^{2i}(\widehat{X},\mathbb{Q}) = \sum_{j=0}^{i} (-1)^j \tbinom{r}{j} \tbinom{k(i-j)+r-1}{r-1}.$$

Note 5.7. (i) For $r=4$, $k=2$, Chiang & Roan [**18**, Thm. 4.1], [**19**, §4], proved that the Hilbert scheme $G(4;2)$-Hilb$(\mathbb{C}^4)$ is a four-dimensional non-singular toric variety with *non-trivial canonical divisor*. The dualizing sheaf $\omega_{G(4;2)\text{-Hilb}(\mathbb{C}^4)}$ is $\cong \mathcal{O}_{G(4;2)\text{-Hilb}(\mathbb{C}^4)}(\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}})$. There are three different ways to blow down this divisor and pass to crepant desingularizations of $\mathbb{C}^4/G(4;2)$, corresponding to the three different projections $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$. The first blow-down leads to the crepant desingularization of $\mathbb{C}^4/G(4;2)$ described in 5.5. The other two are obtained by flops, and belong to the 192 mentioned in 5.6 (ii).

(ii) For $r=5$, $k=2$, the situation becomes worse. $G(5;2)$-Hilb$(\mathbb{C}^5)$ is a five-dimensional *singular* toric variety with *non-trivial canonical divisor*. In this case, among all crepant $\mathbb{T}_{N_{G(5;2)}}$-equivariant desingularizations of $\mathbb{C}^5/G(5;2)$ there are only 12 dominated by $G(5;2)$-Hilb$(\mathbb{C}^5)$ (see [**19**, §5]).

6. First Existence Criterion via Hilbert basis

A *necessary* condition for an *arbitrary* Gorenstein AQS $(\mathbb{C}^r/G,[\mathbf{0}])$ to admit a crepant resolution is described as follows:

Theorem 6.1 (First Necessary Existence Condition). *Let $(\mathbb{C}^r/G,[\mathbf{0}])$ be a Gorenstein AQS. If $\mathfrak{s}_G$ has a basic triangulation, then*

$$\boxed{\mathbf{Hlb}_{N_G}(\sigma_0) = \mathfrak{s}_G \cap N_G,} \tag{6.1}$$

i.e., each of the members of the Hilbert basis of σ_0 has to be either a "junior" element or a vertex of $\mathfrak{s}_G$.

Proof. The inclusion "$\supseteq$" is always true (without any further assumption about the existence or non-existence of such a triangulation) and is obvious by the definition of Hilbert basis. Now if there were an element $n\in \mathbf{Hlb}_{N_G}(\sigma_0)\smallsetminus(\mathfrak{s}_G\cap N_G)$, then by Lemma 2.1 this would be written as a non-negative integer linear combination

$$n=\lambda_1 n_1+\cdots+\lambda_r n_r$$

of r elements of $\mathfrak{s}_G\cap N_G$. Since $\mathbf{0}\notin \mathbf{Hlb}_{N_G}(\sigma_0)$, if there were at least one index $j_\bullet\in\{1,\ldots,r\}$, for which $\lambda_{j_\bullet}\neq 0$. If $\lambda_{j_\bullet}=1$ and $\lambda_j=0$ for all $j\in\{1,\ldots,r\}\smallsetminus\{\,j_\bullet\}$, then $n=n_{j_\bullet}\in\mathfrak{s}_G\cap N_G$ which would contradict our assumption. But even the cases in which either $\lambda_{j_\bullet}=1$ and some other λ_j's were $\neq 0$, or $\lambda_{j_\bullet}\geq 2$, would be exluded as impossible because of the characterization (2.1) of the Hilbert basis $\mathbf{Hlb}_{N_G}(\sigma_0)$ as the set of additively irreducible vectors of $\sigma_0\cap(N_G\smallsetminus\{\mathbf{0}\})$. Hence, $\mathbf{Hlb}_{N_G}(\sigma_0)\subseteq\mathfrak{s}_G\cap N_G$. $\square$

NOTE 6.2. (i) For $r = 2$ and $r = 3$, condition (6.1) is automatically satisfied.

(ii) For $r \geq 4$ there is a plethora of AQS for which (6.1) is violated. A simple example is the (non-terminal) CQS $\left(\mathbb{C}^4/G, [\mathbf{0}]\right)$ of type $\frac{1}{7}\,(1,1,2,3)$. This singularity cannot have any crepant, $\mathbb{T}_{N_G}$-equivariant resolution, because setting

$$\begin{cases} n_1 := \frac{1}{7}\,(1,1,2,3)^{\intercal}\,, \quad n_2 := \frac{1}{7}\,(2,2,4,6)^{\intercal}\,, \quad n_3 := \frac{1}{7}\,(3,3,6,2)^{\intercal}\,, \\ \\ n_4 := \frac{1}{7}\,(4,4,1,5)^{\intercal}\,, \quad n_5 := \frac{1}{7}\,(5,5,3,1)^{\intercal}\,, \quad n_6 := \frac{1}{7}\,(6,6,5,4)^{\intercal}\,, \end{cases}$$

we get

$$\mathfrak{s}_G \cap N_G = \{e_1, e_2, e_3, e_4, n_1\} \underset{\neq}{\subsetneq} \mathbf{Hlb}_{N_G}(\sigma_0) = \left\{ \begin{array}{c} e_1, e_2, e_3, e_4, \\ n_1, n_2, n_3, n_4, n_5 \end{array} \right\}.$$

(iii) In §7 we shall present certain cyclic quotient singularity series of arbitrary dimension for which condition (6.1) turns out to be also *sufficient*. Nevertheless, this is not true *in general* for $r \geq 4$. As it has been shown in [**39**, §4.2, pp. 65-66] and [**40**, Ex. 10, p. 213], there are exactly 10 four-dimensional Gorenstein cyclic quotient singularities with acting group order < 100 which fulfil (6.1) and possess no crepant, $\mathbb{T}_{N_G}$-equivariant resolutions. Among them, the CQS with the smallest possible acting group order is that one having the type $\frac{1}{39}\,(1,5,8,25)$.

7. Non-C.I.'s I: 1- and 2-Parameter CQS-Series

Asking whether non-c.i. Gorenstein cyclic quotient singularities of given type (3.7) can be resolved as desired, we begin with the examination of those CQS whose junior simplex contains lattice points living in a convenient geometric locus (in order to be able to keep track of how the possible maximal lattice triangulations are built). More precisely, we consider:

- 1-*parameter CQS-series* $(\mathbb{C}^r/G, [\mathbf{0}])$, for which the lattice points belonging to $(\mathfrak{s}_G \smallsetminus \{e_1, \ldots, e_r\}) \cap N_G$ are *collinear*, so that the maximal lattice triangulations of the junior simplex $\mathfrak{s}_G$ are uniquely determined.
- 2-*parameter CQS-series* $(\mathbb{C}^r/G, [\mathbf{0}])$, for which the lattice points belonging to $(\mathfrak{s}_G \smallsetminus \{e_1, \ldots, e_r\}) \cap N_G$ are *coplanar*, so that each of the simplices of the required maximal lattice triangulations of $\mathfrak{s}_G$ is to be described as *join* of a lattice polygon (resp., a lattice segment) with an $(r-4)$-dimensional (resp., an $(r-3)$-dimensional) lattice simplex.

(For the somewhat lengthy proofs of Theorems 7.1, 7.2, 7.3, and 7.4, see [**27, 28**].)

THEOREM 7.1 (1-parameter CQS). *If* $(\mathbb{C}^r/G, [\mathbf{0}])$ *is a Gorenstein CQS, such that* $r-1$ *weights in its type are equal* (*with* $r \geq 3$), *then it is isomorphic to the CQS of type*

$$\boxed{\frac{1}{l}\,(\underbrace{1, 1, \ldots, 1, 1}_{(r-1)\text{-times}},\ l-(r-1))} \tag{7.1}$$

with $l = |G| \geq r$. *Moreover, we have*:

(i) *This msc-singularity is isolated if and only if* $\gcd(l, r-1) = 1$.

(ii) *There exists a* unique *maximal, coherent triangulation* $\mathcal{T}$ *of* $\mathfrak{s}_G$ *w.r.t.* N_G *inducing a unique crepant* $\mathbb{T}_{N_G}$*-equivariant partial projective desingularization* (4.1).

This is a (full) desingularization (i.e., $\mathcal{T}$ is basic w.r.t. N_G) if and only if condition (6.1) *is satisfied. In particular,* (6.1) *is equivalent to the following:*

$$\boxed{\textbf{Either}\ \ l \equiv 0 \bmod (r-1) \quad \textbf{or} \quad l \equiv 1 \bmod (r-1)\,.} \tag{7.2}$$

Theorem 7.2 (Exceptional prime divisors). *Suppose that* $(\mathbb{C}^r/G, [\mathbf{0}])$, $r \geq 3$, *is a Gorenstein CQS of type* (7.1). *If l satisfies* (7.2), *then the exceptional locus of* (4.1) *consists of* $\lfloor \frac{l}{r-1} \rfloor$ *prime divisors* $\left\{ D_j \;\middle|\; 1 \leq j \leq \lfloor \frac{l}{r-1} \rfloor \right\}$ *on* $X(N_G, \widehat{\Delta}_G(\mathcal{T}))$, *having the following structure*:

$$D_j \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^{r-2}_{\mathbb{C}}} \oplus \mathcal{O}_{\mathbb{P}^{r-2}_{\mathbb{C}}}(l-(r-1)\,j)) \qquad (\text{as } \mathbb{P}^1_{\mathbb{C}}\text{-bundles over } \mathbb{P}^{r-2}_{\mathbb{C}})$$

for all $j \in \left\{ 1, 2, \ldots, \lfloor \frac{l}{r-1} \rfloor - 1 \right\}$, *and*

$$D_{\lfloor \frac{l}{r-1} \rfloor} \cong \begin{cases} \mathbb{P}^{r-1}_{\mathbb{C}} & , \ \text{if} \quad l \equiv 1 \bmod (r-1)\,, \\ \mathbb{P}^{r-2}_{\mathbb{C}} \times \mathbb{C} & , \ \text{if} \quad l \equiv 0 \bmod (r-1)\,. \end{cases}$$

Theorem 7.3 (2-parameter CQS). *Let* $(\mathbb{C}^r/G, [\mathbf{0}])$ *be a Gorenstein msc-CQS of type* (3.7) *with* $l = |G| \geq r \geq 3$, *for which at least* $r-2$ *of its defining weights are equal. Then* $X(N_G, \Delta_G) = \mathbb{C}^r/G$ *has crepant,* $\mathbb{T}_{N_G}$*-equivariant desingularizations* $f_{\mathcal{T}} : X(N_G, \widehat{\Delta}_G(\mathcal{T})) \longrightarrow X(N_G, \Delta_G)$ *if and only if* (6.1) *is satisfied. Moreover, at least one of these desingularizations is projective.*

Conditions equivalent to (6.1) which can be *directly* expressed in terms of the defining weights occur in the following case:

Theorem 7.4 (Arithmetic conditions for certain 2-parameter CQS). *Let r be an integer* ≥ 3 *and l an integer* $\geq r$. *Write* $l-(r-2) = a+b$, *where a, b are integers* ≥ 1. *Furthermore, set* $t := \gcd(b,l) = \gcd(a+(r-2), l)$, $t' := \gcd(a,l)$, *and consider* $\nu_1, \nu_2 \in \mathbb{N}$, *such that* $\nu_2\,(a+(r-2)) - \nu_1 l = t$. *Next, define*

$$\overline{p} := \frac{\nu_2 \cdot a - \nu_1 \cdot l}{t'}, \quad q := \frac{l}{t \cdot t'}, \quad p := [\overline{p}]_q$$

and write q/p as regular continued fraction

$$\frac{q}{p} = \lambda_1 + \cfrac{1}{\lambda_2 + \cfrac{1}{\lambda_3 + \ddots + \cfrac{1}{\lambda_\kappa}}}$$

with $\lambda_i \geq 2$, $\forall i$, $1 \leq i \leq \kappa$. *Then, for the Gorenstein CQS of type*

$$\boxed{\frac{1}{l}\ (\underbrace{1, 1, \ldots, 1, 1}_{(r-2)\text{-times}},\ a, b),} \tag{7.3}$$

(6.1) *is equivalent to the following*:

$$
\boxed{\begin{array}{ll}
\textbf{Either} & \gcd(a,b,l) = r-2, \\[2ex]
\textbf{or} & \begin{cases} \gcd(a,b,l) = 1, \quad [t]_{r-2} = [t']_{r-2} = 1, \\[1ex] \dfrac{p-\overline{p}}{q} \equiv 0 \bmod (r-2)\,, \\[1ex] \lambda_i \equiv 0 \bmod (r-2)\,, \forall i,\ \ i \in ([2,\kappa-1] \cap 2\mathbb{Z})\,, \\[1ex] \text{and } \lambda_\kappa \equiv 1 \bmod (r-2)\,,\ \text{whenever } \kappa \text{ is even.} \end{cases}
\end{array}} \tag{7.4}
$$

REMARK 7.5. (i) The method of building maximal triangulations $\mathcal{T}$ of $\mathfrak{s}_G$ can be roughly explained by means of Figure 4 (in which $r = 4$).

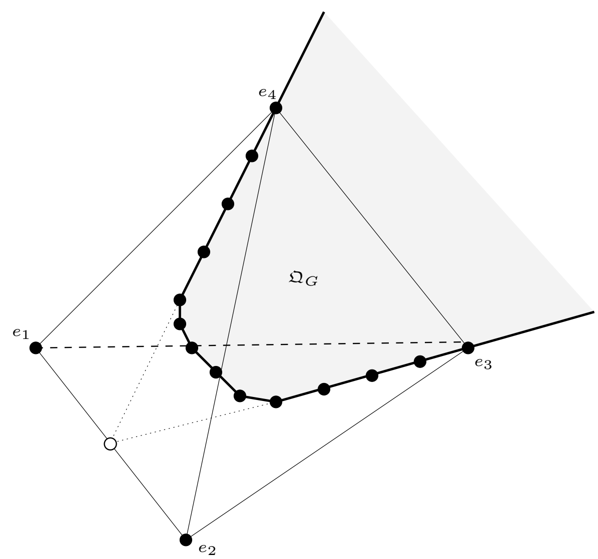

FIGURE 4.

We consider an arbitrary maximal (necessarily basic) triangulation of the lattice polygon $\mathfrak{Q}_G$, and then we construct $\mathcal{T}$ by forming the joins of e_1 and e_2 with all of its triangles. The white point belongs to N_G, and $\mathfrak{Q}_G$ itself becomes the triangle having e_3, e_4, and this point as its vertices, if and only if the first of conditions (7.4) is satisfied. In this case, such a maximal triangulation $\mathcal{T}$ of the entire $\mathfrak{s}_G$ is automatically basic (w.r.t. N_G). If the white point *does not* belong to N_G, then the basicness of such a $\mathcal{T}$ amounts to the second of conditions (7.4).

(ii) If one of the conditions (7.4) is satisfied, all compactly supported exceptional prime divisors w.r.t. $f_{\mathcal{T}}$ are the total spaces of fibrations having basis $\mathbb{P}_{\mathbb{C}}^{r-3}$, and typical fiber isomorphic either to $\mathbb{P}_{\mathbb{C}}^{1}$ or to a non-singular compact toric surface (i.e., to a $\mathbb{P}_{\mathbb{C}}^{2}$ or to an $\mathbb{F}_{\varkappa} = \mathbb{P}(\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^{1}} \oplus \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^{1}}(\varkappa))$, probably blown up at finitely many points, cf. [**96**, Thm. 1.28, p. 42]).

(iii) For the (rather tricky) computation of the cohomology group dimensions (4.3) of the underlying space of any crepant desingularization of these cyclic quotient singularities we refer to [**27**, §7].

EXAMPLES 7.6. (i) The subseries of non-isolated CQS with defining types

$$\frac{1}{(\xi+\xi'+1)\cdot(r-2)}\,(\underbrace{1,1,\ldots,1,1}_{(r-2)\text{-times}},\xi\cdot(r-2)\,,\xi'\cdot(r-2)),\quad \xi,\,\xi'\ \in\mathbb{N},$$

and $\gcd(\xi,\xi')=1$, $r\geq 4$, satisfies obviously the first of the conditions (7.4).
(ii) The subseries of isolated CQS with defining types

$$\frac{1}{2(r-1)^i+r-2}\,(\underbrace{1,1,\ldots,1,1}_{(r-2)\text{-times}},(r-1)^i\,,(r-1)^i)$$

and $i\in\mathbb{N}$, $r\geq 4$, satisfies the second of the conditions (7.4).
(iii) The example of 4-dimensional subseries due to Mohri [**89**]:

$$\tfrac{1}{4\,\xi}\ (1,1,2\,\xi-1,2\,\xi-1)\,,\quad \xi\in\mathbb{N},$$

satisfies the second of the conditions (7.4) and contains only isolated singularities. Note that also the single suitably resolvable CQS of type $\frac{1}{11}\,(1,1,3,6)$ found in [**89**] belongs to the subseries of isolated cyclic quotient singularities of type

$$\tfrac{1}{4r-5}(\underbrace{1,1,\ldots,1,1}_{(r-2)\text{-times}},r-1,2r-2)$$

satisfying obviously the second of the conditions (7.4). Moreover, there are examples like $1/28\,(1,1,1,4,21)$ for which $p=0$, $q=1$.

8. Non-C.I.'s II: The GP-Singularity Series

Another Gorenstein non-c.i. cyclic quotient singularity series of particular interest, admitting the required resolutions, is the so-called *geometric progress singularity series* (GPSS$(r;k)$, for short, with type (8.1)). The purpose of this section is to give a proof of the following Theorem (appearing as Conjecture 10.2 in [**28**]):

THEOREM 8.1. *All Gorenstein CQS* $(\mathbb{C}^r/G,[\mathbf{0}])$ *of type*

$$\boxed{\frac{1}{\left(\dfrac{k^r-1}{k-1}\right)}\quad\left(1,k,k^2,k^3,\ldots,k^{r-2},k^{r-1}\right)} \tag{8.1}$$

admit $\mathbb{T}_{N_G}$*-equivariant projective, crepant resolutions for* ***all*** $r\geq 3$ *and* ***all*** $k\geq 2$. *In particular, for* $k=2$*, there is a unique resolution of this sort.*

REMARK 8.2. (i) Setting $l:=\sum\limits_{i=0}^{r-1}k^i=\frac{k^r-1}{k-1}$ for the order of G acting on $\mathbb{C}^r$, we see that

$$N_G=\mathbb{Z}^r+\tfrac{1}{l}\left(1,k,k^2,\ldots,k^{r-1}\right)^{\intercal},\ \text{with}\ \det(N_G)=\tfrac{1}{l}.$$

(ii) Since $\gcd(k^i,l)=1$, $\forall i\in\{0,\ldots,r-1\}$, all members of the GPSS$(r;k)$ are *isolated* (and therefore *msc*-) singularities (cf. Proposition 3.15 and Corollary 3.17).

LEMMA 8.3. *If we denote by* $W(r;k) = (w_{ij})_{1\leq i,j\leq r}$ *the* $(r\times r)$*-matrix with*

$$w_{ij} := \left[k^{i-1}\cdot k^{j-1}\right]_l = \left[k^{i+j-2}\right]_l$$

as its entries, then

$$|\det(W(r;k))| = (k^r-1)^{r-1} = l^{r-1}(k-1)^{r-1}. \tag{8.2}$$

PROOF. Since

$$k^r = (k-1)\,l + k^0 \Longrightarrow k^{r+\beta} = k^\beta (k-1)\,l + k^\beta, \quad \forall \beta \in \mathbb{Z}_{\geq 0},$$

we have $w_{ij} = \left[k^{i+j-2}\right]_l = k^{[i+j-2]_r}$. On the other hand, performing the elementary operations

$$(i\text{-th row}) \rightsquigarrow (i\text{-th row}) - k^{(i-1)}\cdot(\text{first row}), \ \forall i \in \{2,3,\ldots,r\},$$

we transfer $W(r;k)$ into a matrix of the form

$$\begin{pmatrix} 1 & k & k^2 & \cdots & k^{r-1} \\ 0 & 0 & 0 & \cdots & -k^r+1 \\ 0 & 0 & \cdots & -k^r+1 & \star \\ \vdots & \vdots & \iddots & \star & \star \\ \vdots & \iddots & \star & \vdots & \vdots \\ -k^r+1 & \star & \cdots & \cdots & \star \end{pmatrix}.$$

Hence, $|\det(W(r;k))|$ is given by the formula (8.2). $\square$

LEMMA 8.4. (i) *Setting* $\breve{W}(r;k) := \left(\frac{1}{l(k-1)}W(r;k)\right)^{-1}$, *we have*

$$\breve{W}(r;k) = \begin{pmatrix} -1 & 0 & \cdots & 0 & k \\ 0 & 0 & \cdots & k & -1 \\ \vdots & 0 & k & -1 & 0 \\ 0 & \iddots & \iddots & \cdots & \vdots \\ k & -1 & 0 & \cdots & 0 \end{pmatrix}, \tag{8.3}$$

with

$$\left|\det(\breve{W}(r;k))\right| = l(k-1). \tag{8.4}$$

(ii) *The regular linear transformation* $\Phi : \mathbb{R}^r \longrightarrow \mathbb{R}^r$, *with*

$$\Phi(\mathbf{x}) := \breve{W}(r;k)\,\mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^r, \tag{8.5}$$

maps N_G *onto*

$$\breve{N}_G = \left\{ (\lambda_1,\ldots,\lambda_r)^\intercal \in \mathbb{Z}^r \;\middle|\; \sum_{i=1}^r \lambda_i \equiv 0 (\mathrm{mod}\ (k-1)) \right\}, \tag{8.6}$$

and the junior simplex $\mathfrak{s}_G$ *onto*

$$\boxed{\breve{\mathfrak{s}}_G = \mathrm{conv}(\{\,\breve{w}_j \mid 1 \leq j \leq r\}),} \tag{8.7}$$

where

$$\breve{w}_j := \begin{cases} -e_1 + ke_r, & \text{if } j = 1, \\ -e_j + ke_{j-1}, & \text{if } j \in \{2,\ldots,r\}, \end{cases}$$

denotes the j*-th column vector of the matrix obtained by* $\breve{W}(r;k)$ *by interchanging its* j*-th with its* $(r+2-j)$*-th column for all* $j \in \{2,\ldots,r\}$.* (This rearrangement of the index set for enumerating the vertices of $\breve{\mathfrak{s}}_G$ will turn out to be convenient in the subsequent Lemmas.)

PROOF. (i) Let w_i denote the i-th row of $W(r;k)$. For all $i \in \{1,\dots,r\}$ we have

$$k \cdot w_{ij} - w_{i+1\ j} = k \cdot k^{[i+j-2]_r} - k^{[i+j-1]_r} = \begin{cases} 0, & \text{if } i+j-1 \le r-1 \\ & \text{or } i+j-2 \ge r, \\ k^r - 1, & \text{if } i+j-1 = r. \end{cases}$$

Thus, $\breve{W}(r;k)$ is the matrix (8.3) because

$$k \cdot w_i - w_{i+1} = (k^r - 1)\, e_{r+1-i},\ \forall i \in \{1,\dots,r-1\},\ \text{and } k \cdot w_r - w_1 = (k^r-1)\, e_1,$$

and (8.4) follows directly from (8.2).

(ii) By definition, the determinant of $\breve{N}_G = \Phi(N_G)$ equals $k-1$, and

$$\breve{N}_G = \breve{W}(r;k)\,\mathbb{Z}^r + \mathbb{Z}(k-1,0,0,\dots,0,0)^\intercal.$$

$\breve{N}_G$ is included into $\left\{ (\lambda_1,\dots,\lambda_r)^\intercal \in \mathbb{Z}^r \;\middle|\; \sum_{i=1}^{r} \lambda_i \equiv 0 (\mathrm{mod}\ (k-1)) \right\}$. But also this lattice has determinant $k-1$, leading to equality (8.6). (8.7) is obvious. □

LEMMA 8.5. *We have*

$$\breve{\mathfrak{s}}_G \cap \breve{N}_G = \left\{ (\lambda_1,\dots,\lambda_r)^\intercal \in \mathbb{Z}^r_{\ge 0} \;\middle|\; \sum_{i=1}^{r} \lambda_i = k-1 \right\} \cup \{\, \breve{w}_j \mid 1 \le j \le r \}. \tag{8.8}$$

In particular,

$$\sharp(\breve{\mathfrak{s}}_G \cap \breve{N}_G) = \tbinom{k+r-2}{r-1} + r. \tag{8.9}$$

PROOF. Since $\breve{\mathfrak{s}}_G \subseteq \left\{ \mathbf{x} = (x_1,\dots,x_r)^\intercal \in \mathbb{R}^r \;\middle|\; \sum_{j=1}^{r} x_j = k-1 \right\}$, we have

$$\breve{\mathfrak{s}}_G \cap \breve{N}_G = \{ \mathbf{x} = (x_1,\dots,x_r)^\intercal \in \mathbb{Z}^r \ \mid \mathbf{x} \in \mathrm{conv}(\{\, \breve{w}_j \mid 1 \le j \le r\})\}.$$

We first observe that $(k-1)e_j \in \breve{\mathfrak{s}}_G \cap \breve{N}_G$, for all $j \in \{1,\dots,r\}$. (For instance, $(k-1)e_1 = \sum_{j=1}^{r} \frac{k^{(j-1)}}{l} \breve{w}_j$. The other inclusions follow by symmetry.) Next, we consider an

$$\mathbf{x} = (x_1,\dots,x_r)^\intercal \in \mathrm{conv}(\{\, \breve{w}_j \mid 1 \le j \le r\}) \cap \Big(\breve{N}_G \smallsetminus \{\, \breve{w}_j \mid 1 \le j \le r\} \Big).$$

This can be written as linear combination

$$\mathbf{x} = (x_1,\dots,x_r)^\intercal = \sum_{j=1}^{r} \eta_j \breve{w}_j,\ \text{for suitable } \eta_j\text{'s} \in [0,1).$$

Since $\mathbf{x} \in \mathbb{Z}^r$ we have $x_j \ge 0$, for all $j \in \{1,\dots,r\}$. Hence, $\mathbf{x}$ belongs to

$$\left\{ (x_1,\dots,x_r)^\intercal \in \mathbb{R}^r_{\ge 0} \;\middle|\; \sum_{j=1}^{r} x_j = k-1 \right\} = \ \mathrm{conv}(\{\, (k-1)e_j \mid 1 \le j \le r\}),$$

and both equalities (8.8) and (8.9) are true. □

LEMMA 8.6. *Let* $\mathbf{s}(\varepsilon_1,\dots,\varepsilon_r) \subset \{\, (x_1,\dots,x_r)^\intercal \in \mathbb{R}^r \,|\, \sum_{j=1}^{r} x_j = k-1\}$ *denote the simplices*

$$\mathbf{s}(\varepsilon_1,\dots,\varepsilon_r) :=\ \mathrm{conv}(\{\, \varepsilon_j u_j + (1-\varepsilon_j)\, \breve{w}_j \mid 1 \le j \le r\}),$$

where $\varepsilon_j \in \{0,1\}$ *and* $u_j := (k-1)\, e_j,\ \forall j \in \{1,\ldots,r\}$. *Then there is a unique triangulation* $\mathfrak{T}(r;k)$ *of* $\breve{\mathfrak{s}}_G$ *having* $\{\, u_j, \breve{w}_j \,|\, 1 \leq j \leq r\}$ *as its vertex set, namely*

$$\boxed{\mathfrak{T}(r;k) = \{\, \mathbf{s}(\varepsilon_1,\ldots,\varepsilon_r) \ | \ (\varepsilon_1,\ldots,\varepsilon_r) \in \{0,1\}^r \smallsetminus \{(0,0,\ldots,0,0)\}\}\,.} \tag{8.10}$$

PROOF. First note that $\mathbf{s}(0,0,\ldots,0,0) = \breve{\mathfrak{s}}_G$. Let T be an arbitrary triangulation of the convex hull of the point set $\{\, u_j, \breve{w}_j \,|\, 1 \leq j \leq r\}$. If $\mathbf{t}$ is an $(r-1)$-dimensional simplex belonging to T, then there is no index $j \in \{1,\ldots,r\}$ such that $\{u_j, \breve{w}_j\} \subset \mathbf{t}$. Assuming, in the contrary direction, the existence of such an index j, we would have $\frac{1}{k}u_j + \frac{k-1}{k}\breve{w}_j \in \mathbf{t}$, with

$$\frac{1}{k}u_j + \frac{k-1}{k}\breve{w}_j = \begin{cases} u_r, & \text{if } j=1, \\ u_{j-1}, & \text{if } j \in \{2,\ldots,r\}, \end{cases}$$

which would be absurd (because $\mathbf{t}$ could not be an $(r-1)$-dimensional simplex of a triangulation). Hence, any triangulation of the convex hull of $\{\, u_j, \breve{w}_j \,|\, 1 \leq j \leq r\}$ must have $\{\, \mathbf{s}(\varepsilon_1,\ldots,\varepsilon_r) \ | \ (\varepsilon_1,\ldots,\varepsilon_r) \in \{0,1\}^r \smallsetminus \{(0,0,\ldots,0,0)\}\}$ as maximal dimensional simplices. In fact, it is easy to verify that the intersection of any two simplices of this sort is either a face of both or the empty set, and that

$$\mathrm{Vol}(\mathbf{s}(\varepsilon_1,\ldots,\varepsilon_r)) = \begin{cases} \frac{\sqrt{r}}{(r-1)!}\frac{\left|\det\left(\breve{W}(r;k)\right)\right|}{\det\left(\breve{N}_G\right)} \overset{(8.4)}{=} \frac{\sqrt{r}}{(r-1)!}l, & \text{if } (\varepsilon_1,\ldots,\varepsilon_r) = (0,\ldots,0), \\ \\ \frac{1}{k-1}\frac{\sqrt{r}}{(r-1)!}\ (k-1)^{\sum_{j=1}^r \varepsilon_j}, & \text{if } (\varepsilon_1,\ldots,\varepsilon_r) \neq (0,\ldots,0), \end{cases}$$

(cf. formula (D.7)). Since

$$\begin{aligned} \sum_{(\varepsilon_1,\ldots,\varepsilon_r)\in\{0,1\}^r\smallsetminus\{(0,0,\ldots,0,0)\}} \mathrm{Vol}(\mathbf{s}(\varepsilon_1,\ldots,\varepsilon_r)) &= \tfrac{1}{k-1}\tfrac{\sqrt{r}}{(r-1)!}\left(\sum_{\rho=1}^{r}\tbinom{r}{\rho}(k-1)^{\rho}\right) \\ &= \tfrac{1}{k-1}\tfrac{\sqrt{r}}{(r-1)!}(k^r-1) \\ &= \tfrac{\sqrt{r}}{(r-1)!}l = \mathrm{Vol}(\breve{\mathfrak{s}}_G), \end{aligned}$$

the support of $\mathfrak{T}(r;k)$ given in (8.10) equals $\breve{\mathfrak{s}}_G$, and $\mathfrak{T}(r;k)$ is therefore the *unique* triangulation of $\breve{\mathfrak{s}}_G$ having $\{\, u_j, \breve{w}_j \,|\, 1 \leq j \leq r\}$ as vertex set. $\square$

DEFINITION 8.7. Let $\widetilde{\Phi} : \mathbb{R}^r \longrightarrow \mathbb{R}^r$ be the unimodular transformation

$$\widetilde{\Phi}\,(\mathbf{x}) := L\,\mathbf{x},\ \ \forall \mathbf{x} \in \mathbb{R}^r,$$

where

$$L := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

Then $\widetilde{\Phi}$ maps the hyperplane $\{\mathbf{x} \in \mathbb{R}^r \,|\, \sum_{j=1}^r x_j = k-1\}$ onto $\{\mathbf{x} \in \mathbb{R}^r \,| x_r = k-1\,\}$, with

$$\widetilde{\Phi}(u_j) = \begin{cases} (k-1)e_i + (k-1)e_r, & \text{if } j \in \{1,\ldots,r-1\}, \\ (k-1)e_r \ (= u_r), & \text{if } j = r, \end{cases}$$

and

$$\widetilde{\Phi}(\breve{w}_j) = \begin{cases} -e_1 + (k-1)e_r, & \text{if } j = 1, \\ -e_j + ke_{j-1} + (k-1)e_r, & \text{if } j \in \{2, \ldots, r-1\}. \\ ke_{r-1} + (k-1)e_r, & \text{if } j = r. \end{cases}$$

For all $j \in \{1, \ldots, r\}$ let us use the abbreviations

$$\widetilde{u}_j := \widetilde{\Phi}(u_j), \quad \widetilde{w}_j := \widetilde{\Phi}(\breve{w}_j), \quad \text{and} \quad \overline{w}_j := \widetilde{w}_j - (k-1)e_r.$$

We observe that $\widetilde{\Phi}$ transfers the triangulation $\mathfrak{T}(r;k)$ of $\breve{\mathfrak{s}}_G$ onto the triangulation

$$\widetilde{\Phi}\left(\mathfrak{T}(r;k)\right) = \{\widetilde{\mathbf{s}}(\varepsilon_1, \ldots, \varepsilon_r) \mid (\varepsilon_1, \ldots, \varepsilon_r) \in \{0,1\}^r \smallsetminus \{(0,0,\ldots,0,0)\}\}$$

of $\widetilde{\mathfrak{s}}_G = \widetilde{\Phi}\left(\breve{\mathfrak{s}}_G\right)$, where

$$\widetilde{\mathbf{s}}(\varepsilon_1, \ldots, \varepsilon_r) := \widetilde{\Phi}(\mathbf{s}(\varepsilon_1, \ldots, \varepsilon_r)) = \operatorname{conv}(\{\varepsilon_j \widetilde{u}_j + (1-\varepsilon_j)\, \widetilde{w}_j \mid 1 \le j \le r\}),$$

and we define the r-*dimensional* (!) lattice polytope

$$\boxed{P\,(r;k) := \operatorname{conv}(\{\{\widetilde{u}_j, \overline{w}_j \mid 1 \le j \le r\}) \subset \mathbb{R}^r.}$$

Lemma 8.8. *The facets of* $P\,(r;k)$ *are exactly those belonging to the set*

$$\{\overline{\mathbf{s}}(\varepsilon_1, \ldots, \varepsilon_r) \mid (\varepsilon_1, \ldots, \varepsilon_r) \in \{0,1\}^r\},$$

where

$$\overline{\mathbf{s}}(\varepsilon_1, \ldots, \varepsilon_r) := \operatorname{conv}(\{\varepsilon_j \widetilde{u}_j + (1-\varepsilon_j)\, \overline{w}_j \mid 1 \le j \le r\}).$$

Proof. Since $\sum_{j=1}^{r} k^{r-j}\overline{w}_j = 0$, the origin $\mathbf{0}$ is an interior point of $P\,(r;k)$, and we may assume that the coordinates of each point $\mathbf{x} = (x_1, \ldots, x_r)^\intercal \in P\,(r;k)$ satisfy inequalities of the form

$$\sum_{j=1}^{r} \eta_j x_j \le k-1, \text{ for suitable } r\text{-tuples } (\eta_1, \ldots, \eta_r) \in \mathbb{R}^r. \tag{8.11}$$

Since $\widetilde{u}_j \in P\,(r;k)$ (resp., $\overline{w}_j \in P\,(r;k)$) $\forall j \in \{1, \ldots, r\}$, valid inequalities for $P\,(r;k)$ of the form (8.11) must satisfy

$$\begin{cases} \eta_j + \eta_r \le 1, \forall j \in \{1, \ldots, r-1\}, \\ \eta_r \le 1, \end{cases} \tag{8.12}$$

and

$$-\eta_1 \le k-1, \tag{8.13}$$

$$k\eta_{j-1} - \eta_j \le k-1, \quad \forall j \in \{2, \ldots, r-1\}, \tag{8.14}$$

$$k\eta_{r-1} \le k-1, \tag{8.15}$$

respectively. Let F be a facet of $P\,(r;k)$. Assume that the supporting hyperplane of F is described by an equation $\sum_{j=1}^{r} \eta_j x_j = k-1$. If there were an index, say $j \in \{2, \ldots, r-1\}$, such that both $\widetilde{u}_j$ and $\overline{w}_j$ belong to F, then we would have

$$\left.\begin{array}{l} \eta_j + \eta_r = 1 \\ k\eta_{j-1} - \eta_j = k-1 \end{array}\right\} \Longrightarrow k\eta_{j-1} + \eta_r = k. \tag{8.16}$$

Since (8.16) is also valid for $P\,(r;k)$ itself, all inequalities (8.12), (8.13), (8.14), (8.15) would be satisfied. Hence,

$$\left.\begin{array}{r} \eta_{j-1} + \eta_r \le 1 \stackrel{(8.16)}{\Longrightarrow} \eta_r \le 0 \\ \eta_j + \eta_r = 1 \end{array}\right\} \Longrightarrow \eta_j \ge 1 \stackrel{(8.14)}{\Longrightarrow} \eta_{j+1} \ge k\eta_j - (k-1) \ge 1,$$

and using the same argument,

$$\eta_{j+1} \geq 1 \Longrightarrow \eta_{j+2} \geq 1 \Longrightarrow \cdots \Longrightarrow \eta_{r-1} \geq 1.$$

On the other hand, (8.15) would give $\eta_{r-1} \leq \frac{k-1}{k} < 1$, leading to contradiction. Analogously, by (8.12), (8.13), (8.14) and (8.15) one shows that $\widetilde{u}_j$ and $\overline{w}_j$ cannot simultaneously belong to F even if $j = 1$ or $j = r$. Thus, F is necessarily of the form

$$F = \overline{\mathbf{s}}(\varepsilon_1, \ldots, \varepsilon_r), \text{ for a suitable } r\text{-tuple } (\varepsilon_1, \ldots, \varepsilon_r) \in \{0,1\}^r.$$

It remains to prove that *all* $\overline{\mathbf{s}}(\varepsilon_1, \ldots, \varepsilon_r)$'s are realized as facets of $P(r;k)$. If $(\varepsilon_1, \ldots, \varepsilon_r)$ equals $(0,0,\ldots,0,0)$ (resp., $(1,1,\ldots,1,1)$), then we get obviously the *bottom* (resp., the *top*) *facet* of $P(r;k)$. Next, choose an arbitrary simplex

$$\overline{\mathbf{s}}(\varepsilon_1, \ldots, \varepsilon_r) \text{ with } (\varepsilon_1, \ldots, \varepsilon_r) \in \{0,1\}^r \smallsetminus \{(0,0,\ldots,0,0),(1,1,\ldots,1,1)\}$$

and assume without loss of generality (i.e., up to a permutation of coordinates) that

$$\varepsilon_j = 1, \quad \forall j \in \{1,\ldots,\rho\}, \quad \text{and} \quad \varepsilon_j = 0, \quad \forall j \in \{\rho+1,\ldots,r\},$$

for some $\rho \in \{1,\ldots,r-1\}$. Defining

$$\eta_j := \begin{cases} \frac{k^{r-\rho}-1}{k^{r-\rho}}, & \text{if } j \in \{1,\ldots,\rho-1\}, \\[2ex] \frac{1}{k^{r-\rho}}, & \text{if } j = \rho, \\[2ex] \frac{k^{r-j}-1}{k^{r-j}}, & \text{if } j \in \{\rho+1,\ldots,r\}, \end{cases}$$

one checks easily that η_j's fulfil (8.12), (8.13), (8.14) and (8.15), and that the r affinely independent points

$$\{(k-1)e_j+(k-1)e_r \,|\, 1 \leq j \leq \rho\}, \{-e_j+(k-1)e_{j-1} \,|\, \rho+1 \leq j \leq r-1\}, \{ke_{r-1}\},$$

satisfy the equality

$$\sum_{j=1}^{r} \eta_j x_j = k-1.$$

Hence, their convex hull $\overline{\mathbf{s}}(\varepsilon_1, .., \varepsilon_r)$ constitutes a facet of $P(r;k)$, as asserted. $\square$

Corollary 8.9. *The triangulation $\mathfrak{T}(r;k)$ of $\check{\mathfrak{s}}_G$ is coherent.*

Proof. Using Lemma 8.8 and the projection $\varpi : \mathbb{R}^r \longrightarrow \{\mathbf{x} \in \mathbb{R}^r \mid x_r = k-1\}$, with

$$\varpi(\mathbf{x}) := (x_1, x_2, \ldots, x_{r-1}, k-1)^{\intercal}, \quad \forall \mathbf{x} = (x_1, \ldots, x_r)^{\intercal} \in \mathbb{R}^r,$$

we see that the set $\{\overline{\mathbf{s}}(\varepsilon_1, \ldots, \varepsilon_r) \mid (\varepsilon_1, \ldots, \varepsilon_r) \in \{0,1\}^r \smallsetminus \{(0,0,\ldots,0,0)\}\}$ consisting of the facets of $P(r;k)$ which belong to its "higher envelope" is mapped via ϖ onto the triangulation

$$\varpi(\{\overline{\mathbf{s}}(\varepsilon_1, \ldots, \varepsilon_r) \mid (\varepsilon_1, \ldots, \varepsilon_r) \in \{0,1\}^r \smallsetminus \{(0,0,\ldots,0,0)\}\}) = \widetilde{\Phi}(\mathfrak{T}(r;k))$$

of $\widetilde{\mathfrak{s}}_G$ because

$$\varpi(\overline{\mathbf{s}}(\varepsilon_1, \ldots, \varepsilon_r)) = \widetilde{\mathbf{s}}(\varepsilon_1, \ldots, \varepsilon_r), \quad \forall (\varepsilon_1, \ldots, \varepsilon_r) \in \{0,1\}^r \smallsetminus \{(0,0,\ldots,0,0)\}.$$

Hence, the $\widetilde{\Phi}(\mathfrak{T}(r;k))$-support function $\theta : \widetilde{\mathfrak{s}}_G \longrightarrow (0,k-1] \subset \mathbb{R}$ defined by the formula

$$\theta(\mathbf{x}) := \max\{t \in [0,k-1] \mid (x_1, x_2, \ldots, x_{r-1}, t)^{\intercal} \in P(r;k)\},$$

for all $\mathbf{x} = (x_1, x_2, \ldots, x_{r-1}, k-1)^\intercal \in \widetilde{\mathfrak{s}}_G$, is strictly upper convex. This means that

$$\theta \circ \widetilde{\Phi}\Big|_{\breve{\mathfrak{s}}_G} : \breve{\mathfrak{s}}_G \longrightarrow (0, k-1] \subset \mathbb{R}$$

is a strictly upper convex $\mathfrak{T}(r;k)$-support function on $\breve{\mathfrak{s}}_G$. $\square$

REMARK 8.10. An alternative proof of Corollary 8.9 can be obtained by observing that $\mathfrak{T}(r;k)$ is actually a *lexicographic triangulation*, and by using the fact that lexicographic triangulations are coherent (see Lee [**81**]).

EXAMPLE 8.11. The unique, coherent triangulation $\Phi^{-1}(\mathfrak{T}(3;4))$ (with Φ as defined in (8.5)) of the original junior simplex $\mathfrak{s}_G$ (w.r.t. N_G) which is induced by $\mathfrak{T}(3;4)$ (for $r=3$, $k=4$), and has $\left\{ \Phi^{-1}(u_j), e_j \,\middle|\, 1 \le j \le 3 \right\}$ as its vertex set, is depicted in Figure 5.

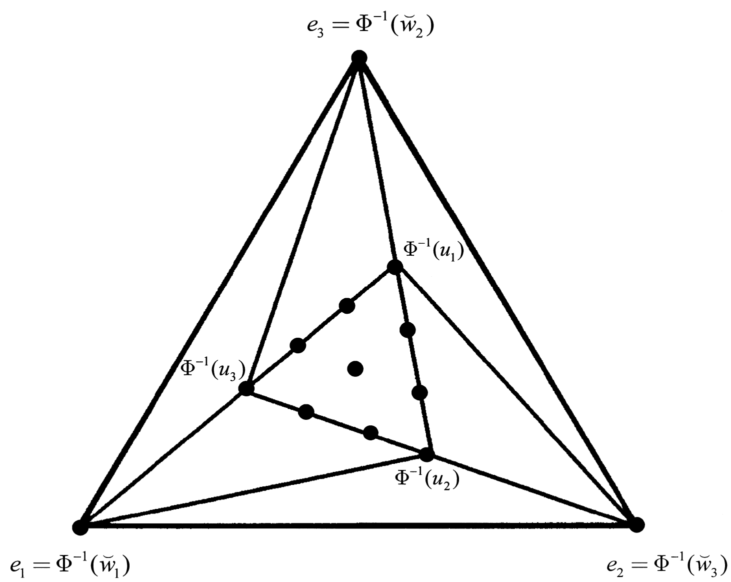

FIGURE 5.

REMARK 8.12. By Remark 8.2 (ii) and Lemma 8.5, the $(r-1)$-dimensional lattice simplex

$$\boxed{\mathring{\mathfrak{s}}_G := \mathbf{s}(1, 1, \ldots, 1, 1) = \operatorname{conv}\left(\breve{\mathfrak{s}}_G \smallsetminus \{\, \breve{w}_j \mid 1 \le j \le r\}\right),}$$

(included in the interior of $\breve{\mathfrak{s}}_G = \operatorname{conv}(\{\, \breve{w}_j \mid 1 \le j \le r\}))$ contains the $\binom{k+r-2}{r-1}$ non-vertex lattice points of $\breve{\mathfrak{s}}_G$. Since

$$\mathring{\mathfrak{s}}_G = \operatorname{conv}(\{\, u_j \mid 1 \le j \le r\}) = (k-1) \cdot \operatorname{conv}(\{\, e_j \mid 1 \le j \le r\})$$

is the dilation of a *basic simplex* (w.r.t. $\breve{N}_G$) by the factor $k-1$, there is an affine transformation $\mathbb{R}^r \longrightarrow \mathbb{R}^{r-1} \times \{0\} \subset \mathbb{R}^r$ whose restriction on the affine hull $\operatorname{aff}(\breve{\mathfrak{s}}_G)$

of $\breve{\mathfrak{s}}_G$ is a bijection, say Υ, mapping the lattice $\mathrm{aff}(\breve{\mathfrak{s}}_G) \cap \breve{N}_G$ onto the standard rectangular lattice $\mathbb{Z}^{r-1} = \mathbb{Z}^{r-1} \times \{0\} \subset \mathbb{R}^{r-1} \times \{0\}$, and $\mathring{\mathfrak{s}}_G$ onto

$$\Upsilon(\mathring{\mathfrak{s}}_G) = (k-1) \cdot \mathrm{conv}(\{\mathbf{0}, e_1, e_1 + e_2, \ldots, e_1 + e_2 + \cdots + e_{r-1}\}).$$

Hence, $\Upsilon^{-1}(\mathbf{T}(r-1;k-1))$ is a basic coherent triangulation of $\mathring{\mathfrak{s}}_G$ (w.r.t. $\breve{N}_G$), where $\mathbf{T}(r-1;k-1)$ denotes the triangulation of $\Upsilon(\mathring{\mathfrak{s}}_G)$ (w.r.t. $\mathbb{Z}^{r-1}$) defined in Example 5.4 (with $d = r-1$).

$\triangleright$ Proof of Theorem 8.1: (i) *Basicness.* Setting

$$\mathcal{E}_{\nu_1,\ldots,\nu_\rho} := \left\{ \begin{array}{l|c} \mathrm{conv}(\{\breve{w}_{\nu_1}, \ldots, \breve{w}_{\nu_\rho}\}) * \mathbf{s} & \mathbf{s} \text{ an } (r-1-\rho)\text{-dimensional} \\ \text{(together with} & \text{simplex of } \Upsilon^{-1}(\mathbf{T}(r-1;k-1)) \\ \text{their faces)} & \text{belonging to the face} \\ & \mathrm{conv}(\{u_j \mid j \in \{1,..,r\} \smallsetminus \{\nu_1,..,\nu_\rho\}\}) \\ & \text{of the simplex } \mathring{\mathfrak{s}}_G \end{array} \right\},$$

for all $\rho \in \{1, \ldots, r-1\}$ and all index subfamilies $1 \leq \nu_1 < \nu_2 < \cdots < \nu_\rho \leq r$, we define the triangulation

$$\mathfrak{E} := \bigcup_{\rho=1}^{r-1} \bigcup_{1 \leq \nu_1 < \nu_2 < \cdots < \nu_\rho \leq r} \mathcal{E}_{\nu_1,\nu_2,\ldots,\nu_\rho}$$

which refines $\mathfrak{T}(r;k)|_{\breve{\mathfrak{s}}_G \smallsetminus \mathrm{int}(\mathring{\mathfrak{s}}_G)}$ (with $\mathfrak{T}(r;k)$ as given in (8.10)). The set of $(r-1)$-dimensional simplices of $\mathfrak{E}$ consists of well-defined joins (cf. the proof of Lemma 8.6). Glueing $\Upsilon^{-1}(\mathbf{T}(r-1;k-1))$ and $\mathfrak{E}$ together we obtain a lattice triangulation (w.r.t. $\breve{N}_G$)

$$\boxed{\mathcal{T}(r;k) := \Upsilon^{-1}(\mathbf{T}(r-1;k-1)) \cup \mathfrak{E}}$$

of the entire simplex $\breve{\mathfrak{s}}_G$. The triangulation $\Upsilon^{-1}(\mathbf{T}(r-1;k-1))$ itself is basic. Since for all $\rho \in \{1, \ldots, r-1\}$,

$$\begin{aligned} &\mathrm{aff}_{\mathbb{Z}}(\{\breve{w}_{\nu_1}, \ldots, \breve{w}_{\nu_\rho}\} \cup \{u_j \mid j \in \{1,..,r\} \smallsetminus \{\nu_1,..,\nu_\rho\}\}) \\ &= \breve{N}_G \cap \mathrm{aff}(\mathrm{conv}(\{\breve{w}_{\nu_1}, \ldots, \breve{w}_{\nu_\rho}\}) \cup \mathrm{conv}(\{u_j \mid j \in \{1,..,r\} \smallsetminus \{\nu_1,..,\nu_\rho\}\})), \end{aligned}$$

$\mathfrak{E}$ is basic by [**29**, Thm. 3.5, pp. 206-207]. Thus, the entire $\mathcal{T}(r;k)$ is also basic. (Alternatively, since

$$\left\{ \begin{array}{l} \sharp\{(r-1)\text{-dimensional simplices of } \mathfrak{T}(r;k)\} = 2^r - 1, \\[2ex] \sharp \left\{ \begin{array}{c} (r-1)\text{-dimensional simplices} \\ \text{of } \Upsilon^{-1}(\mathbf{T}(r-1;k-1)) \end{array} \right\} = (k-1)^{r-1}, \\[2ex] \text{and, analogously, for all } \rho \in \{1, \ldots, r-1\}, \\[2ex] \sharp \left\{ \begin{array}{c} (r-1-\rho)\text{-dimensional simplices} \\ \text{of } \Upsilon^{-1}(\mathbf{T}(r-1;k-1)) \text{ belonging to the face} \\ \mathrm{conv}(\{u_j \mid j \in \{1,..,r\} \smallsetminus \{\nu_1,..,\nu_\rho\}\}) \text{ of } \mathring{\mathfrak{s}}_G \end{array} \right\} = (k-1)^{r-\rho-1}, \end{array} \right.$$

we get

$$\sharp \left\{ \begin{array}{l} (r-1)\text{-dimensional} \\ \text{simplices of } \mathcal{T}(r;k) \end{array} \right\} = (2^r - 1) + \sum_{\rho=0}^{r-1} \left(\binom{r}{\rho} (k-1)^{r-\rho-1} - \binom{r}{\rho} \right)$$

$$= (2^r - 1) - \sum_{\rho=0}^{r-1} \binom{r}{\rho} + \tfrac{1}{k-1} \sum_{\rho=0}^{r-1} \left(\binom{r}{\rho} (k-1)^{r-\rho} \right)$$

$$= \tfrac{1}{k-1} \sum_{\rho=0}^{r-1} \left(\binom{r}{\rho} (k-1)^{r-\rho} \right) = \tfrac{((k-1)+1)^r - 1}{k-1} = l,$$

and $\mathcal{T}(r;k)$ has to be a basic triangulation of $\breve{\mathfrak{s}}_G$ according to Remark 4.9). We conclude that $\Phi^{-1}(\mathcal{T}(r;k))$ is a basic triangulation of the junior simplex $\mathfrak{s}_G$ (w.r.t. N_G), where Φ is the regular linear transformation (8.5). The basic triangulation $\Phi^{-1}(\mathcal{T}(3;4))$ of $\mathfrak{s}_G$ (refining that one of Example 8.11) is shown in Figure 6.

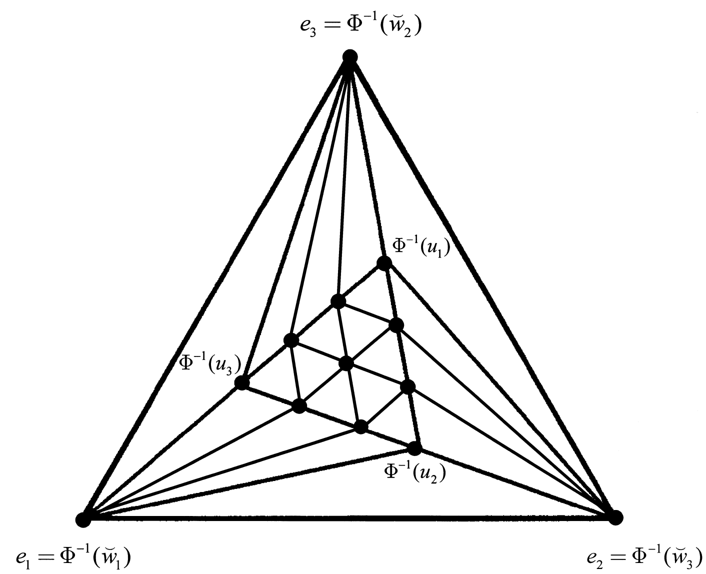

Figure 6.

(ii) *Coherence.* We define the $\mathcal{T}(r;k)$-support function $\mathbf{\Psi} : \breve{\mathfrak{s}}_G \longrightarrow \mathbb{R}$ as follows:

$$\breve{\mathfrak{s}}_G \ni \mathbf{x} \longmapsto \mathbf{\Psi}(\mathbf{x}) := \begin{cases} \psi_{\mathrm{Heav}}^{(r-1)}(\Upsilon(\mathbf{x})), & \text{if } \mathbf{x} \in \mathring{\breve{\mathfrak{s}}}_G, \\ \psi_{\nu_1,\nu_2,\dots,\nu_\rho}(\mathbf{x}), & \text{if } \left\{ \begin{array}{l} \mathbf{x} \in \left| \mathcal{E}_{\nu_1,\nu_2,\dots,\nu_\rho} \right|, \\ \text{for some indices} \\ 1 \le \nu_1 < \cdots < \nu_\rho \le r, \\ \text{with } \rho \in \{1,\dots,r-1\}, \end{array} \right. \end{cases}$$

where $\psi_{\mathrm{Heav}}^{(r-1)}$ is the function defined in 5.4 (with $d = r-1$), and

$$\psi_{\nu_1,\nu_2,\dots,\nu_\rho}(\mathbf{x}) := t \cdot (\theta(\widetilde{\Phi}(\mathbf{x}_1))) + (1-t) \cdot \psi_{\mathrm{Heav}}^{(r-1)}(\Upsilon(\mathbf{x}_2)),$$

for all $\mathbf{x} = t\mathbf{x}_1 + (1-t)\mathbf{x}_2 \in \mathrm{conv}(\{\breve{w}_{\nu_1}, \ldots, \breve{w}_{\nu_\rho}\}) * \mathbf{s}$ belonging to $\left|\mathcal{E}_{\nu_1,\ldots,\nu_\rho}\right|$, with $t \in [0,1]$,

$$\mathbf{x}_1 \in \mathrm{conv}(\{\breve{w}_{\nu_1}, \ldots, \breve{w}_{\nu_\rho}\}) \text{ and } \mathbf{x}_2 \in \mathbf{s} \in \left| \mathcal{T}\,(r;k)|_{\mathrm{conv}(\{\,u_j | j \in \{1,..,r\} \smallsetminus \{\nu_1,..,\nu_\rho\}\})} \right| .$$

Since θ (by Corollary 8.9), as well as $\left.\psi^{(r-1)}_{\mathrm{Heav}}\right|_{\mathring{\mathfrak{s}}_G}$ and $\psi_{\nu_1,\ldots,\nu_\rho}$'s, are strictly upper convex, and the latter ones coincide on their common domains of linearity, we deduce by the Patching Lemma B.3 that also $\mathbf{\Psi}$ is strictly upper convex. This means that $\mathcal{T}\,(r;k)$ is a coherent triangulation of $\breve{\mathfrak{s}}_G$. Consequently, $\Phi^{-1}\,(\mathcal{T}\,(r;k))$ is a coherent triangulation of the junior simplex $\mathfrak{s}_G$.

(iii) *The special case in which* $k = 2$. In this case, $\mathring{\mathfrak{s}}_G$ itself is already basic (w.r.t. $\breve{N}_G$), and the only maximal (and necessarily basic) triangulation of the junior simplex $\mathfrak{s}_G$ (w.r.t. N_G) is $\Phi^{-1}\,(\mathfrak{T}(r;k))$. Its uniqueness and coherence follow from Lemma 8.6 and Corollary 8.9, respectively. □

Note 8.13. (i) For $k \geq 3$, besides $\Phi^{-1}\,(\mathcal{T}\,(r;k))$, there are lots of other basic triangulations of $\mathfrak{s}_G$, due to those of $\Phi^{-1}\,(\mathring{\mathfrak{s}}_G)$; cf. Remark 5.6 (ii).

(ii) *Open Problem*: As it was proven recently by Sebestean [**112**] (for $r = 4$, $k = 2$), the smooth fourfold obtained by the unique projective crepant resolution of the (non-symplectic) CQS $(\mathbb{C}^4/G, [\mathbf{0}])$ of type $\frac{1}{15}(1,2,4,8)$ coincides with the Hilbert scheme G-Hilb$(\mathbb{C}^4)$ of G-clusters. It is therefore natural to ask if this is in general true for *all* the members of the series GPSS$(r;2)$ (or not) whenever $r \geq 5$.

Exercise 8.14. Compute the non-trivial cohomology dimensions (4.2) of any crepant resolution space of any member of the geometric progress singularity series GPSS$(r;k)$. (*Hint.* Consider $\Phi^{-1}\,(\mathcal{T}\,(r;k))$ as a composite of geometrically more "elementary" triangulations, and use the inclusion-exclusion property of lattice point enumerators, combined with the multiplicative property of the polynomial generating the **h**-vectors of joins of triangulations [**16**, p. 466], and with Theorem C.9 and formula (4.3).)

Remark 8.15. Concerning the Existence Problem, it is worthwhile stressing the qualitative difference between the behaviour of the 1- and 2-parameter singularity series discussed in §7 and that one of the geometric progress singularity series GPSS$(r;k)$. The one or two parameters in the types of the first mentioned singularities have to obay to *restrictive* arithmetic conditions in order to lead to crepant resolutions (cf. (7.2) and (7.4)), whereas the parameter $k \geq 2$ in the GP-singularity series is *unconditionally* free in this respect.

9. Second Existence Criterion via UBT

Let $(\mathbb{C}^r/G, [\mathbf{0}])$ be a Gorenstein AQS with $l = |G|$ and $r \geq 4$. The presence of *basic* triangulations $\mathcal{T}$ of $\mathfrak{s}_G$ (w.r.t. N_G) implies the equality $l = (r-1)!\mathrm{Vol}(\mathfrak{s}_G) = \mathfrak{f}_{r-1}(\mathcal{T})$ (by (4.4)). Bounding the cardinality $\mathfrak{f}_{r-1}(\mathcal{T})$ of the facets of any such $\mathcal{T}$ from above by a number depending only on the number of the available lattice points in $\mathfrak{s}_G$, it is possible to obtain a second *necessary existence condition* which is highly effective and of purely geometric nature. It comes as no surprise to learn that such a number involves the cardinality $\mathfrak{f}_{r-1}\,(\mathrm{CycP}_r\,(\sharp\,(\mathfrak{s}_G \cap N_G)))$ of the facets of the r-dimensional cyclic polytope with $\sharp\,(\mathfrak{s}_G \cap N_G)$ vertices, because it reminds you of the celebrated *UBT* A.3 *for simplicial spheres.* Nevertheless, this has first to be suitably modified to be valid for *simplicial balls* (like $\mathcal{T}$); see Theorem A.5. Unfortunately, even if we use the latter upper bound, we do lose some information whenever our singularity is *non-isolated*, because we are throwing away a considerable part of the individual contributions of lattice points which belong to the boundary of $\mathfrak{s}_G$. In fact, our expectation concerning a general, *tight* upper bound for $l = \mathfrak{f}_{r-1}(\mathcal{T})$ is expressed in the following:

Conjecture 9.1. *Let* $(\mathbb{C}^r/G, [\mathbf{0}])$, $r \geq 4$, *be a Gorenstein AQS with* $l = |G|$, *and* $\mathfrak{s}_G$ *the corresponding junior simplex. If* $\mathfrak{s}_G$ *has a basic triangulation, then* l *has the following upper bound*:

$$\boxed{l \leq \mathfrak{f}_{r-1}\left(\mathrm{CycP}_r\left(\sharp\left(\mathfrak{s}_G \cap N_G\right)\right)\right) - \sum_{k=2}^{r-1}(r-k)\left(\sharp\left(\mathfrak{B}_G\left(1,k\right)\right)\right) - 1,} \tag{9.1}$$

with $\sharp\left(\mathfrak{B}_G\left(1,k\right)\right)$'s *as given in* (3.20).

Note 9.2. For the proof of (9.1) it would suffice to show that UBT-Conjecture A.20 is true. In Theorem 9.3 we prove (9.1) only for $r = 4$, and give the weaker upper bound for $r \geq 5$.

Theorem 9.3 (Second Necessary Existence Condition). *Let* $(\mathbb{C}^r/G, [\mathbf{0}])$, $r \geq 4$, *be a Gorenstein AQS with* $l = |G|$, *and* $\mathfrak{s}_G$ *the corresponding junior simplex. If* $\mathfrak{s}_G$ *has a basic triangulation* $\mathcal{T}$, *then* l *has as upper bound*

$$\boxed{l \leq \mathfrak{f}_3\left(\mathrm{CycP}_4\left(\sharp\left(\mathfrak{s}_G \cap N_G\right)\right)\right) - 2\ \left(\sharp\left(\mathfrak{B}_G\left(1,2\right)\right)\right) - \left(\sharp\left(\mathfrak{B}_G\left(1,3\right)\right)\right) - 1} \tag{9.2}$$

for $r = 4$, *and*

$$\boxed{l \leq \mathfrak{f}_{r-1}\left(\mathrm{CycP}_r\left(\sharp\left(\mathfrak{s}_G \cap N_G\right)\right)\right) - \sum_{k=2}^{r-1}\left(\sharp\left(\mathfrak{B}_G\left(1,k\right)\right)\right) - 1} \tag{9.3}$$

for $r \geq 5$. (The number $\sharp\left(\mathfrak{s}_G \cap N_G\right)$ can be calculated by the formulae (D.16) and (D.17) given in Appendix D. The numbers $\sharp\left(\mathfrak{B}_G\left(1,k\right)\right)$, $2 \leq k \leq r-1$, occuring in (9.2), (9.3), are computable *either* by (3.20) and (3.21) *or* by using (3.20) and then counting the lattice points lying in the relative interior of each of the $(k-1)$-dimensional faces of $\mathfrak{s}_G$, by (C.2), (D.16) and (D.17), applied for these faces instead for $\mathfrak{s}_G$ itself).

Proof. Using the notation of Appendix A, apply (A.5) (to get (9.3) for $r \geq 5$) just by setting $d = r-1$, $\mathcal{S} = \mathcal{T}$, $\mathfrak{b} = \sharp\left(\mathfrak{s}_G \cap N_G\right)$, $\mathfrak{b}' = \sum_{k=2}^{r-1}\left(\sharp\left(\mathfrak{B}_G\left(1,k\right)\right)\right) + r$. Correspondingly, to get (9.2) for $r = 4$, apply Theorem A.22 by setting $d = 3$, $\mathbf{s} = \mathfrak{s}_G$, $\mathcal{S} = \mathcal{T}$, $b_1 = 4$, and $b_k = \sharp\left(\mathfrak{B}_G\left(1,k\right)\right)$, for $k \in \{2,3\}$. Of course, for the desingularizing space $X(N_G, \widehat{\Delta_G\left(\mathcal{T}\right)})$ of $X\left(N_G, \Delta_G\right) = \mathbb{C}^r/G$ being induced by $\mathcal{T}$, we have $l = \mathfrak{f}_{r-1}\left(\mathcal{T}\right)$ by (4.4). $\square$

COROLLARY 9.4. *Let* $(\mathbb{C}^4/G, [\mathbf{0}])$ *be a Gorenstein cyclic quotient msc-singularity of type* $\frac{1}{l}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. *Then the inequality* (9.2) *can be written as follows*:

$$\boxed{l \le \tfrac{\sharp(\mathfrak{s}_G \cap N_G)(\sharp(\mathfrak{s}_G \cap N_G)-3)}{2} - \sum_{i=1}^{4} \tfrac{\gcd(\alpha_i, l)}{2} - \sum_{1 \le i < j \le 4} \gcd(\alpha_i, \alpha_j, l) + 7,} \tag{9.4}$$

where $\sharp(\mathfrak{s}_G \cap N_G)$ *is known by the formulae* (D.3), (D.4), (D.5).

PROOF. Obviously,

$$\sharp(\mathfrak{B}_G(1,2)) = \sum_{1 \le \nu_1 < \nu_2 \le 4} \sharp(\mathfrak{B}_G(1,2;\nu_1,\nu_2)).$$

If for any pair of indices ν_1, ν_2, with $1 \le \nu_1 < \nu_2 \le 4$, we define $\{\nu_3, \nu_4\}$ to be the complement set $\{1,2,3,4\} \smallsetminus \{\nu_1, \nu_2\}$, then

$$\sharp(\mathfrak{B}_G(1,2;\nu_1,\nu_2)) = \gcd(\alpha_{\nu_3}, \alpha_{\nu_4}, l) - 1 \tag{9.5}$$

by (3.22). Analogously,

$$\sharp(\mathfrak{B}_G(1,3)) = \sum_{1 \le \nu_1 < \nu_2 < \nu_3 \le 4} \sharp(\mathfrak{B}_G(1,3;\nu_1,\nu_2,\nu_3)),$$

and if for any triple of indices ν_1, ν_2, ν_3, with $1 \le \nu_1 < \nu_2 < \nu_3 \le 4$, $\{\nu_4\}$ denotes the complement set $\{1,2,3,4\} \smallsetminus \{\nu_1, \nu_2, \nu_3\}$, then the number of the interior points of each 2-face of the junior tetrahedron $\mathfrak{s}_G$ equals

$$\begin{aligned} \sharp(\mathfrak{B}_G(1,3;\nu_1,\nu_2,\nu_3)) &= \frac{1}{2}\left[\gcd(\alpha_{\nu_4}, l) - 1 - \left(\sum_{j=1}^{3} \gcd\left(\alpha_{\nu_4}, \alpha_{\nu_j}, l\right) - 1\right)\right] \\ &= \frac{1}{2}\left[\gcd(\alpha_{\nu_4}, l) - \sum_{j=1}^{3} \gcd\left(\alpha_{\nu_4}, \alpha_{\nu_j}, l\right)\right] + 1 \end{aligned} \tag{9.6}$$

by the refined Ping-Pong Lemma 3.13 (see Remark 3.20). Substituting (9.5), (9.6) into formula (9.2) we obtain (9.4). □

EXAMPLE 9.5. Let $(\mathbb{C}^4/G, [\mathbf{0}])$ be the CQS of type $\frac{1}{12}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, with $\alpha_1 = \alpha_2 = 2$, $\alpha_3 = 3$, and $\alpha_4 = 5$. Since $\sharp(\mathfrak{s}_G \cap N_G) = 7$, and the right-hand side of (9.4) equals

$$\tfrac{7(7-3)}{2} - \tfrac{1}{2}\left[\sum_{i=1}^{4} \gcd(\alpha_i, l)\right] - \sum_{1 \le i < j \le 4} \gcd(\alpha_i, \alpha_j, l) + 7 = 14 - 4 = 10,$$

it does not admit any crepant resolution because $10 < 12 = l$.

REMARK 9.6 (Comparison of the two Existence Criteria). Which of the necessary conditions (6.1) and (9.2)-(9.3) given in Theorems 6.1 and 9.3, respectively, is better? The answer to this question depends on how one would like to interpret the adjective "better". Undoubtedly, (6.1) "kills" more candidates for having crepant resolutions. For instance, for the CQS $(\mathbb{C}^4/G, [\mathbf{0}])$ of type $\frac{1}{9}(1,2,3,3)$ (9.4) holds as equality but $\frac{1}{9}(5,1,6,6)^{\intercal} \in \mathbf{Hlb}_{N_G}(\sigma_0) \smallsetminus (\mathfrak{s}_G \cap N_G)$. (Hence, this CQS does not have any crepant resolution.) On the other hand, in view of Theorem 2.4, the determination of the Hilbert basis is a time-consuming procedure compared with the lattice point enumeration of the junior simplex (in particular, in high dimensions and for acting groups with big orders).

EXERCISE 9.7. For the Gorenstein CQS of type $\frac{1}{12}(1,3,3,5)$ show that (9.4) holds as strict inequality, though $\frac{1}{12}(3,9,9,3)^{\intercal} \in \mathbf{Hlb}_{N_G}(\sigma_0) \smallsetminus (\mathfrak{s}_G \cap N_G)$.

10. Sketching an Auxiliary Algorithm

Taking into account what we have discussed so far, it is possible, for given AQS $(\mathbb{C}^r/G, [\mathbf{0}])$ of type (3.6), to outline an algorithm in order to examine whether it admits the desired resolutions, but at the cost of *increasing computational complexity* (in the consecutive steps). More precisely, the auxiliary algorithm we have in mind (summarized in Figure 7) is built up as follows:

$\triangleright$ STEP 1. If $\mathfrak{s}_G$ is lattice equivalent to a Watanabe simplex, then $\mathbb{C}^r/G$ admits of projective crepant desingularizations according to Theorem 5.1. If not, we go to Step 2.

$\triangleright$ STEP 2. If $(\mathbb{C}^r/G, [\mathbf{0}])$ is non-c.i but belongs to "special" singularity series (like those 1- and 2-parameter series of §7 having weights satisfying conditions (7.2), (7.4), or even the entire GP-singularities series of §8), which have projective crepant resolutions *by construction*, we stop; otherwise we proceed. (To continue increasing our stock of "special" singularity series of this kind would be a real challenge for future work.)

$\triangleright$ STEP 3. We *count the lattice points* of the junior simplex $\mathfrak{s}_G$ involved in (9.2), resp., (9.3), by the formulae given in Appendix D, and then we check if these inequalities for $l = |G|$ are valid or not. We proceed only if (9.2) (for $r = 4$), resp., (9.3) (for $r \geq 5$), are indeed valid; otherwise $\mathbb{C}^r/G$ does not admit any crepant desingularization by Theorem 9.3.

$\triangleright$ STEP 4. We determine the Hilbert basis $\mathbf{Hlb}_{N_G}(\sigma_0)$ (see Remark 2.5), and control if it satisfies condition (6.1). We proceed to the next (final) step only if (6.1) is satisfied; otherwise the quotient space $\mathbb{C}^r/G$ does not have any crepant desingularization by Theorem 6.1.

$\triangleright$ STEP 5. If $(\mathbb{C}^r/G, [\mathbf{0}])$ happens to pass all the above tests without stop in the one or the other stage, we have to find out *all the junior lattice points*

$$\{\, n_g \in N_G \text{ (as in (3.9))} \mid \text{ age}(g) = 1\} = \mathfrak{s}_G \cap N_G$$

(not just their cardinality!), to run `Puntos` [**31**] or `TOPCOM` [**100**] for the point configuration $\mathcal{V} = \mathfrak{s}_G \cap N_G$ in order to specify the coherent lattice triangulations of $\mathfrak{s}_G$, to separate the *maximal* ones, and then to count the number of $(r-1)$-dimensional simplices in each of them; cf. Note B.5. Projective crepant desingularizations of $\mathbb{C}^r/G$ are present as long as this number equals l for at least one of them. ("Sporadic" counterexamples, like the isolated CQS of type $\frac{1}{39}(1,5,8,25)$ mentioned in Remark 6.2 (iii), indicate why Step 5 or any similar computer-assisted procedure seems -as yet- to be unavoidable.)

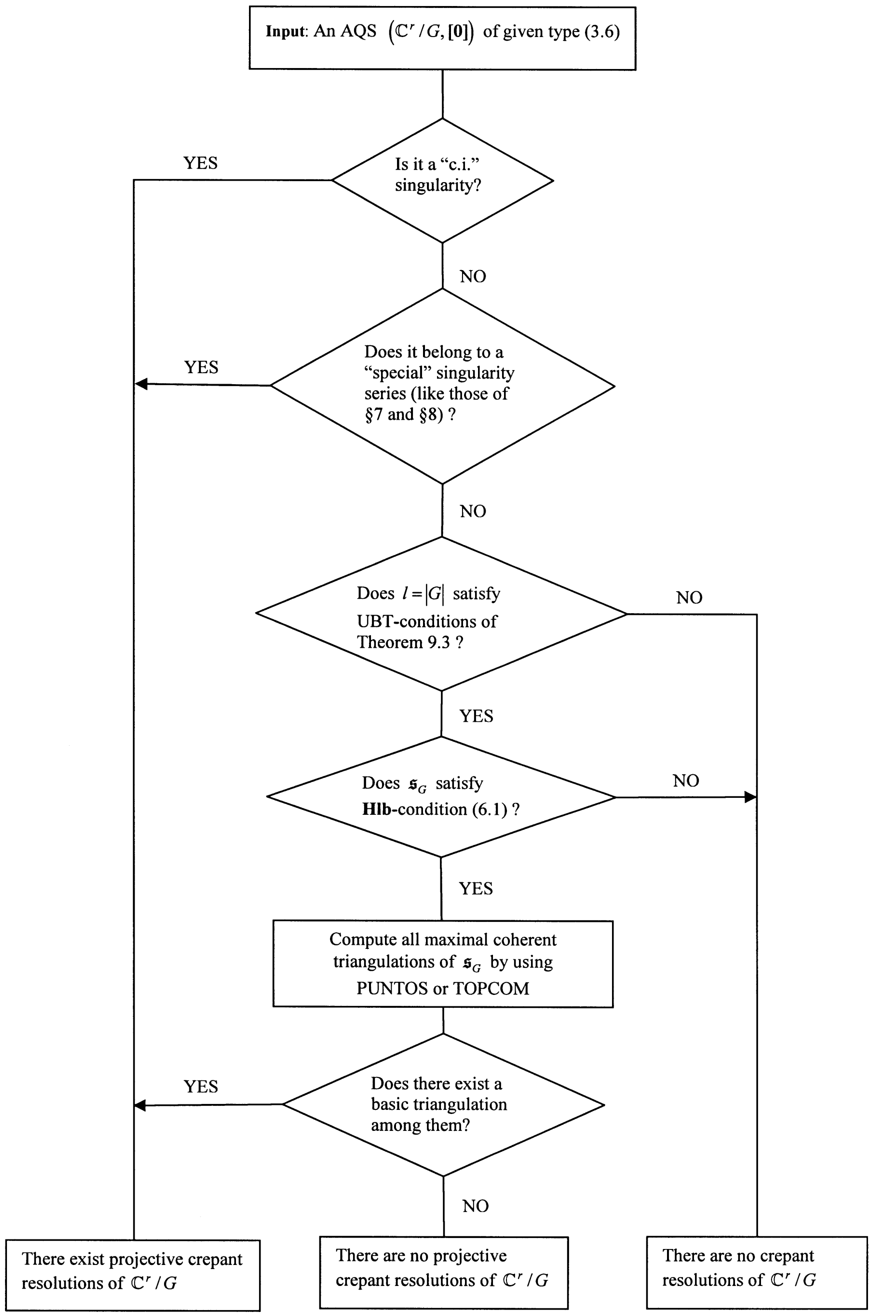
Input: An AQS $(\mathbb{C}^r/G, [\mathbf{0}])$ of given type (3.6)
Is it a "c.i." singularity?
YES
NO
Does it belong to a "special" singularity series (like those of §7 and §8) ?
YES
NO
Does $l = |G|$ satisfy UBT-conditions of Theorem 9.3 ?
NO
YES
Does $\mathfrak{s}_G$ satisfy **Hlb**-condition (6.1) ?
NO
YES
Compute all maximal coherent triangulations of $\mathfrak{s}_G$ by using PUNTOS or TOPCOM
Does there exist a basic triangulation among them?
YES
NO
There exist projective crepant resolutions of $\mathbb{C}^r/G$
There are no projective crepant resolutions of $\mathbb{C}^r/G$
There are no crepant resolutions of $\mathbb{C}^r/G$

FIGURE 7.

Appendix A. Triangulations and Upper Bound Theorems

Triangulations (as geometric simplicial subdivisions of polytopes or polytopal complexes) are treated in the classical framework of the categorial inclusions:

$$\left\{\begin{array}{c}\text{geometric}\\ \text{simplicial complexes}\end{array}\right\} \subset \left\{\begin{array}{c}\text{polytopal}\\ \text{complexes}\end{array}\right\} \subset \left\{\begin{array}{c}\text{“regular cell}\\ \text{complexes”}\\ \text{(i.e. regular finite}\\ \text{CW complexes)}\end{array}\right\}.$$

• **Notation**. The symbol "$\approx$" between two topological spaces indicates the existence of an homeomorphism from the one onto the other. A topological space X is called a *sphere* (resp., a *ball*) if $X \approx \mathbb{S}^k$ (resp., $X \approx \mathbb{B}^k$), for some k, where $\mathbb{S}^k$ and $\mathbb{B}^k$ denote the standard k-dimensional sphere $\mathbb{S}^k = \partial\mathbb{B}^{k+1} = \left\{\mathbf{x} \in \mathbb{R}^{k+1} \mid \|\mathbf{x}\| = 1\right\}$ and the standard k-dimensional ball $\mathbb{B}^k = \left\{\mathbf{x} \in \mathbb{R}^k \mid \|\mathbf{x}\| \leq 1\right\}$, respectively. If k is assumed to be fixed, then we simply say that such an X is a *k-sphere* (resp. a *k-ball*). *Polytopes* will be always convex, defined as in [**126**, Lecture 1].

• **Regular cell complexes**. A *regular cell complex* $\mathcal{K}$ is a finite collection of balls $\mathbf{c}$ in a Hausdorff topological space $|\mathcal{K}| = \bigcup\{\mathbf{c} \mid \mathbf{c} \in \mathcal{K}\}$ such that
(i) the relative interiors $\operatorname{int}(\mathbf{c})$ of all $\mathbf{c}$'s partition $|\mathcal{K}|$, i.e., each element of $|\mathcal{K}|$ lies in exactly one $\operatorname{int}(\mathbf{c})$, and
(ii) the relative boundary $\partial\mathbf{c}$ of every $\mathbf{c} \in \mathcal{K}$ is a union of some members of $\mathcal{K}$.

The balls $\mathbf{c} \in \mathcal{K}$ are called the *closed cells* of $\mathcal{K}$ and their interiors $\operatorname{int}(\mathbf{c})$ the *open cells* of $\mathcal{K}$. $|\mathcal{K}|$ is called the *underlying space* (or *the support*) of $\mathcal{K}$. The dimension of a (closed) cell $\mathbf{c}$, for which $\mathbf{c} \approx \mathbb{B}^k$, is defined to be k. (Such a cell is particularly called a *k-cell*). If $\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{K}$ and $\mathbf{c}_1 \subset \mathbf{c}_2$, then $\mathbf{c}_1$ is said to be a *face* of $\mathbf{c}_2$. (We use the notation: $\mathbf{c}_1 \prec \mathbf{c}_2$). 0-and 1-cells are called *vertices* and *edges*, respectively. $\mathcal{K}$ is defined to be *pure* if all maximal cells have the same dimension. $\mathcal{K}'$ is a *subcomplex* of $\mathcal{K}$ if $\mathbf{c} \in \mathcal{K}'$ implies that every face of $\mathbf{c}$ belongs to $\mathcal{K}'$. Note that a regular cell complex is homeomorphic to the order complex of its face poset.

• **Polytopal complexes**. A *polytopal complex* $\mathcal{S}$ consists of a finite family of polytopes in $\mathbb{R}^d$ such that
(i) if $P \in \mathcal{S}$ and $F \prec P$, then $F \in \mathcal{S}$, and
(ii) if $P_1, P_2 \in \mathcal{S}$ have non-empty intersection, then $P_1 \cap P_2 \prec P_1$, and $P_1 \cap P_2 \prec P_2$.

Since every polytope is topologically a ball, a polytopal complex $\mathcal{S}$ is a regular cell complex whose (closed) cells (called also *faces*) are the participating polytopes, and whose underlying space $|\mathcal{S}|$ is the union of these polytopes. (If $\mathcal{S}$ is a polytopal complex, we denote by $\operatorname{vert}(\mathcal{S})$ the set of its vertices. If $\mathcal{S}$ is, in addition, pure, we call the $\dim(\mathcal{S})$-faces *facets* of $\mathcal{S}$.)

• **Geometric simplicial complexes**. A *geometric simplicial complex* is by definition a polytopal complex all of whose (closed) cells are simplices. We frequently denote the simplices of such an $\mathcal{S}$ by F or $\mathbf{s}$ instead of $\mathbf{c}$. If $|\mathcal{S}| \approx \mathbb{S}^k$ (resp., if $|\mathcal{S}| \approx \mathbb{B}^k$), then $\mathcal{S}$ is called a *simplicial k-sphere* (resp., a *simplicial k-ball*).

Example A.1. Every d-polytope P together with all of its faces forms a polytopal d-complex $\mathcal{S}_P$. For a d-polytope P the *boundary complex* $\mathcal{S}_{\partial P}$ of P is defined to be the $(d-1)$-dimensional polytopal complex consisting of the proper faces of P together with $\varnothing$ and having support $|\mathcal{S}_{\partial P}| = \partial P$. The *facets* of P are defined to be the facets of $\mathcal{S}_{\partial P}$. Obviously, $\mathcal{S}_{\partial P}$ is a geometric pure simplicial complex (in fact, a simplicial $(d-1)$-sphere) if and only if P is a simplicial polytope.

• **Abstract simplicial complexes**. Geometric simplicial complexes can be obtained as *geometric realizations* of "abstract" simplicial (finite) complexes. An *abstract simplicial (finite) complex* $\mathcal{S}(\mathcal{V})$ with vertex set $\mathcal{V}$ is a finite collection of subsets F of $\mathcal{V}$ having the properties :

(i) If $v \in \mathcal{V}$, then $\{v\} \in \mathcal{S}(\mathcal{V})$, and (ii) if $F \in \mathcal{S}(\mathcal{V})$ and $F' \subset F$, then $F' \in \mathcal{S}(\mathcal{V})$.

The elements $F \in \mathcal{S}(\mathcal{V})$ are called *abstract simplices* or *faces*. For an $F \in \mathcal{S}(\mathcal{V})$ one defines $\dim(F) := \sharp(F) - 1$ and $\dim(\mathcal{S}(\mathcal{V})) := \max\{\dim(F) \,|\, F \in \mathcal{S}(\mathcal{V})\}$ as the *dimension* of $\mathcal{S}(\mathcal{V})$. (If $\dim(F) = k+1$, then F is said to be an *abstract k-simplex* or a *k-face*). A *subcomplex* of $\mathcal{S}(\mathcal{V})$ is an abstract simplicial complex contained in $\mathcal{S}(\mathcal{V})$ whose vertex-set is a subset of $\mathcal{V}$. (Sometimes, for F a subset of $\mathcal{V}$, one denotes the abstract simplex with vertex set F by 2^F.)

Definition A.2. Let $\mathcal{S}(\mathcal{V})$ be an abstract simplicial complex with vertex set $\mathcal{V}$ and $\iota : \mathcal{V} \to \mathbb{R}^d$ an injective map, such that

(i) the elements of the images $\iota(F)$ are affinely independent for all $F \in \mathcal{S}(\mathcal{V})$, and

(ii) $\mathrm{int}(\mathrm{conv}(\iota(F))) \cap \mathrm{int}(\mathrm{conv}(\iota(F'))) = \varnothing$, for all $F, F' \in \mathcal{S}(\mathcal{V})$, $F \neq F'$.

Then $\bigcup_{F \in \mathcal{S}(\mathcal{V})} \mathrm{int}(\mathrm{conv}(\iota(F)))$ is called a *geometric realization* of $\mathcal{S}(\mathcal{V})$ w.r.t. ι.

Geometric realizations always exist, the underlying spaces of any two geometric realizations of $\mathcal{S}(\mathcal{V})$ are homeomorphic to each other, and therefore any *support* $|\mathcal{S}(\mathcal{V})|$ "realizing" $\mathcal{S}(\mathcal{V})$ is well-defined in the topological category. In many cases, we shall denote both geometric and abstract simplicial complexes by the letter $\mathcal{S}$. When, for some reason, our intention is to stress what kind of complexes is meant (if this is not clear from the context), and our starting point is a geometric simplicial complex $\mathcal{S}$, we denote by $\mathcal{S}^{\mathrm{abs}}$ the corresponding abstract simplicial complex; and conversely, when our starting point is an abstract simplicial complex $\mathcal{S}$, we consider a fixed realization $|\mathcal{S}|$. Moreover, we mostly use $\mathcal{V}$ and $\mathrm{vert}(\mathcal{S})$ (for the vertex set) interchangeably.

• **f- and h-vectors**. The **f**-*vector* $\mathbf{f}(\mathcal{S}) = (\mathfrak{f}_{-1}(\mathcal{S}), \mathfrak{f}_0(\mathcal{S}), \ldots, \mathfrak{f}_d(\mathcal{S}))$ of a d-dimensional geometric or abstract simplicial complex $\mathcal{S}$ is defined by setting

$$\mathfrak{f}_{-1}(\mathcal{S}) := -1, \text{ and } \mathfrak{f}_i(\mathcal{S}) := \sharp\{i\text{-dimensional faces of } \mathcal{S}\}, \ \forall i \in \{0, \ldots, d\}.$$

The **h**-*vector* $\mathbf{h}(\mathcal{S}) = (\mathfrak{h}_0(\mathcal{S}), \mathfrak{h}_1(\mathcal{S}), \ldots, \mathfrak{h}_{d+1}(\mathcal{S}))$ of $\mathcal{S}$ is defined by the equation

$$\mathbf{h}(\mathcal{S}; t) = (1-t)^d \ \mathbf{f}(\mathcal{S}; \tfrac{t}{1-t}), \tag{A.1}$$

where $\mathbf{f}(\mathcal{S}; t) := \sum_{i=0}^{d+1} \mathfrak{f}_{i-1}(\mathcal{S})\, t^i \in \mathbb{Z}_{\geq 0}[t]$ and $\mathbf{h}(\mathcal{S}; t) := \sum_{i=0}^{d+1} \mathfrak{h}_i(\mathcal{S})\, t^i \in \mathbb{Z}[t]$. Note, in particular, that (A.1) gives

$$\mathfrak{f}_{j-1}(\mathcal{S}) = \sum_{i=0}^{j} \binom{d-i}{d-j} \mathfrak{h}_i(\mathcal{S}), \ \forall j, \ 0 \leq j \leq d+1. \tag{A.2}$$

The **f**-vector $\mathbf{f}(P)$ of a simplicial polytope P is by definition the **f**-vector $\mathbf{f}(\mathcal{S}_{\partial P})$ of $\mathcal{S}_{\partial P}$ (as in Example A.1).

• **UBT for the facets of simplicial balls**. We denote by $\mathrm{CycP}_d\,(k)$ the *cyclic d-polytope* with k vertices. As it is known, the number of its facets equals

$$\mathfrak{f}_{d-1}\,(\mathrm{CycP}_d\,(k)) = \binom{k-\left\lceil \frac{d}{2}\right\rceil}{\left\lfloor \frac{d}{2}\right\rfloor} + \binom{k-1-\left\lceil \frac{d-1}{2}\right\rceil}{\left\lfloor \frac{d-1}{2}\right\rfloor}. \tag{A.3}$$

This is due to Gale's evenness condition and to the fact that $\mathrm{CycP}_d\,(k)$ is $\lfloor d/2 \rfloor$-neighbourly (cf. [**126**, p. 24]). Let us first recall the classical UBT and LBT for simplicial *spheres*, and then explain how one obtains an UBT for the facets of simplicial *balls*.

Theorem A.3 (Upper Bound Theorem for Simplicial Spheres, [**119**, II.4.5]). *The* $\mathbf{f}$*-vector coordinates of a simplicial $(d-1)$-sphere $\mathcal{S}$ with $\mathfrak{f}_0\,(\mathcal{S}) = k$ vertices satisfy the following inequalities*:

$$\mathfrak{f}_i\,(\mathcal{S}) \leq \mathfrak{f}_i\,(\mathrm{CycP}_d\,(k)) \ \ , \ \forall \ i \ , \ 0 \leq i \leq d-1.$$

Theorem A.4 (Lower Bound Theorem for Simplicial Spheres, [**69**]). *The* $\mathbf{h}$-*vector coordinates of a simplicial $(d-1)$-sphere $\mathcal{S}$ with $\mathfrak{f}_0\,(\mathcal{S}) = k$ vertices satisfy the following inequalities*:

$$\mathfrak{h}_1\,(P) = k-d \leq \mathfrak{h}_i\,(P) \ \ , \ \forall \ i \ , \ 2 \leq i \leq d. \tag{A.4}$$

Theorem A.5 (UBT for the Facets of Simplicial Balls, [**25**]). *Let $\mathcal{S}$ be a simplicial d-ball with $\mathfrak{f}_0\,(\mathcal{S}) = \mathfrak{b}$ vertices. Suppose that $\mathfrak{f}_0\,(\partial\mathcal{S}) = \mathfrak{b}'$. Then*:

$$\mathfrak{f}_d\,(\mathcal{S}) \leq \mathfrak{f}_d\left(\mathrm{CycP}_{d+1}\,(\mathfrak{b})\right) - (\mathfrak{b}'-d)\,. \tag{A.5}$$

Sketch of Proof. Introduce the auxiliary vector $\widetilde{\mathbf{h}}\,(\mathcal{S}) = (\widetilde{\mathfrak{h}}_0\,(\mathcal{S})\,,\ldots,\widetilde{\mathfrak{h}}_{d+1}\,(\mathcal{S}))$ with

$$\widetilde{\mathfrak{h}}_i\,(\mathcal{S}) := \begin{cases} \mathfrak{h}_i\,(\mathcal{S}) & , \quad \text{for} \quad 0 \leq i \leq \ \frac{d+1}{2} \ , \\ \\ \mathfrak{h}_i\,(\mathcal{S}) - (\mathfrak{h}_{d-i}\,(\partial\mathcal{S}) - \mathfrak{h}_{d+1-i}\,(\partial\mathcal{S})) & , \quad \text{for} \quad \frac{d+1}{2} \ +1 \leq i \leq d+1. \end{cases}$$

By [**110**, Thm. 4.3, p. 136] we see that

$$\mathfrak{h}_i\,(\mathcal{S}) \leq \mathfrak{h}_i\left(\mathrm{CycP}_{d+1}\,(\mathfrak{b})\right) = \binom{\mathfrak{b}-d+i-2}{i}, \ \ \forall i, \ \ 0 \leq i \leq \left\lfloor \tfrac{d+1}{2} \right\rfloor.$$

On the other hand, using Stanley's "$\mathfrak{h}$ of ∂"-Lemma [**118**, Lemma 2.3, p. 253], i.e.,

$$\mathfrak{h}_{i-1}\,(\partial\mathcal{S}) - \mathfrak{h}_i\,(\partial\mathcal{S}) = \mathfrak{h}_{(d+1)-i}\,(\mathcal{S}) - \mathfrak{h}_i\,(\mathcal{S})\,, \ \ \forall i, \ 0 \leq i \leq d+1,$$

we get $\widetilde{\mathfrak{h}}_i\,(\mathcal{S}) = \widetilde{\mathfrak{h}}_{(d+1)-i}\,(\mathcal{S})\,, \ \forall i \in \{0,1,\ldots,d+1\}$, and therefore

$$\widetilde{\mathfrak{h}}_i\,(\mathcal{S}) \leq \mathfrak{h}_i\left(\mathrm{CycP}_{d+1}\,(\mathfrak{b})\right), \ \ \forall i, \ \ 0 \leq i \leq d+1. \tag{A.6}$$

Passing to the $\mathbf{f}$-vector, and using Dehn-Sommerville relations for $\mathbf{h}(\partial\mathcal{S})$, we verify easily via (A.6) that

$$\mathfrak{f}_i\,(\mathcal{S}) \leq \mathfrak{f}_i\left(\mathrm{CycP}_{d+1}\,(\mathfrak{b})\right) - \sum_{j=d-i}^{\left\lfloor \frac{d}{2}\right\rfloor} \binom{j}{d-i} \ (\mathfrak{h}_j\,(\partial\mathcal{S}) - \mathfrak{h}_{j-1}\,(\partial\mathcal{S}))\,. \tag{A.7}$$

For $i = d$, (A.7) gives

$$\begin{aligned} \mathfrak{f}_d\,(\mathcal{S}) &\leq \mathfrak{f}_d \ \ \mathrm{CycP}_{d+1}\,(\mathfrak{b}) \ \ - \sum_{j=0}^{\left\lfloor \frac{d}{2}\right\rfloor} (\mathfrak{h}_j\,(\partial\mathcal{S}) - \mathfrak{h}_{j-1}\,(\partial\mathcal{S})) \\ &= \mathfrak{f}_d \ \ \mathrm{CycP}_{d+1}\,(\mathfrak{b}) \ \ - \mathfrak{h}_{\left\lfloor \frac{d}{2}\right\rfloor}\,(\partial\mathcal{S}) \leq \mathfrak{f}_d \ \ \mathrm{CycP}_{d+1}\,(\mathfrak{b}) \ \ - \mathfrak{h}_1\,(\partial\mathcal{S})\,, \end{aligned}$$

where the latter inequality comes from (A.4) for the simplicial sphere $\partial\mathcal{S}$. Now obviously, $\mathfrak{h}_1(\partial\mathcal{S}) = \mathfrak{b}' - d$. □

• **Subdivisions**. There are several kinds of *subdivisions* of simplicial and polytopal complexes which correspond to the different distinctive features of the partitioning objects and depend on the way they have to fit together. (Here we follow Stanley's terminology from [**117, 119**].)

Definition A.6. Let $\mathcal{S}$ be an abstract simplicial complex. A *topological simplicial subdivision* of $\mathcal{S}$ is a pair $(\mathcal{S}', \varphi)$ consisting of an abstract simplicial complex $\mathcal{S}'$ together with a map $\varphi : \mathcal{S}' \to \mathcal{S}$ satisfying the following conditions:

(i) For every $F \in \mathcal{S}$, the *restriction* $\mathcal{S}'_F := \varphi^{-1}\left(2^F\right)$ of $\mathcal{S}'$ to F is a subcomplex of $\mathcal{S}'$ having a geometric realization $|\mathcal{S}_F|$ which is a $(\dim(F))$-ball.

(ii) For each $F' \in \mathcal{S}'$, $F = \varphi(F') \in \mathcal{S}$ if and only if F' is an *interior face* of $\mathcal{S}'_F$, i.e., if and only if $|F'|$ is not contained in the boundary $\partial|\mathcal{S}'_F| = |\mathcal{S}'_F| \smallsetminus \mathrm{int}(|\mathcal{S}'_F|)$.

(Often one calls $\mathcal{S}'$ a topological simplicial subdivision of $\mathcal{S}$ and omits φ if it is self-evident from the context.)

Definition A.7. Let $\mathcal{S}$ be an abstract simplicial complex. A topological simplicial subdivision $(\mathcal{S}', \varphi)$ of S is called *quasigeometric* if for every face F' of $\mathcal{S}'$ there does not exist a face $F \in \mathcal{S}$ for which

(i) $\dim(F) < \dim(F')$ and

(ii) each vertex v of F' lies on some subset of F (depending on v).

Definition A.8. Let $\mathcal{S}$ be a geometric simplicial complex and $\mathcal{S}^{\mathrm{abs}}$ the corresponding abstract simplicial complex. A geometric simplicial complex $\mathcal{S}'$ is called a *geometric simplicial subdivision* or a *geometric triangulation* or simply a *triangulation* of $\mathcal{S}$ (and, respectively, $\mathcal{S}'^{\,abs}$ a *geometric simplicial subdivision* or *a geometric triangulation* of $\mathcal{S}^{abs}$) if $|\mathcal{S}| = |\mathcal{S}'|$ and every simplex in $\mathcal{S}'$ is contained in some simplex in $\mathcal{S}$.

Definition A.9. Let $\mathcal{S}$ be a polytopal complex. A polytopal complex $\mathcal{S}'$ is called a *polytopal subdivision* of $\mathcal{S}$ if $|\mathcal{S}| = |\mathcal{S}'|$ and every polytope in $\mathcal{S}'$ is contained in some polytope in $\mathcal{S}$. If $\mathcal{S}'$ is a simplicial complex, then we again say that $\mathcal{S}'$ is a *geometric simplicial subdivision* or *a geometric triangulation* or simply a *triangulation of* $\mathcal{S}$.

Note A.10. Quasigeometric simplicial subdivisions are geometric (because of the affine independence of the vertices of a geometric simplex) but the converse is not always true. Moreover, not every topological simplicial subdivision is quasigeometric. For counterexamples we refer to [**16**, p. 468] and [**117**, p. 814].

• **Working in the PL-category**. For the proof of the Upper Bound Theorem A.22 we shall make use of notions and propositions from "PL-topology". Working in the PL-category, i.e., in the category of simplicial complexes with piecewise linear maps (as morphisms), we can take full advantage of the fact that most of the theoretic arguments do not depend on specific geometric realizations and that many topological operations with PL balls or PL spheres (like starring, linking, gluing etc.) produce again other PL balls or PL spheres (i.e., something which is by no means true in general within the *usual* topological category). Standard references for the "PL-topology" are the books of Glaser [**50**], Hudson [**62**], and Rourke & Sanderson [**108**].

Definition A.11. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two abstract simplicial complexes. A *simplicial map* $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ is a function $\varphi : \mathrm{vert}(\mathcal{S}_1) \to \mathrm{vert}(\mathcal{S}_2)$, such that whenever $\{v_0, \ldots, v_k\}$ is an (abstract) simplex, then $\{\varphi(v_0), \ldots, \varphi(v_k)\}$ is an (abstract) simplex too. If such a φ is, in addition, a homeomorphism (i.e. $|\varphi|(|\mathcal{S}_1|) \approx |\mathcal{S}_2|$), then φ is called a *simplicial homeomorphism.* Passing to geometric realizations, a simplicial map $|\varphi| : |\mathcal{S}_1| \to |\mathcal{S}_2|$ carries the vertices of $\mathcal{S}_1$ to the vertices of $\mathcal{S}_2$ and the geometrically realized simplices of $\mathcal{S}_1$ *linearly* onto those of $\mathcal{S}_2$. (For $|\varphi|$ to be *linear* means that for each point $v \in |\mathcal{S}_1|$, which is uniquely expressible as a convex linear combination $v = \sum_{i=0}^{k} \lambda_i v_i$ by the *barycentric coordinates* $\lambda_0, \ldots, \lambda_k$ with respect to an ambient simplex $\mathrm{conv}(\{v_0, \ldots, v_k\})$, one has $|\varphi|(v) = \sum_{i=0}^{k} \lambda_i \, |\varphi|(v_i)$).

Definition A.12. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two abstract simplicial complexes. A map $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ is called *piecewise linear* (or PL *map*) if for some geometric realizations of $\mathcal{S}_1$ and $\mathcal{S}_2$, the corresponding map $|\varphi| : |\mathcal{S}_1| \to |\mathcal{S}_2|$ satisfies anyone of the following equivalent conditions:

(i) There exist geometric triangulations $\mathcal{S}_1'$, $\mathcal{S}_2'$ of the complexes $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, relative to which $|\varphi| : |\mathcal{S}_1'| \to |\mathcal{S}_2'|$ is simplicial.

(ii) There is a geometric triangulation $\mathcal{S}_1'$ *of* $\mathcal{S}_1$, relative to which $|\varphi| : |\mathcal{S}_1'| \to |\mathcal{S}_2|$ is linear.

(It can be shown that this definition depends neither on the particular choice of the geometric realizations of S_1 and S_2 nor on the particular choice of the geometric triangulations in (i), (ii). Note that a simplicial map is a PL map but the converse is not always true).

Definition A.13. (i) Two abstract simplicial complexes $\mathcal{S}_1$, $\mathcal{S}_2$ are called *PL homeomorphic* (denoted by $\mathcal{S}_1 \underset{\mathrm{PL}}{\approx} \mathcal{S}_2$) if there is a PL map $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ which is also a homeomorphism.

(ii) In particular, an abstract k-dimensional simplicial complex $\mathcal{S}$ is called a (*simplicial*) PL k-*ball* (resp., PL k-*sphere*) if $\mathcal{S}$ is PL homeomorphic to the k-simplex (resp., to the boundary of the $(k+1)$-simplex).

[Geometric triangulations of topological spheres (resp., balls) are not necessarily PL spheres (resp., PL balls). It is well-known, for instance, that all geometric triangulations of a k-sphere are PL k-spheres for $k \leq 3$, whereas there exist non-PL geometric triangulations of a k-sphere for $k \geq 5$.]

(iii) A regular cell complex $\mathcal{S}$ is called a PL k-*ball* (resp., a PL k-*sphere*) if the (simplicial) order complex of its face poset is a simplicial PL k-ball (resp., k-sphere).

• **Joins, stars and links**. Let $\mathcal{S}$ be an abstract simplicial complex, $v \in \mathrm{vert}(\mathcal{S})$, and F a face of $\mathcal{S}$. For $v \notin \mathrm{vert}(F)$, $v * F$ is defined to be the simplex with vertex set $\mathrm{vert}(F) \cup \{v\}$, i.e. the so-called *join* of v with F. In general, if $\mathcal{S}_1$, $\mathcal{S}_2$ are two abstract simplicial complexes on disjoint vertex sets $\mathcal{V}_1$, $\mathcal{V}_2$, the *join* of $\mathcal{S}_1$ and $\mathcal{S}_2$ is defined to be the simplicial complex

$$\mathcal{S}_1 * \mathcal{S}_2 := \{F \in \mathcal{V}_1 \cup \mathcal{V}_2 \ | \ F \cap \mathcal{V}_1 \in \mathcal{S}_1 \text{ and } F \cap \mathcal{V}_2 \in \mathcal{S}_2\}$$

For $w \notin \mathrm{vert}(\mathcal{S})$, $w * \mathcal{S}$ is nothing but the simplicial complex whose faces are $\{\varnothing\} \cup \{w * F \ | \ F \in \mathcal{S}\} \cup \mathcal{S}$, i.e. the *cone* (with apex w) over $\mathcal{S}$. If $w' \notin \mathcal{S}$ and w' is different from w, then the double joining

$$\{w, w'\} * \mathcal{S} := w * (w' * \mathcal{S})$$

is the *suspension* of $\mathcal{S}$ w.r.t. the additional vertices w, w'.

• For $v \in \mathrm{vert}(\mathcal{S})$, let

$$\begin{cases} \mathrm{star}_v(\mathcal{S}) := \{F \in \mathcal{S} \mid v \in \mathrm{vert}(F)\}, \\ \\ \mathrm{ast}_v(\mathcal{S}) := \{F \in \mathcal{S} \mid v \notin \mathrm{vert}(F)\}, \\ \\ \overline{\mathrm{star}_v}(\mathcal{S}) := \left\{ F' \in \mathcal{S} \;\middle|\; \begin{array}{c} F' \text{ faces of all } F \in \mathcal{S} \\ \text{for which } v \in \mathrm{vert}(F) \end{array} \right\}, \\ \\ \overline{\mathrm{ast}_v}(\mathcal{S}) := \left\{ F' \in \mathcal{S} \;\middle|\; \begin{array}{c} F' \text{ faces of all } F \in \mathcal{S} \\ \text{for which } v \notin \mathrm{vert}(F) \end{array} \right\}, \\ \\ \mathrm{link}_v(\mathcal{S}) := \{F \in \mathcal{S} \mid v \notin \mathrm{vert}(F),\ v * F \in \mathcal{S}\}, \end{cases}$$

denote the *star*, the *antistar*, the *closed star*, the *closed antistar,* and the *link* of v in $\mathcal{S}$, respectively. The last three form subcomplexes of $\mathcal{S}$ and are related as follows:

$$\overline{\mathrm{star}_v}(\mathcal{S}) \cap \overline{\mathrm{ast}_v}(\mathcal{S}) = \mathrm{link}_v(\mathcal{S}), \text{ and } \overline{\mathrm{star}_v}(\mathcal{S}) = v * \mathrm{link}_v(\mathcal{S}).$$

Proposition A.14. *For* simplicial PL *spheres and* PL *balls $\mathcal{S}$ we have the following implications*:

(i) $\mathcal{S}$ *is a* PL *k-ball* $\Longrightarrow$ $\partial\mathcal{S}$ *is a* PL $(k-1)$*-sphere*

(ii) $\left.\begin{array}{l} \mathcal{S} \textit{ is a } \mathrm{PL}\ k\textit{-ball} \\ (\textit{or a } \mathrm{PL}\ k\textit{-sphere}) \\ \textit{and } w \notin \mathcal{S} \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \textit{the cone } w * \mathcal{S} \textit{ is a } \mathrm{PL}\ (k+1)\textit{-ball} \\ \textit{with boundary } = \mathcal{S} \cup (w * \partial\mathcal{S}) \end{array}\right.$

(iii) $\mathcal{S}$ *is a* PL *k-sphere* $\Longrightarrow$ $\mathrm{link}_v(\mathcal{S})$ *is a* PL $(k-1)$*-sphere*

(iv) $\mathcal{S}$ *is a* PL *k-sphere* $\Longrightarrow$ $\overline{\mathrm{star}_v}(\mathcal{S})$ *is a* PL *k-ball*

(v) $\mathcal{S}$ *is a* PL *k-sphere* $\Longrightarrow$ $\overline{\mathrm{ast}_v}(\mathcal{S})$ *is a* PL *k-ball*

(vi) $\left.\begin{array}{l} \mathcal{S} \textit{ is a } \mathrm{PL}\ k\textit{-sphere,} \\ v \in \mathrm{vert}(\mathcal{S}) \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} v * \overline{\mathrm{ast}_v}(\mathcal{S}) \textit{ is a } \mathrm{PL}\ (k+1)\textit{-ball} \\ \textit{with boundary } = \mathcal{S} \end{array}\right.$

Proof. (i) This is obvious because the boundary of any simplicial subdivision of $\mathcal{S}$ is the restriction of this subdivision to the boundaries of the participating simplices.
(ii) Since, in general, the join of two abstract simplicial complexes is PL homeomorphic to the join of the images of these complexes under any PL homeomorphisms (see e.g. [**50**, II.5, p. 22, and II.17, p. 41]), we have $w * \mathcal{S} = w * (\text{a } k\text{-simplex})$ (or $= w * (\text{the boundary of a } (k+1)\text{-simplex})$).
(iii) See [**62**, proof of the Corollary 1.16, p. 24].
(iv) By (iii) and (ii) we get

$$\overline{\mathrm{star}_v}(\mathcal{S}) = v * \mathrm{link}_v(\mathcal{S}) \underset{\mathrm{PL}}{\approx} v * (\text{the boundary of a } k\text{-simplex}) \underset{\mathrm{PL}}{\approx} (\text{a } k\text{-simplex}).$$

(v) Since $\overline{\mathrm{ast}_v}(\mathcal{S}) = \mathcal{S} \smallsetminus \mathrm{star}_v(\mathcal{S})$ and $\mathrm{star}_v(\mathcal{S})$ has a PL k-ball (and consequently a stellar k-ball) as closure (by (iv)), $\overline{\operatorname{ast}_v}(\mathcal{S})$ is a stellar k-ball according to [**50**, II.15, p. 37]. From the equivalence of stellar- and PL-homeomorphism-property (cf. [**50**, II.17, p. 41]) we conclude that $\overline{\operatorname{ast}_v}(\mathcal{S})$ is indeed a PL k-ball.

(vi) That $v * \overline{\operatorname{ast}_v}(\mathcal{S})$ is a PL $(k+1)$-ball follows from (ii). Its boundary equals

$$\overline{\operatorname{ast}_v}(\mathcal{S}) \cup \left(v * \partial\left(\overline{\operatorname{ast}_v}(\mathcal{S})\right)\right) = \overline{\operatorname{ast}_v}(\mathcal{S}) \cup \overline{\operatorname{star}_v}(\mathcal{S}) = \mathcal{S}$$

and the proof is completed. □

Corollary A.15. *Let $\mathcal{S}$ be a (simplicial) PL k-ball, $k \geq 2$, $v \in \operatorname{vert}(\partial\mathcal{S})$ a vertex of its boundary, and $w \in \operatorname{vert}(\operatorname{link}_v(\mathcal{S}) \cap \partial\mathcal{S})$ a boundary vertex of its link. Then the suspension*

$$\mathcal{S}_{v,w} := \{v, w\} * \overline{\operatorname{ast}_w}(\operatorname{link}_v(\partial\mathcal{S}))$$

is a (simplicial) PL k-ball and $\overline{\operatorname{star}_v}(\partial\mathcal{S})$ is a subcomplex of the boundary $\partial\mathcal{S}_{v,w}$.

Proof. By Proposition A.14,

$\partial\mathcal{S}$ is a PL $(k-1)$-sphere	(using (i)),
$\operatorname{link}_v(\partial\mathcal{S})$ is a PL $(k-2)$-sphere	(using (iii)),
$\overline{\operatorname{ast}_w}(\operatorname{link}_v(\partial\mathcal{S}))$ is a PL $(k-2)$-ball	(using (v)),
$w * \overline{\operatorname{ast}_w}(\operatorname{link}_v(\partial\mathcal{S}))$ is a PL $(k-1)$-ball	(using (ii)).

Hence, the first claim for $\mathcal{S}_{v,w} = v * (w * \overline{\operatorname{ast}_w}(\operatorname{link}_v(\partial\mathcal{S})))$ is true by (ii). The verification of the second claim is a consequence of the fact that $w * \overline{\operatorname{ast}_w}(\operatorname{link}_v(\partial\mathcal{S}))$ is a PL $(k-1)$-ball whose boundary is the $\operatorname{link}_v(\partial\mathcal{S})$ (by using (vi)). □

Definition A.16. Let $\mathcal{S}$ be an abstract simplicial complex. A subcomplex $\mathcal{S}'$ of $\mathcal{S}$ is called *induced* if for any face $F \in \mathcal{S}$, $\operatorname{vert}(F) \subset \operatorname{vert}(\mathcal{S}')$ implies $F \in \mathcal{S}'$ (i.e., if the vertex set of a face lies in the subcomplex, then the whole face lies in the subcomplex).

Definition A.17. Let $\mathbf{s}$ be an abstract d-simplex (considered as simplicial complex consisting of itself together with all of its faces). An abstract simplicial complex $\mathcal{S}$ is called an *induced simplicial subdivision* or an *induced triangulation* of $\mathbf{s}$ if there is a PL homeomorphism $\varphi : \mathbf{s} \to \mathcal{S}$ such that for every face F of $\mathbf{s}$, the image $\varphi(F)$ is an induced subcomplex of $\mathcal{S}$.

Remark A.18. If $\mathbf{s}$ is an abstract d-simplex (viewed as simplicial complex), then every induced simplicial subdivision $\mathcal{S}$ of $\mathbf{s}$ is quasigeometric (see A.7).

Proposition A.19 (Gluing PL Balls). *Let $\mathcal{S}$, $\mathcal{S}'$ be two simplicial* PL *k-balls and $\mathcal{S}'' := \mathcal{S} \cap \mathcal{S}'$. Suppose that $\mathcal{S}'' \subset \partial\mathcal{S} \cap \partial\mathcal{S}'$.*

(i) *If $\mathcal{S}''$ is a (simplicial)* PL *$(k-1)$-ball, then the regular cell complex $\mathcal{S} \bigcup_{\mathcal{S}''} \mathcal{S}'$ which is obtained by gluing $\mathcal{S}$ with $\mathcal{S}'$ along $\mathcal{S}''$ is a* PL *k-ball.*

(ii) *If $\mathcal{S} \bigcup_{\mathcal{S}''} \mathcal{S}'$ is a subcomplex of both $\mathcal{S}$, $\mathcal{S}'$, and for at least one of $\mathcal{S}$, $\mathcal{S}'$, the subcomplex $\mathcal{S}''$ is* induced*, then the glued regular cell complex $\mathcal{S} \bigcup_{\mathcal{S}''} \mathcal{S}'$ is a* simplicial *complex.*

Proof. (i) It follows directly from [**62**, Corollary 1.28, p. 39].

(ii) Suppose that $\mathcal{S}''$ is an induced subcomplex of $\mathcal{S}$. If $\mathcal{S} \bigcup_{\mathcal{S}''} \mathcal{S}'$ were a non-simplicial complex, then, w.l.o.g. we may assume that there were a face F of $\mathcal{S}$ and a face F' of $\mathcal{S}'$, such that $F \cap F'$ is not a *single simplex* (considered itself as simplicial complex together with all its faces). But the vertex set of $F \cap F' \subset \mathcal{S}''$ is contained in F. Thus, there is one single face $\widetilde{F}$ of F with $\mathrm{vert}(\widetilde{F}) = \mathrm{vert}(F \cap F')$. On the other hand, $\widetilde{F} \in \mathcal{S}''$, because $\mathcal{S}''$ contains all the vertices of $\widetilde{F}$. But this would mean that $F \cap F' = \widetilde{F}$, which would lead to contradiction. $\square$

Conjecture A.20 (UBC for the facets of geometric simplex triangulations). *Let* $\mathbf{s}$ *be an* d*-dimensional simplex (considered as an abstract simplicial complex), and let* $\mathcal{V} \subset \mathbf{s}$ *be a finite set of* $\sharp(\mathcal{V}) = \mathfrak{b}$ *points in* $\mathbf{s}$*, so that* b_k *of them are contained in the relative interiors of the* $(k-1)$*-dimensional faces of* $\mathbf{s}$*, with*

$$b_1 = d+1, \quad \mathfrak{b}' := \mathfrak{b} - b_{d+1} \quad \textit{and} \quad \mathfrak{b} = \sum_{k=1}^{d+1} b_k > d+1.$$

Then a geometric triangulation $\mathcal{S}$ *of* $\mathbf{s}$ *with vertex set* $\mathcal{V}$ *has not more than*

$$\mathfrak{f}_d\left(\mathrm{CycP}_{d+1}(\mathfrak{b})\right) - \sum_{k=2}^{d} \; (d-(k-1)) \; b_k - 1$$

$$\overset{(\mathrm{A.3})}{=} \begin{pmatrix} \mathfrak{b} - \left\lceil \frac{d+1}{2} \right\rceil \\ \\ \left\lfloor \frac{d+1}{2} \right\rfloor \end{pmatrix} + \begin{pmatrix} \mathfrak{b} - 1 - \left\lceil \frac{d}{2} \right\rceil \\ \\ \left\lfloor \frac{d}{2} \right\rfloor \end{pmatrix} - \sum_{k=2}^{d-1} \; (d-k) \; b_k - (\mathfrak{b}' - (d+1)) - 1$$

facets $(= d$*-faces*$)$.

Remark A.21. (i) Conjecture A.20 is true in dimension $d=1$ because we have $\mathfrak{f}_1(\mathrm{CycP}_2(\mathfrak{b})) = \mathfrak{b}$, and it follows from Euler's polyhedron formula for $d=2$ (where $\mathfrak{f}_2(\mathrm{CycP}_3(\mathfrak{b})) = 2\mathfrak{b} - 4$). Thus, the first "interesting" case coming into question is that for $d=3$ (where $\mathfrak{f}_3(\mathrm{CycP}_4(\mathfrak{b})) = \frac{1}{2}\mathfrak{b}(\mathfrak{b}-3)$), corresponding to triangulations of the tetrahedron using b_2 additional vertices on its edges, b_3 extra vertices in its 2-faces, and b_4 more vertices in its relative interior.

(ii) A.20 is also true (and tight) in the case in which all "additional" vertices lie in the relative interior of $\mathbf{s}$, that is, if $b_2 = b_3 = \cdots = b_d = 0$, because the upper bound (A.5) can be written as

$$\mathfrak{f}_d\left(\mathrm{CycP}_{d+1}(\mathfrak{b})\right) - (b_2 + \cdots + b_d) - 1 = \mathfrak{f}_d\left(\mathrm{CycP}_{d+1}(\mathfrak{b})\right) - (\mathfrak{b}' - (d+1)) - 1 \; .$$

But whenever there *are* additional vertices on the boundary, the above "new" upper bound would obviously improve (A.5) by subtracting the "extra" summand $\sum_{k=2}^{d-1} (d-k) \; b_k$. This would correspond to a better estimation of a part of a "missing correction term" involving $\mathbf{h}$-vector components of $\partial\mathcal{S}$.

(iii) We believe that the "right" setting for a proof of Conjecture A.20 is provided by Stanley's theory [**117**] of "local $\mathbf{h}$-vectors": If $\mathcal{S}$ is a topological simplicial subdivision of an abstract d-simplex $\mathbf{s}$, and $\mathcal{V}$ the vertex set of $\mathbf{s}$, then the *local* $\mathbf{h}$*-vector* $\ell_{\mathcal{V}}(\mathcal{S}) := (\ell_0(\mathcal{S}), \ldots, \ell_{d+1}(\mathcal{S}))$ of $\mathcal{S}$ is defined by expanding

$$\ell_{\mathcal{V}}(\mathcal{S}; t) := \sum_{\mathcal{W} \subseteq \mathcal{V}} \; (-1)^{\#(\mathcal{V} \smallsetminus \mathcal{W})} \; \mathbf{h}(\mathcal{S}_{\mathcal{W}}; t) \in \mathbb{Z}[t]$$

w.r.t. t,

$$\ell_{\mathcal{V}}(\mathcal{S}; t) = \ell_0(\mathcal{S}) + \ell_1(\mathcal{S})\, t + \cdots + \ell_d(\mathcal{S})\, t^d + \ell_{d+1}(\mathcal{S})\, t^{d+1},$$

and has the following properties: **a)** *Reciprocity*: $\ell_i(\mathcal{S}) = \ell_{d-i}(\mathcal{S}), \forall i \in \{0, .., d+1\}$, **b)** *Positivity*: $\ell_i(\mathcal{S}) \geq 0, \ \forall i \in \{0, \ldots, d+1\}$, whenever is a quasigeometric subdivision of $\mathbf{s}$ (in the sense of A.7), and **c)** *Locality*:

$$\mathbf{h}(\mathcal{S}'; t) = \sum_{F \in \mathcal{S}} \ell_F(\mathcal{S}'_F; t) \ \mathbf{h}(\mathrm{link}_F(\mathcal{S}); t),$$

with $\mathcal{S}'$ a topological subdivision of a pure abstract simplicial d-complex $\mathcal{S}$, and

$$\mathrm{link}_F(\mathcal{S}) := \{F' \in \mathcal{S} \mid F \cup F' \in \mathcal{S}, \, F \cap F' = \varnothing\}.$$

However, even for $d = 3$, this is not "for free" in our case: we would need to establish the following upper bound for the difference between the second and the first coordinate of the local $\mathbf{h}$-vector $\ell_{\mathcal{V}}(\mathcal{S})$ of $\mathcal{S}$: $\ell_2(\mathcal{S}) - \ell_1(\mathcal{S}) \leq \binom{\mathfrak{h}_1(\mathcal{S})}{2}$, whereas from the results in [**117, 15, 16**] we only get

$$\ell_2(\mathcal{S}) \leq \mathfrak{h}_2(\mathcal{S}) \leq \binom{\mathfrak{h}_1(\mathcal{S}) + 1}{2} \quad \left(\text{i.e.,} \quad \ell_2(\mathcal{S}) - \mathfrak{h}_1(\mathcal{S}) \leq \binom{\mathfrak{h}_1(\mathcal{S})}{2}\right)$$

which is weaker than that we would like to have whenever there are boundary vertices. In any case, to proceed along these lines depends certainly on a deep understanding of how **a)-c)** could be applicable to our specific situation. Here we restrict ourselves to present another proof for dimension $d = 3$ by passing to triangulations in the category of PL subdivisions of $\mathbf{s}$ within which the gluing of balls is able to work without essential obstructions and leads to the desired result.

THEOREM A.22. *For $d = 3$, Conjecture* A.20 *holds in greater generality: any* induced *triangulation $\mathcal{S}$* (cf. A.17) *of a tetrahedron* $\mathbf{s}$ (*considered as abstract simplicial complex*) *using $b_2 + b_3 + b_4$ additional vertices within the edges/2-faces/interior of* $\mathbf{s}$ *possesses at most $\mathfrak{f}_3(\mathrm{CycP}_4(\mathfrak{b})) - 2b_2 - b_3 - 1$ facets.*

PROOF. If $b_2 = b_3 = 0$, then we have nothing to show. The proof will use induction on $b_2 + b_3$. For fixed b_2, b_3, with $b_2 + b_3 > 0$, assume that all induced triangulations of tetrahedra whose number of vertices lying in the relative interior of their edges and of their 2-faces is $< b_2 + b_3$ enjoy the desired property. We shall distinguish two cases.

▷ FIRST CASE. Suppose that $b_3 > 0$, i.e., that there are vertices of $\mathcal{S}$ in the relative interior of at least one 2-face, say F, of the tetrahedron $\mathbf{s}$. For every vertex v of F (in the simplicial PL 2-sphere $\partial\mathcal{S}$) the closed star $\overline{\mathrm{star}_v}(\partial\mathcal{S})$ of v in $\partial\mathcal{S}$ is a simplicial PL 2-ball (that is, a simplicial PL disc[9]), cf. Proposition A.14 (i) and (iv). $\overline{\mathrm{star}_v}(\partial\mathcal{S})$ is not necessarily an induced subcomplex of $\partial\mathcal{S}$ (cf. A.16). This first difficulty will be removed as follows.

▶ CLAIM. *Considering as "starting-point" an arbitrary vertex v_0 of the relative interior of F, we can determine another vertex v_μ (also lying in the relative interior of F), such that $\overline{\mathrm{star}_{v_\mu}}(\partial\mathcal{S})$ forms an* induced *subcomplex of $\partial\mathcal{S}$.*

▶ PROOF OF THE CLAIM. At first define

$$\mathfrak{U}(v_0) := \left\{ \begin{array}{c} \text{all simplicial PL discs } \mathbf{D} \text{ with vertex sets belonging} \\ \text{exclusively to } F \text{, such that } v_0 \text{ lies in their} \\ \text{relative interior, and all } w \in \mathrm{vert}(\partial\mathbf{D}) \text{ are adjacent to } v_0 \end{array} \right\},$$

[9] We may freely identify all simplicial discs which will occur in the arguments of our proof with the planar graphs consisting only of their vertices and edges (i.e. with their 1-skeletons).

and

$$\mathfrak{U}(v_0;\lambda) := \{\mathbf{D} \in \mathfrak{U}(v_0) \mid \sharp(\mathrm{vert}(\mathbf{D})) = \lambda\},$$

$$\lambda_0 := \max\{\lambda \mid 4 \leq \lambda \leq \sharp(\mathrm{vert}(\partial\mathcal{S}|_F)), \text{ such that } \mathfrak{U}(v_0;\lambda) \neq \varnothing\}.$$

Fix a simplicial PL disc $\mathbf{D}_0 \in \mathfrak{U}(v_0;\lambda_0)$. If $\mathbf{D}_0$ is not an induced subcomplex of $\partial\mathcal{S}$, choose a vertex $v_1 \neq v_0$ of the relative interior of $\mathbf{D}_0$. After that, define analogously

$$\mathfrak{U}(v_1) := \left\{ \begin{array}{c} \text{all simplicial PL discs } \mathbf{D} \text{ with vertex sets belonging} \\ \text{exclusively to } \mathbf{D}_0\text{, such that } v_1 \text{ lies in their} \\ \text{relative interior, and all } w \in \mathrm{vert}(\partial\mathbf{D}) \text{ are adjacent to } v_1 \end{array} \right\},$$

and

$$\mathfrak{U}(v_1;\lambda) := \{\mathbf{D} \in \mathfrak{U}(v_1) \mid \sharp(\mathrm{vert}(\mathbf{D})) = \lambda\},$$

$$\lambda_1 := \max\{\lambda \mid 4 \leq \lambda \leq \sharp(\mathrm{vert}(\mathbf{D}_0)), \text{ such that } \mathfrak{U}(v_1;\lambda) \neq \varnothing\}.$$

Fix again a simplicial PL disc $\mathbf{D}_1 \in \mathfrak{U}(v_1;\lambda_1)$. If $\mathbf{D}_1$ is not an induced subcomplex of $\partial\mathcal{S}$, choose a vertex $v_2 \neq v_1$ of the relative interior of $\mathbf{D}_1$, and repeat this construction for v_2 ... etc. We shall call the occuring numbers $\lambda_0, \lambda_1, \lambda_2, \ldots$ *ring sizes* of $v_0, v_1, v_2, \ldots$ *with respect to* v_0.

We have:

(i) $\overline{\mathrm{star}_{v_i}}(\partial\mathcal{S}) \subseteq \mathbf{D}_{i-1}$, for all i, $i = 1, 2, \ldots$ (This is immediate by definition).

(ii) $\lambda_0 > \lambda_1 > \lambda_2 > \cdots > \lambda_{i-1} > \lambda_i > \cdots$ (Since v_i is contained in the triangle formed by v_{i-1} together with two neighbours whose connecting edge does not belong to $\overline{\mathrm{star}_{v_{i-1}}}(\partial\mathcal{S})$, we have $\lambda_{i-1} > \lambda_i$.)

(iii) This procedure leads (after μ steps) to a vertex v_μ, such that all members of $\mathfrak{U}(v_\mu)$ are PL homeomorphic to the closed star $\overline{\mathrm{star}_{v_\mu}}(\partial\mathcal{S})$ (i.e., they have no vertices in their relative interiors besides v_μ itself), and are induced subcomplexes of $\partial\mathcal{S}$ (and hence of $\mathcal{S}$); moreover, $\lambda_\mu - 1$ equals the (graph-theoretic) *degree*[10] $\deg(v_\mu)$ of v_μ within $\overline{\mathrm{star}_{v_\mu}}(\partial\mathcal{S})$. (This is clear because all λ_i's are integers ≥ 4).

Figure 8 illustrates the above construction for an example in which $\mu = 2$ and v_0, v_1, v_2 have ring sizes (w.r.t. v_0) $\lambda_0 = 12$, $\lambda_1 = 6$ and $\lambda_2 = 4$, respectively.

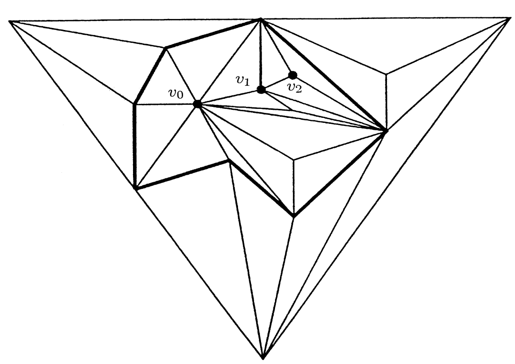

FIGURE 8.

[10]The *degree* of a vertex in a graph (without loops) is defined to be the number of the edges containing this vertex.

$\triangleright$ PROOF OF THEOREM FOR THE 1ST CASE (CONTINUED). From now on, let v denote an (always existing, as verified above) vertex of the relative interior of F, such that $\overline{\mathrm{star}_v}\,(\partial\mathcal{S})$ is an induced subcomplex of the simplicial PL 3-ball $\mathcal{S}$. Take a $w \in \mathrm{vert}(F)$ which is adjacent to v (and hence lying on the boundary of $\overline{\mathrm{star}_v}\,(\partial\mathcal{S})$). Then $\mathrm{link}_v\,(\partial\mathcal{S})$ is a (graph-theoretic) *circuit*[11] with *size* $= \deg(v)$, its closed antistar $\overline{\mathrm{ast}_w}\,(\mathrm{link}_v\,(\partial\mathcal{S}))$ (obtained by deleting w) is a path of $\deg(v) - 2$ edges, and its join $\mathcal{S}_{v,w} := \{v, w\} * \overline{\mathrm{ast}_w}\,(\mathrm{link}_v\,(\partial\mathcal{S}))$ with the edge $\{v, w\}$ is a simplicial PL 3−ball consisting of $\deg(v) - 2 > 0$ tetrahedra (by Corollary A.15). Gluing $\mathcal{S}$ and $\mathcal{S}_{v,w}$ along their intersection $\mathcal{S} \cap \mathcal{S}_{v,w} = \overline{\mathrm{star}_v}\,(\partial\mathcal{S})$ we obtain a *simplicial* PL 3-ball

$$\mathcal{S}' := \mathcal{S} \bigcup_{\overline{\mathrm{star}_v}(\partial\mathcal{S})} \mathcal{S}_{v,w}$$

according to Proposition A.19 (i), (ii). On the other hand,

$$\mathfrak{f}_3\,(\mathcal{S}') = \mathfrak{f}_3\,(\mathcal{S}) + 2 > \mathfrak{f}_3\,(\mathcal{S})\,,$$

and the number of the vertices of $\mathcal{S}'$ lying in the relative interior of the 2-faces of $\mathbf{s}$ equals $b_3 - 1$. So we are done by induction.

$\triangleright$ SECOND CASE. Assume that $b_3 = 0$ but $b_2 > 0$, and let v denote a subdivision vertex on an edge $F_1 \cap F_2$, for F_1, F_2 two 2−faces of the original tetrahedron $\mathbf{s}$ (which are adjacent to v). Consider a vertex w adjacent to v on the same edge $F_1 \cap F_2$ (w may be a vertex of the original tetrahedron). In this case $\overline{\mathrm{star}_v}\,(\partial\mathcal{S}\,|_{F_i})$ is obviously an induced subcomplex of $\mathcal{S}$ for $i = 1, 2$ (see Figure 9).

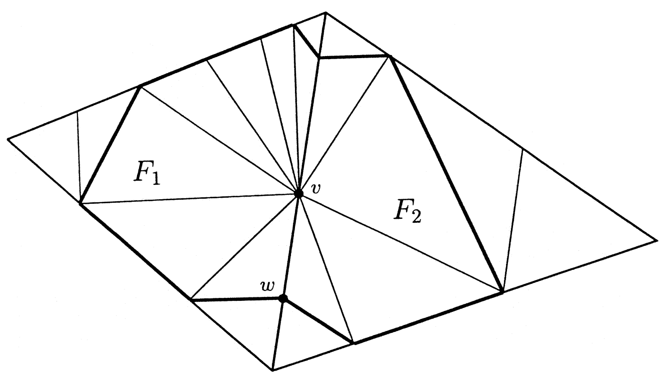

FIGURE 9.

As above one proves that

$$\mathcal{S}^{[i]}_{v,w} := \{v, w\} * \overline{\mathrm{ast}_w}\,(\mathrm{link}_v\,(\partial\mathcal{S}\,|_{F_i}))$$

is a simplicial PL 3-ball for $i = 1, 2$; gluing $\mathcal{S}$ and $\mathcal{S}^{[i]}_{v,w}$ along $\mathcal{S} \cap \mathcal{S}^{[i]}_{v,w} = \overline{\mathrm{star}_v}\,(\partial\mathcal{S}\,|_{F_i})$ we obtain a new *simplicial* PL 3-ball

$$\mathcal{S}' := \left(\mathcal{S} \bigcup_{\overline{\mathrm{star}_v}(\partial\mathcal{S}|_{F_1})} \mathcal{S}^{[1]}_{v,w} \right) \bigcup \left(\mathcal{S} \bigcup_{\overline{\mathrm{star}_v}(\partial\mathcal{S}|_{F_2})} \mathcal{S}^{[2]}_{v,w} \right)$$

[11]In our special case this is synonymous to a simple closed path with the number ($=$ *size*) of its edges being equal to the number of its vertices.

with $\mathfrak{f}_3(\mathcal{S}') \geq \mathfrak{f}_3(\mathcal{S}) + 2 > \mathfrak{f}_3(\mathcal{S})$, and the number of the vertices of $\mathcal{S}'$ lying in the relative interior of the edges of $\mathbf{s}$ equals $b_2 - 1$. Thus, the proof is finished by induction. □

Appendix B. Coherent Triangulations and Secondary Polytopes

Coherent triangulations of simplices are exactly what one needs to express the "projectivity condition" for crepant birational morphisms desingularizing (partially or fully) Gorenstein AQS in the language of geometric combinatorics.

DEFINITION B.1. A triangulation $\mathcal{T}$ of a polytope P is called *coherent* (or *regular*) if there exists *a strictly upper convex $\mathcal{T}$-support function* $\psi : |\mathcal{T}| \to \mathbb{R}$, i.e., a piecewise-linear real function defined on the underlying space $|\mathcal{T}|$, for which

$$\psi(t\ \mathbf{x} + (1-t)\ \ \mathbf{y}) \geq t\ \psi(\mathbf{x}) + (1-t)\ \ \psi(\mathbf{y}),\ \text{ for all }\ \mathbf{x}, \mathbf{y} \in |\mathcal{T}|,\ t \in [0,1],$$

so that its domains of linearity are the simplices of $\mathcal{T}$ having maximal dimension.

NOTE B.2. (i) For a polytope P one can always construct coherent triangulations $\mathcal{T}$ with given vertex set (e.g., lexicographic, reverse lexicographic etc.; cf. [**81**, §2-§4] and [**121**, Ch. 8]).
(ii) Already in dimension 2 there are lots of examples of non-coherent triangulations. Figure 10 shows two triangulations of a triangle with the same vertex set. Triangulation (a) is coherent (in fact, affinely isomorphic to the triangulation given in Figure 3), whereas (b) is non-coherent. (Its "mirror image" is a different triangulation with the same vertex set; though, the number of the simplices belonging to the star of each vertex remains invariant. This is enough to prove non-coherence; compare [**7**, Example 2.4, p. 161].)

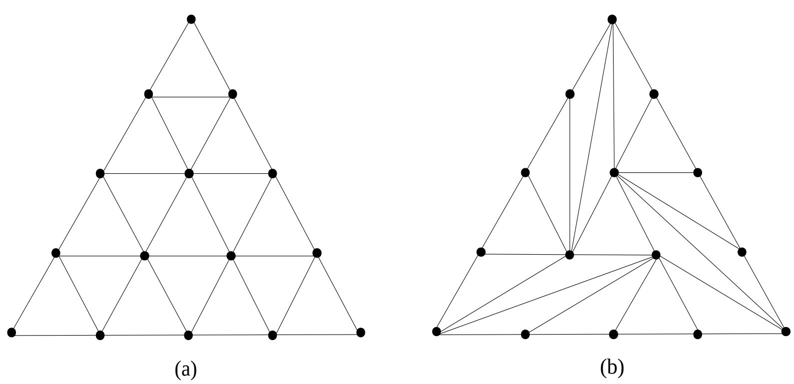

FIGURE 10.

Next Lemma is used essentially in the proof of Theorem 8.1.

LEMMA B.3 (Patching Lemma, [**12**, 2.2.2, pp. 143-145], [**74**, 5.12, p. 115]). *Let P be a polytope, $\mathcal{T} = \{\mathbf{s}_i \,|i \in I\}$ (with I a finite set) a coherent triangulation of P, and $\mathcal{T}_i = \{\mathbf{s}_{i,j} \,|j \in J_i\}$ (J_i finite, for all $i \in I$) a coherent triangulation of $\mathbf{s}_i$, for all $i \in I$. If $\psi_i : |\mathcal{T}_i| \to \mathbb{R}$ denote strictly upper convex $\mathcal{T}_i$-support functions, such that*

$$\psi_i \left|_{\mathbf{s}_i \cap \mathbf{s}_{i'}} = \psi_{i'} \right|_{\mathbf{s}_i \cap \mathbf{s}_{i'}}$$

for all $(i,i') \in I \times I$, *then* $\widetilde{\mathcal{T}} := \{\mathbf{s}_{i,j} \mid j \in J_i \text{ and } i \in I\}$ *forms a* coherent triangulation *of the initial polytope* P (*because the above* ψ_i*'s can be canonically "patched together" to construct a strictly upper convex* $\widetilde{\mathcal{T}}$*-support function* ψ).

• **Secondary polytope**. For any finite set of points $\mathcal{V}$ in $\mathbb{R}^d$, *all triangulations* $\mathcal{T}$ of a polytope $P = \text{conv}(\mathcal{V})$ with $\text{vert}(\mathcal{T}) \subseteq \mathcal{V}$ are parametrized by the vertices of a "gigantic" polytope $\text{Un}(\mathcal{V})$, the so-called *universal polytope* of P (see [**7**, §3] and [**32**, §1-§4]). $\text{Un}(\mathcal{V})$ projects onto a polytope $\text{Sec}(\mathcal{V})$ whose vertices parametrize only the coherent $\mathcal{T}$'s.

Definition B.4. If $\mathcal{V} = \{a_1, \ldots, a_k\}$ and P is d-dimensional, the *secondary polytope* $\text{Sec}(\mathcal{V})$ of P is the $(k-d-1)$-dimensional polytope

$$\text{Sec}(\mathcal{V}) := \text{conv}\left(\{\mathbf{v}_{\mathcal{T}} \mid \mathcal{T} \text{ a triangulation of } P \text{ with } \text{vert}(\mathcal{T}) \subseteq \mathcal{V}\}\right) \subset \mathbb{R}^k,$$

defined as the convex hull of the points

$$\mathbf{v}_{\mathcal{T}} := \sum_{i=1}^{k} \left[\sum_{j=1}^{\nu} \{\text{Vol}(\mathbf{s}_j) : a_i \in \mathbf{s}_j\} \right] \cdot e_i,$$

where $\{\mathbf{s}_1, \ldots, \mathbf{s}_\nu\}$ denotes an enumeration of the d-simplices of $\mathcal{T}$ and $\{e_1, \ldots, e_k\}$ the standard unit vector basis of $\mathbb{R}^k$. (For the main concepts of the theory of secondary polytopes we refer to [**7**], [**33**], [**48**, Ch. 7], [**97**], and [**126**, Lecture 9].)

Note B.5. $\text{Sec}(\mathcal{V})$ is in most of the cases also considerably "big". In practice, working with examples for which k is relatively small, the vertices of $\text{Sec}(\mathcal{V})$ can be easily determined by using De Loera's `Puntos`[12] [**31**] or Rambau's `TOPCOM`[13] [**100**].

• **Circuits and bistellar flips**. "Flipping" along circuits in a given finite set of points $\mathcal{V} \subset \mathbb{R}^d$ is an elementary geometric procedure which enable us to move from one vertex of $\text{Sec}(\mathcal{V})$ to another.

Definition B.6 (Circuits). A non-empty subset $\mathcal{C} \subset \mathcal{V}$ is called a *circuit* if any $\mathcal{C}' \subsetneqq \mathcal{C}$ is affinely independent but $\mathcal{C}$ itself is affinely dependent. Up to a real scalar multiple, there is a unique real affine dependence relation among the elements of a circuit $\mathcal{C}$. In fact, one can decompose $\mathcal{C}$ into those elements $\mathcal{C}_+$ occuring with positive coefficients in this affine linear relation, and $\mathcal{C}_- := \mathcal{C} \smallsetminus \mathcal{C}_+$, so that

$$\text{int}(\text{conv}(\mathcal{C}_+)) \cap \text{int}(\text{conv}(\mathcal{C}_-)) \neq \varnothing .$$

Lemma B.7 ([**30**, 1.2.1]). *Every circuit* $\mathcal{C}$ *has only two triangulations, namely*

$$\mathcal{Y}_+(\mathcal{C}) := \{\text{conv}(\mathcal{C} \smallsetminus \{v\}) \,|\, v \in \mathcal{C}_+\} \quad \text{and} \quad \mathcal{Y}_-(\mathcal{C}) := \{\text{conv}(\mathcal{C} \smallsetminus \{v\}) \,|\, v \in \mathcal{C}_-\}.$$

Definition B.8. Let $\mathcal{C}$ be a circuit and $\mathcal{T}$ a triangulation of $P = \text{conv}(\mathcal{V})$ with $\text{vert}(\mathcal{T}) \subseteq \mathcal{V}$. Suppose that there is a sign $\odot \in \{+,-\}$ such that the following conditions are satisfied:
(i) The triangulation $\mathcal{Y}_\odot(\mathcal{C})$ is a simplicial subcomplex of $\mathcal{T}$.

[12] As it is pointed out in [**30**, Thm. 1.5.12], the computation of all coherent triangulations of a d-polytope $P = \text{conv}(\mathcal{V})$ with `Puntos` requires $O(dRk^{d+\lfloor d/2 \rfloor} L(k-d-1, (k-d)k^{\lfloor d/2 \rfloor}), \eta)$ arithmetic operations. The symbol R denotes the number of coherent triangulations of the point configuration $\mathcal{V}$ (i.e., $R = \mathfrak{f}_0(\text{Sec}(\mathcal{V}))$), η is the size of the matrix encoding $\mathcal{V}$, and $L(\beta_1, \beta_2, \beta_3)$ denotes the number of arithmetic operations used to solve a linear system of inequalities of β_1 variables, β_2 constraints and input size β_3.

[13] `TOPCOM` is much faster (as its written in C^{++}) and has more functionalities than `Puntos`.

(ii) The links of all maximal-dimensional simplices of $\mathcal{Y}_{\odot}(\mathcal{C})$ within $\mathcal{T}$ coincide, i.e., they form the same simplicial subcomplex, say $\mathcal{T}^{[\mathcal{C}]}$, of $\mathcal{T}$.

Then we say that $\mathcal{T}$ is said to be *supported on* $\mathcal{C}$.

Definition B.9 (Bistellar Flips). Let $\mathcal{C} \subset \mathcal{V}$ be a circuit and $\mathcal{T}$ a triangulation of P which is supported on $\mathcal{C}$. The triangulation

$$\mathrm{FL}_{\mathcal{C}}(\mathcal{T}) := \left\{ \text{all faces of } \mathcal{T} \smallsetminus (\mathcal{Y}_{\odot}(\mathcal{C}) * \mathcal{T}^{[\mathcal{C}]}) \right\} \cup \left\{ \text{all faces of } \mathcal{Y}_{\boxtimes}(\mathcal{C}) * \mathcal{T}^{[\mathcal{C}]} \right\}$$

of P, where $\{\boxtimes\} = \{+-\} \smallsetminus \{\odot\}$, is called the *bistellar flip*[14] *of* $\mathcal{T}$ *along* $\mathcal{C}$.

Theorem B.10 (Flipping Property, [**48**, Thm. 2.10, p. 233]). *Let* $\mathcal{V} \subset \mathbb{R}^d$ *be a finite set of points and* $P = \mathrm{conv}(\mathcal{V})$. *For two (different) coherent triangulations* $\mathcal{T}, \mathcal{T}'$ *with vertex set* $\mathcal{V}$, *the vertices* $\mathbf{v}_{\mathcal{T}}, \mathbf{v}_{\mathcal{T}'}$ *of the secondary polytope* $\mathrm{Sec}(\mathcal{V})$ *corresponding to* $\mathcal{T}$ *and* $\mathcal{T}'$ *are joined by an edge of* $\mathrm{Sec}(\mathcal{V})$ *if and only if there is a circuit* $\mathcal{C}$ *of* $\mathcal{V}$ *on which both* $\mathcal{T}$ *and* $\mathcal{T}'$ *are supported, and* $\mathcal{T}' = \mathrm{FL}_{\mathcal{C}}(\mathcal{T})$, *i.e.,* $\mathcal{T}'$ *is the bistellar flip of* $\mathcal{T}$ *along* $\mathcal{C}$.

Appendix C. Ehrhart Polynomials and Lattice Triangulations

A polytope $P \subset \mathbb{R}^d$ is a *lattice polytope* w.r.t. a lattice $N \subset \mathbb{R}^d$ if $\mathrm{aff}(P) \cap N \neq \varnothing$ and $\mathrm{vert}(P) \subset N$. We denote by N_P the sublattice of N generated (as subgroup) by $\mathrm{aff}(P) \cap N$, and by[15] $\mathrm{Vol}_{N_P}(P) := \mathrm{Vol}(P)/\det(N_P)$ its *relative volume*.

Two lattice polytopes $P_i \subset \mathbb{R}^d$ w.r.t. N, $i = 1, 2$, are called *lattice equivalent* to each other if there exists an affine map $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ such that its restriction $\Phi|_{\mathrm{aff}(P_1)} : \mathrm{aff}(P_1) \to \mathrm{aff}(P_2)$ is a bijection mapping P_1 onto the (necessarily equidimensional) P_2, every j-dimensional face of P_1 onto a j-dimensional face of P_2, for all $j = 0, 1, \ldots, \dim(P_1) = \dim(P_2)$, and N_{P_1} onto N_{P_2}. (If $\mathrm{rank}(N) = \dim(P_1) = \dim(P_2)$, then these Φ's are exactly the *affine integral transformations*, composed of unimodular transformations and lattice translations.)

• **Ehrhart polynomials**. If $P \subset \mathbb{R}^{d'}$ is a lattice d-dimensional polytope w.r.t. $N \subset \mathbb{R}^{d'}$, $d \leq d'$, and $\nu P = \left\{ \nu x \in \mathbb{R}^{d'} \mid x \in P \right\}$ the ν times dilated polytope P (for $\nu \in \mathbb{N}$), then we denote the enumerating function of its lattice points by

$$\mathbf{Ehr}_N(P, \nu) := \sharp\,(\nu P \cap N).$$

Theorem C.1 ([**4**, §3.3], [**58**, 28.3], [**120**, 4.6.28]). $\mathbf{Ehr}_N(P,\nu)$ *can be expressed as a polynomial*

$$\mathbf{Ehr}_N(P,\nu) = \mathbf{a}_0(P) + \mathbf{a}_1(P)\ \nu + \cdots + \mathbf{a}_{d-1}(P)\ \nu^{d-1} + \mathbf{a}_d(P)\ \nu^d \ \in\ \mathbb{Q}[\nu] \quad \text{(C.1)}$$

of degree d (*the so-called* Ehrhart polynomial *of* P).

Note C.2. (i) To find the number of lattice points of the interior of νP one uses the reciprocity law:

$$\sharp\,(\mathrm{int}(\nu P) \cap N) = (-1)^d\, \mathbf{Ehr}_N(P, -\nu). \quad \text{(C.2)}$$

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[14] Other alternative names used in the literature are: *elementary transformation, modification, geometric bistellar operation, surgery with respect to* Z etc.

[15] In the main text, working with the junior simplex $\mathfrak{s}_G$ (or similar $(r-1)$-dimensional lattice simplices) we often write $\mathrm{Vol}(\mathfrak{s}_G)$ instead of $\mathrm{Vol}_{(N_G)_{\mathfrak{s}_G}}(\mathfrak{s}_G)$ (by abuse of notation), but it is always clear from the context what we mean in each case.

(ii) As it is well-known (cf. [**4**, §3.4-3.5]), the first, the last but one and the last coefficient of $\mathbf{Ehr}_N(P,\nu)$ are equal to $\mathbf{a}_0(P)=1$,

$$\mathbf{a}_{d-1}(P)=\frac{1}{2}\sum_{\text{facets } F\prec P}\mathrm{Vol}_{N_F}(F),\quad \text{and}\quad \mathbf{a}_{d-1}(P)=\mathrm{Vol}_{N_P}(P), \tag{C.3}$$

respectively. (For the remaining coefficients for simplices, see below Theorem D.8).
(iii) $\mathbf{a}_i(P_1)=\mathbf{a}_i(P_2)$, for all $i\in\{0,1,\ldots,\dim(P_1)=\dim(P_2)\}$, whenever P_1 and P_2 are lattice equivalent.

• **Ehrhart series**. W.l.o.g.[16] we shall henceforth assume that $d=d'$. Let

$$\sigma_P:=\left\{(\lambda,\lambda p)\in\mathbb{R}\oplus\mathbb{R}^d\ \mid p\in P,\ \lambda\in\mathbb{R}_{\geq 0}\right\}$$

be the $(d+1)$-dimensional rational s.c.p. cone supporting P within $\mathbb{R}^{d+1}$, and let $\mathfrak{R}_P$ denote the subring of $\mathbb{C}[\mathfrak{x}_0,\mathfrak{x}_1^{\pm1},\ldots,\mathfrak{x}_d^{\pm1}]$ spanned over $\mathbb{C}$ by all Laurent monomials of the form $\mathfrak{x}_0^{\nu}\,\mathfrak{x}_1^{\mu_1}\cdots\mathfrak{x}_d^{\mu_d}$, where $\nu\in\mathbb{Z}_{\geq 0}$ and $(\mu_1,\ldots,\mu_d)\in\nu P\cap N$. $\mathfrak{R}_P=\mathbb{C}[\sigma_P\cap\mathbb{Z}^{d+1}]$ is graded by setting $\deg(\mathfrak{x}_0^{\nu}\,\mathfrak{x}_1^{\mu_1}\cdots\mathfrak{x}_d^{\mu_d}):=\nu$. Hence, the so-called *Ehrhart power series*

$$\mathbf{Ehr}_N(P;t):=1+\sum_{\nu=1}^{\infty}\mathbf{Ehr}_N(P,\nu)\ t^{\nu}\in\mathbb{Q}\,[\![t]\!]$$

of P is the Hilbert series of the graded normal semigroup ring $\mathfrak{R}_P=\bigoplus_{\nu\geq 0}(\mathfrak{R}_P)_{\nu}$, and can be therefore written in the form

$$\mathbf{Ehr}_N(P;t)=\frac{\mathfrak{h}_0^*(P)+\ \mathfrak{h}_1^*(P)\ t+\cdots+\ \mathfrak{h}_{d-1}^*(P)\ t^{d-1}+\ \mathfrak{h}_d^*(P)\ t^d}{(1-t)^{d+1}},$$

having the *Eulerian* $\mathbf{Ehr}_N(P,\cdot)$*-polynomial* as its numerator (see [**120**, §4.3]).

Definition C.3. $\mathbf{h}^*(P):=(\mathfrak{h}_0^*(P),\mathfrak{h}_1^*(P),\ldots,\mathfrak{h}_d^*(P))\in\mathbb{Z}^{d+1}$ is called the $\mathbf{h}^*$-*vector*[17] of P.

Theorem C.4 (cf. [**58**, §28], [**116, 118**]). (i) $\mathfrak{h}_j^*(P)\geq 0,\forall j\in\{0,1,\ldots,d\}$.
(ii) $\mathfrak{h}_0^*(P)=1,\ \mathfrak{h}_1^*(P)=\mathbf{Ehr}_N(P,1)-(d+1)$, and $\mathfrak{h}_d^*(P)=\#\,(\mathrm{int}(P)\cap N)$.
(iii) *The sum of all coordinates of the* $\mathbf{h}^*$*-vector of* P *equals*

$$\sum_{j=0}^{d}\mathfrak{h}_j^*(P)=d!\,\mathrm{Vol}(P). \tag{C.4}$$

Proposition C.5. *Each coordinate of the* $\mathbf{h}^*$*-vector of* P *can be expressed as integer linear combination of the coefficients of its Ehrhart polynomial* (C.1) *as follows*:

$$\mathfrak{h}_i^*(P)=\sum_{j=0}^{d}\left(\sum_{\kappa=0}^{i}(-1)^{\kappa}\tbinom{d+1}{\kappa}(i-\kappa)^j\right)\mathbf{a}_j(P),\quad \forall i\in\{0,1,\ldots,d\}. \tag{C.5}$$

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[16] If $d<d'$, then we work with N_P instead of N.

[17] We adopt here Stanley's notation from [**118**], but it appears in the literature also under different names (e.g., as "ψ-vector" in [**3**, §4], as "δ-vector" in [**58**, Ch. IX] etc).

PROOF. We expand

$$\mathbf{Ehr}_N(P,\nu) = \dim_{\mathbb{C}}(\mathfrak{R}_P)_\nu = \frac{1}{\nu!}\left[\frac{d^\nu}{dx^\nu}\left(\frac{\sum_{\iota=0}^{d}\mathfrak{h}_\iota^*(P)\ x^\iota}{(1-x)^{d+1}}\right)\Bigg|_{x=0}\right]$$

$$= \sum_{\iota=0}^{d}\ \mathfrak{h}_\iota^*(P)\ \binom{d+1+(\nu-\iota)-1}{\nu-\iota} = \sum_{\iota=0}^{d}\ \mathfrak{h}_\iota^*(P)\ \binom{\nu-\iota+d}{d}$$

$$= \frac{1}{d!}\left[\sum_{j=0}^{d}\mathfrak{h}_j^*(P)\right]\ \nu^d + \cdots\cdots + 1,$$

as polynomial in the variable ν and compare coefficients. □

• **Lattice triangulations**. These are one of our main tools in §4 and thereafter.

DEFINITION C.6 (Lattice subdivisions and triangulations). A *lattice subdivision* $\mathcal{S}$ of a lattice polytope[18] P is a polytopal subdivision of P, such that the set $\mathrm{vert}(\mathcal{S})$ of the vertices of $\mathcal{S}$ belongs to the reference lattice, and $\mathrm{vert}(P) \subseteq \mathrm{vert}(\mathcal{S})$. A *lattice triangulation* of a lattice polytope P is a lattice subdivision of P which, in addition, is a triangulation (in the sense of A.8).

DEFINITION C.7 (Basic triangulations). (i) A lattice polytope is called *elementary* if the lattice points belonging to it are exactly its vertices. A lattice simplex $\mathbf{s}$ is said to be *basic* (or *unimodular*) if its vertices constitute a part of a $\mathbb{Z}$-basis of the reference lattice (that is, if its relative volume equals $1/\dim(\mathbf{s})!$).

(ii) A lattice triangulation $\mathcal{T}$ of a lattice polytope P is defined to be *basic* if it consists only of elementary basic simplices.

NOTE C.8. (i) Obviously, all basic simplices are elementary. On the other hand, all elementary triangles are basic, but in dimensions $d \geq 3$ there exist lots of elementary simplices which are non-basic. For instance, the so-called *Reeve's simplices* [**101**]:

$$\mathrm{RS}(k) := \ \mathrm{conv}(\{\mathbf{0}, e_1, e_2, \ldots, e_{d-1}, (1,1,\ldots,1,k)^\intercal\}) \subset \mathbb{R}^d$$

are elementary but non-basic (w.r.t. $\mathbb{Z}^d$) for $d \geq 3$ and $k \geq 2$ because

$$d!\mathrm{Vol}(\mathrm{RS}(k)) = |\det(e_1, e_2, \ldots, e_{d-1}, (1,1,\ldots,1,k)^\intercal)| = k \neq 1.$$

(ii) "Basicness" is a property preserved by lattice equivalence.

(iii) *Open problem*: Hibi and Ohsugi [**59**] discovered a 9-dimensional 0/1-polytope (with 15 vertices) having basic triangulations, but none of whose coherent triangulations is basic. It is not known if such high-dimensional "pathological counterexamples" can be also found in the class of lattice *simplices*.

[18] Lattice subdivisions, lattice triangulations and basic triangulations of the undelying point set of *simplicial complexes* whose vertices are lattice points are defined similarly. (We use this generalization only for $\mathbf{T}_{\mathbb{H}_d}$ in Example 5.4.)

THEOREM C.9 (Betke & McMullen [**5**, Thm. 2], Stanley [**116**, Corollary 2.5]). *If $\mathcal{T}$ is a lattice triangulation of a d-dimensional lattice polytope P, then*[19]

$$\mathfrak{h}_j^*(P) \geq \mathfrak{h}_j(\mathcal{T}), \ \forall j \in \{0,1,\ldots,d\}, \tag{C.6}$$

and $\mathcal{T}$ is basic if and only if (C.6) *hold* (*simultaneously*) *as equations.*

Appendix D. Counting the Lattice Points of the Junior Simplex

In this Appendix we explain how one can count, for a given Gorenstein AQS $(\mathbb{C}^r/G, [\mathbf{0}])$ of type (3.6), the number of lattice points of the junior simplex $\mathfrak{s}_G$ (w.r.t. N_G) by making use of Mordell-Pommersheim and Diaz-Robins formulae.

• **Mordell-Pommersheim formula**. By (C.3) the first coefficient of the Ehrhart polynomial (C.1), whose expression (as rational linear combination of relative volumes of faces of P) turns out to be relatively "difficult", is $\mathbf{a}_1(\mathbf{s})$, arising already in the case in which $d = 3$ and $P = \mathbf{s}$ is a lattice tetrahedron. In fact, by Theorem D.2, the corresponding formula (D.2) for $\mathbf{a}_1(\mathbf{s})$ involves *Dedekind measures* of the "dihedral angles" of $\mathbf{s}$.

DEFINITION D.1. (i) If $x \in \mathbb{Q}$, we define

$$(\!(x)\!) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & \text{if} \quad x \notin \mathbb{Z}, \\ 0, & \text{if} \quad x \in \mathbb{Z}. \end{cases}$$

(ii) Let p, q be two integers with $q > 0$ and $\gcd(p,q) = 1$. The *Dedekind sum* $\mathrm{DS}(p,q)$ of p and q is defined to be

$$\mathrm{DS}(p,q) := \sum_{j=1}^{q-1} \left(\!\!\left(\frac{j}{q}\right)\!\!\right) \left(\!\!\left(\frac{pj}{q}\right)\!\!\right).$$

THEOREM D.2 (Mordell-Pommersheim formula, [**90, 98**]). *Let $N \subset \mathbb{R}^3$ be a lattice of rank* 3, *$n_1, n_2, n_3, n_4 \in N$ four affinely independent points, and $\mathbf{s}$ the tetrahedron $\mathbf{s} = \mathrm{conv}(\{n_1, n_2, n_3, n_4\})$. Denote by*

$$E_{i,j} := \mathrm{conv}(\{n_i, n_j\}) \ , \quad \forall\ i, j \ , \quad 1 \leq i < j \leq 4,$$

its six edges, and by

$$F_i := \mathrm{conv}(\{n_1, n_2, n_3, n_4\} \smallsetminus \{n_i\}) \ , \quad \forall i, \ 1 \leq i \leq 4 \ ,$$

its four facets. For any fixed pair i, j, $1 \leq i < j \leq 4$, let $F_{i'}$, $F_{j'}$ be the two facets of $\mathbf{s}$ containing its edge $E_{i,j}$ (with $\{i', j'\} = \{1,2,3,4\} \smallsetminus \{i, j\}$, $i' < j'$), $\widetilde{n}_{i'}, \widetilde{n}_{j'}$ the images of $n_{i'}, n_{j'}$ in $N(E_{i,j}) := N/N_{E_{i,j}}$ under the canonical projection map $N \to N(E_{i,j})$, and $\widetilde{\widetilde{n}}_{i'}$, $\widetilde{\widetilde{n}}_{j'}$ the primitive vectors of $\mathrm{conv}(\{\mathbf{0}, n_i\})$ *and* $\mathrm{conv}(\{\mathbf{0}, n_j\})$, *respectively, lying within the lattice $N(E_{i,j})$. Set*

$$\mathbf{q}_{i,j} := \frac{\det(\widetilde{\widetilde{n}}_{i'}, \widetilde{\widetilde{n}}_{j'})}{\det(N(E_{i,j}))}.$$

Moreover, choosing a $\mathbb{Z}$-basis, say $\{\widetilde{\widetilde{n}}_{i'}, \widehat{n}\}$, of $N(E_{i,j})$, and expressing $\widetilde{\widetilde{n}}_{j'}$ as an integer linear combination of its elements in the form $\widetilde{\widetilde{n}}_{j'} = \lambda \cdot \widetilde{\widetilde{n}}_{i'} + \mathbf{q}_{i,j} \cdot \widehat{n}$, define

[19]Note that $\mathfrak{h}_{d+1}(\mathcal{T}) = 0$ because P is homeomorphic to a d-ball.

$\mathbf{p}_{i,j} := [\lambda]_{\mathbf{q}_{i,j}}$. *Then the number of lattice points of* $\mathbf{s}$ *is given by the formula*

$$\sharp\left(\mathbf{s} \cap N\right) = \mathbf{Ehr}_N\left(\mathbf{s}, 1\right) = 1 + \mathbf{a}_1\left(\mathbf{s}\right) + \frac{1}{2}\sum_{i=1}^{4} \mathrm{Vol}_{N_{F_i}}\left(F_i\right) + \mathrm{Vol}_{N_{\mathbf{s}}}\left(\mathbf{s}\right), \tag{D.1}$$

and

$$\mathbf{a}_1\left(\mathbf{s}\right) = \sum_{1 \le i < j \le 4}\left(\frac{1}{36\mathbf{q}_{i,j}}\,\mathfrak{Y}_{i,i'} + \widetilde{\mathrm{DS}}(\mathbf{p}_{i,j}, \mathbf{q}_{i,j})\right)\mathrm{Vol}_{N_{E_{i,j}}}\left(E_{i,j}\right), \tag{D.2}$$

where

$$\mathfrak{Y}_{i,i'} := \frac{\mathrm{Vol}_{N_{F_i}}\left(F_j\right)}{\mathrm{Vol}_{N_{F_{i'}}}\left(F_{j'}\right)} + \frac{\mathrm{Vol}_{N_{F_{i'}}}\left(F_{j'}\right)}{\mathrm{Vol}_{N_{F_i}}\left(F_j\right)},$$

and

$$\widetilde{\mathrm{DS}}(\mathbf{p}_{i,j}, \mathbf{q}_{i,j}) := \frac{1}{4} - \mathrm{DS}(\mathbf{p}_{i,j}, \mathbf{q}_{i,j})$$

denotes the so-called "Dedekind measure" *of* $\mathbf{p}_{i,j}$ *and* $\mathbf{q}_{i,j}$.

Corollary D.3. *Let* $(\mathbb{C}^4/G, [\mathbf{0}])$ *be a 4-dimensional Gorenstein CQS of type* $\frac{1}{l}\left(\alpha_1, \alpha_2, \alpha_3, \alpha_4\right)$, *and* $\mathfrak{s}_G = \mathrm{conv}(\{e_1, e_2, e_3, e_4\})$ *the corresponding junior tetrahedron. For any* i, $1 \le i \le 4$, *consider an (arbitrary) representation of* $\gcd(\alpha_i, l)$ *as integer linear combination*

$$\gcd\left(\alpha_i, l\right) = \gamma_i \cdot \alpha_i + \widehat{\gamma}_i \cdot l.$$

Moreover, for any fixed pair i,j, $1 \le i < j \le 4$, *let* $\{i', j'\} = \{1,2,3,4\} \smallsetminus \{i,j\}$ *denote the complement (with* $i' < j'$*). Then the number of lattice points of* $\mathfrak{s}_G$ *is given by the formula*

$$\sharp\left(\mathfrak{s}_G \cap N_G\right) = \sharp\left\{\lambda \in \mathbb{Z} \;\middle|\; 1 \le \lambda \le l-1 \;\text{ with }\; \sum_{i=1}^{4}\left[\lambda\alpha_i\right]_l = l\right\} + 4$$

$$= \mathbf{Ehr}_{N_G}\left(\mathfrak{s}_G, 1\right) = 1 + \mathbf{a}_1\left(\mathfrak{s}_G\right) + \mathbf{a}_2\left(\mathfrak{s}_G\right) + \mathbf{a}_3\left(\mathfrak{s}_G\right) \tag{D.3}$$

with

$$\mathbf{a}_1\left(\mathfrak{s}_G\right) = \sum_{1 \le i < j \le 4}\left[\tfrac{(\gcd(\alpha_{i'}, l))^2 + (\gcd(\alpha_{j'}, l))^2}{36\ l} + \widetilde{\mathrm{DS}}(\mathbf{p}_{i,j}, \mathbf{q}_{i,j})\right]\cdot \gcd(\alpha_{i'}, \alpha_{j'}, l)$$

$$= \frac{1}{12\,l}\left(\sum_{i=1}^{4}\gcd\left(\alpha_i, l\right)\right) + \sum_{1 \le i < j \le 4}\left[\widetilde{\mathrm{DS}}\left(\mathbf{p}_{i,j}, \mathbf{q}_{i,j}\right)\cdot \gcd\left(\alpha_{i'}, \alpha_{j'}, l\right)\right] \tag{D.4}$$

and

$$\mathbf{a}_2\left(\mathfrak{s}_G\right) = \frac{1}{4}\sum_{i=1}^{4}\gcd\left(\alpha_i, l\right)\ , \quad \mathbf{a}_3\left(\mathfrak{s}_G\right) = \frac{l}{6}, \tag{D.5}$$

where now

$$\mathbf{q}_{i,j} = \frac{l \cdot \gcd(\alpha_{i'}, \alpha_{j'}, l)}{\gcd\left(\alpha_{i'}, l\right)\cdot \gcd(\alpha_{j'}, l)}, \quad \mathbf{p}_{i,j} = \left[\frac{(-\gamma_{i'})\cdot \alpha_{i'}}{\gcd\left(\alpha_{i'}, l\right)}\right]_{\mathbf{q}_{i,j}}.$$

Since our lattice $N_G = \sum_{i=1}^4 \mathbb{Z}\, e_i + \mathbb{Z}\, \frac{1}{l}\, (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^{\intercal}$ is "skew" and of rank 4, to apply Mordell-Pommersheim formula we have to modify appropriately our "lattice data". As we shall see below, it is more convenient to consider $l\, \mathfrak{s}_G$ and $l\, N_G$ instead of $\mathfrak{s}_G$ and N_G (to avoid bothersome denominators), to work with the fundamental half-open parallelotope of the cone supporting $l\, \mathfrak{s}_G$, and to evaluate (after that) the relative volumes of the simplex faces by passing to the intersection with the l-times dilated affine hyperplane $l\, \mathcal{H}_1$ of level 1.

Lemma D.4. *Let $N \subset \mathbb{R}^d$ be a lattice and $\mathcal{H}$ a $(d-1)$-dimensional linear hyperplane in $\mathbb{R}^d$, so that $\overline{N} := N \cap \mathcal{H}$ is a lattice of rank $d-1$. Then there exists an element $n \in N$ such that $N = \overline{N} + \mathbb{Z}\, n$, and*

$$\det(N) = \left(\begin{array}{c} \text{euclidean distance} \\ \text{between } \mathbf{0} \text{ and } n + \mathcal{H} \end{array} \right) \cdot \det\left(\overline{N}\right). \tag{D.6}$$

Proof. Let $\{n_1, \ldots, n_{d-1}, n_d\}$ be a $\mathbb{Z}$-basis of N such that $\{n_1, \ldots, n_{d-1}\}$ is a $\mathbb{Z}$-basis of the lattice $\overline{N}$, $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ the dual lattice of N, and let v denote the normal vector of $\mathcal{H}$. Then $\frac{v}{\langle v, n_d \rangle} \in M$ because

$$\left\langle \frac{v}{\langle v, n_d \rangle}, n_i \right\rangle = 0, \ \forall i, \ 1 \le i \le d-1, \quad \text{and} \quad \left\langle \frac{v}{\langle v, n_d \rangle}, n_d \right\rangle = 1.$$

Furthermore, it is primitive and by definition we have

$$\left(\begin{array}{c} \text{euclidean distance} \\ \text{between } \ \mathbf{0} \ \text{ and } \ n_d + \mathcal{H} \end{array} \right) = \frac{\langle v, n_d \rangle}{\|v\|} = \frac{1}{\left\| \frac{v}{\langle v, n_d \rangle} \right\|}.$$

Thus, setting $n = n_d$ and applying [**14**, Corollary of p. 25], we get (D.6). $\square$

Lemma D.5 ([**20**, Ch. 21, 2.E, formula (12) on p. 453]). *The (standard) volume of a* regular *d-dimensional simplex* $\mathbf{s}$ *of edge length $\sqrt{2}$ equals*

$$\mathrm{Vol}(\mathbf{s}) = \frac{\sqrt{d+1}}{d!} \tag{D.7}$$

Proof of Corollary D.3. Let $l\, \mathfrak{s}_G$ in $(l\, N_G)_{\mathbb{R}}$ be the l-times dilated junior lattice simplex with $l\, N_G = \sum_{i=1}^4 \mathbb{Z}\, (l\, e_i) + \mathbb{Z}\, (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^{\intercal}$, and

$$\begin{aligned} Q &:= \mathbf{Par}\left(\mathbb{R}_{\ge 0}\, (l\, e_1) + \mathbb{R}_{\ge 0}\, (l\, e_2) + \mathbb{R}_{\ge 0}\, (l\, e_3) + \mathbb{R}_{\ge 0}\, (l\, e_4)\right) \cap (l\, N_G) \\ &= \left\{ n \in (l\, N_G) \;\middle|\; n = \sum_{i=1}^4 \varepsilon_i\, (l\, e_i), \text{ with } 0 \le \varepsilon_i < 1, \ \forall i, \ 1 \le i \le 4 \right\} \\ &= \left\{ \left([j\, \alpha_1]_l, [j\, \alpha_2]_l, [j\, \alpha_3]_l, [j\, \alpha_4]_l\right)^{\intercal} \;\middle|\; j \in \{0, 1, \ldots, l-1\} \right\}. \end{aligned}$$

Defining

$$Q_i := \left\{ n = \left(n^{(1)}, n^{(2)}, n^{(3)}, n^{(4)}\right)^{\intercal} \in Q \;\middle|\; n^{(i)} = 0 \right\}, \ \forall i, \ 1 \le i \le 4,$$

it is easy to verify that

$$\left\{ \begin{array}{ll} \sharp(Q_i) = & \gcd(\alpha_i, l), \ \forall i, \ 1 \le i \le 4, \text{ and} \\ \sharp(Q_i \cap Q_j) = & \gcd(\alpha_i, \alpha_j, l), \ \forall i, j, \ 1 \le i < j \le 4 \end{array} \right\} \tag{D.8}$$

To apply D.2 it suffices to consider $l\, \mathfrak{s}_G = \mathrm{conv}(\{(l\, e_1), (l\, e_2), (l\, e_3), (l\, e_4)\})$ w.r.t. the lattice $\overline{l\, N_G} := l\, N_G \cap l\, \mathcal{H}_1$ of rank 3. Note that the euclidean distance between $\mathbf{0}$ and $le_1 + l\, \mathcal{H}_1$ is equal to $\frac{l}{2}$. Thus, by (D.6) and (2.3), we deduce that

$$\det\left(\overline{l\,N_G}\right)\cdot\frac{l}{2}=\ \det\left(l\,N_G\right),\quad \tfrac{\det\left(l\,\mathbb{Z}^4\right)}{\det(l\,N_G)}=\tfrac{l^4}{\det(l\,N_G)}=\sharp\left(Q\right)=l$$

$$\Rightarrow \det\left(l\,N_G\right)=\det((l\,N_G)_{l\,\mathfrak{s}_G})=l^3,\ \det\left(\overline{l\,N_G}\right)=\det(\left(\overline{l\,N_G}\right)_{l\,\mathfrak{s}_G})=2\,l^2. \tag{D.9}$$

$\triangleright$ First step. Let $F_i=\mathrm{conv}(\{(l\,e_1)\,,\,(l\,e_2)\,,\,(l\,e_3)\,,\,(l\,e_4)\}\smallsetminus\{(l\,e_i)\})$, $1\le i\le 4$, be the facets and $E_{i,j}=\mathrm{conv}(\{(l\,e_i)\,,\,(l\,e_j)\})\,,\ 1\le i<j\le 4$, the edges of the tetrahedron $l\,\mathfrak{s}_G$. How does one compute their relative volumes with respect to $\overline{l\,N_G}$? By (D.7) the standard volumes are the following:

$$\mathrm{Vol}\left(l\,\mathfrak{s}_G\right)=\frac{2\,l^3}{3!}=\frac{l^3}{3}\ ,\quad \mathrm{Vol}\left(F_i\right)=\frac{\sqrt{3}}{2}\,l^2\ ,\quad \mathrm{Vol}\left(E_{i,j}\right)=\sqrt{2}\,l. \tag{D.10}$$

On the other hand, applying again Lemma D.4 and (2.3) for the facets and the edges of $l\,\mathfrak{s}_G$ we get similarly:

$$\det\left(\mathrm{aff}\left(F_i\right)\cap\overline{l\,N_G}\right)\cdot\tfrac{l}{\sqrt{3}}=\ \det\left(\mathrm{lin}\left(F_i\right)\cap l\,N_G\right),$$

$$\frac{\det(\sum_{j\in\{1,2,3,4\}\smallsetminus\{i\}}\mathbb{Z}\ (l\,e_j))}{\det\left(\mathrm{lin}\left(F_i\right)\cap l\,N_G\right)}=\frac{l^3}{\det\left(\mathrm{lin}\left(F_i\right)\cap l\,N_G\right)}=\sharp\left(Q_i\right)\overset{(\mathrm{D}.8)}{=}\ \gcd\left(\alpha_i,l\right)$$

$$\Longrightarrow \det\left(\mathrm{lin}\left(F_i\right)\cap l\,N_G\right)=\tfrac{l^3}{\gcd(\alpha_i,l)}\ ,\ \det\left(\mathrm{aff}\left(F_i\right)\cap\overline{l\,N_G}\right)=\tfrac{\sqrt{3}\,l^2}{\gcd(\alpha_i,l)}, \tag{D.11}$$

and

$$\det\left(\mathrm{aff}\left(E_{i,j}\right)\cap\overline{l\,N_G}\right)\cdot\tfrac{l}{\sqrt{2}}=\det\left(\mathrm{lin}\left(E_{i,j}\right)\cap l\,N_G\right)\ ,$$

$$\begin{aligned}\frac{\det\left(\mathbb{Z}\ (l\,e_{i'})+\mathbb{Z}\ (l\,e_{j'})\right)}{\det\left(\mathrm{lin}\left(E_{i,j}\right)\cap l\,N_G\right)}&=\frac{l^2}{\det\left(\mathrm{lin}\left(E_{i,j}\right)\cap l\,N_G\right)}\\&=\sharp\left(Q_{i'}\cap Q_{j'}\right)\overset{(\mathrm{D}.8)}{=}\ \gcd\left(\alpha_{i'},\alpha_{j'},l\right)\end{aligned}$$

$$\Longrightarrow\left\{\begin{array}{l}\det\left(\mathrm{lin}\left(E_{i,j}\right)\cap l\,N_G\right)=\frac{l^2}{\gcd(\alpha_{i'},\alpha_{j'},l)},\ \text{and}\\ \det\left(\mathrm{aff}\left(E_{i,j}\right)\cap\overline{l\,N_G}\right)=\frac{\sqrt{2}\,l}{\gcd(\alpha_{i'},\alpha_{j'},l)}\end{array}\right\}. \tag{D.12}$$

Combining (D.10) with (D.9), (D.11), (D.12), we finally obtain

$$\mathrm{Vol}_{(\overline{l\,N_G})_{l\,\mathfrak{s}_G}}\left(l\,\mathfrak{s}_G\right)=\tfrac{\mathrm{Vol}(l\,\mathfrak{s}_G)}{\det(\left(\overline{l\,N_G}\right)_{l\,\mathfrak{s}_G})}=\tfrac{l}{6}\ , \tag{D.13}$$

$$\mathrm{Vol}_{\left(\overline{l\,N_G}\right)_{F_i}}\left(F_i\right)=\tfrac{\mathrm{Vol}(F_i)}{\det\left(\mathrm{aff}(F_i)\cap\overline{l\,N_G}\right)}=\tfrac{\gcd(\alpha_i,l)}{2}, \tag{D.14}$$

$$\mathrm{Vol}_{(\overline{l\,N_G})_{E_{i,j}}}\left(E_{i,j}\right)=\tfrac{\mathrm{Vol}(E_{i,j})}{\det(\mathrm{aff}(E_{i,j})\cap\overline{l\,N_G})}=\gcd(\alpha_{i'},\alpha_{j'},l). \tag{D.15}$$

$\triangleright$ Second step. The procedure of the determination of $\mathbf{q}_{i,j}$'s and $\mathbf{p}_{i,j}$'s (whose Dedekind measures lead to the evaluation of the contributions of the "dihedral angles" to the counting of lattice points) is a little bit more complicated. To simplify it, we shall this time transform $l\,\mathfrak{s}_G$. For all indices $i,j,\ 1\le i<j\le 4$, we define an integer translation

$$\mathbf{s}_{i,j}:=\ l\,\mathfrak{s}_G-l\,e_i=\ \mathrm{conv}\left(\{\mathbf{0},\,l\left(e_{j'}-e_i\right),\,l\left(e_{i'}-e_i\right),\,l\left(e_j-e_i\right)\}\right)$$

and

$$N_{i,j}:=\overline{l\,N_G}-l\,e_i$$

$$= \mathbb{Z}\, l\, (e_{j'} - e_i) + \mathbb{Z}\, l\, (e_{i'} - e_i) + \mathbb{Z}\, l\, (e_j - e_i) + \mathbb{Z}\, l\, ((\alpha_1, \alpha_2, \alpha_3, \alpha_4)^{\mathsf{T}} - (\sum_{\iota=1}^{4} \alpha_\iota)\, e_i)$$

Furthermore, we define a unimodular transformation $\Phi_{i,j} : (N_{i,j})_{\mathbb{R}} \longrightarrow (N_{i,j})_{\mathbb{R}}$ by

$$\Phi_{i,j}\,(e_j) = e_i\,, \quad \Phi_{i,j}\,(e_i) = e_i - e_j\,, \quad \Phi_{i,j}\,(e_{i'}) = e_i + e_{i'} \quad \text{and} \quad \Phi_{i,j}\,(e_j) = e_i + e_{j'}\,,$$

and linear extension. $\Phi_{i,j}$ transfers $\mathbf{s}_{i,j}$ onto

$$\Phi_{i,j}\,(\mathbf{s}_{i,j}) = \operatorname{conv}\left(\left\{ \begin{array}{c} \mathbf{0}\,, \quad \Phi_{i,j}\,\left(l\,(e_{j'} - e_i)\right) = l(e_j + e_{j'}), \\ \Phi_{i,j}\,\left(l\,(e_{i'} - e_i)\right) = l\,(e_{i'} + e_j)\,,\, \Phi_{i,j}\,(l\,(e_j - e_i)) = l e_j \end{array} \right\}\right),$$

the lattice $N_{i,j}$ onto

$$\widehat{N_{i,j}} := \Phi_{i,j}\,(N_{i,j})$$

$$= \mathbb{Z}\, l\,(e_{j'} + e_j) + \mathbb{Z}\, l\, e_j + \mathbb{Z}\, l\, (e_{i'} + e_j) + \mathbb{Z}\left(\left(\sum_{\iota \in \{1,2,3,4\} \smallsetminus \{i\}} \alpha_\iota\right) e_j + \alpha_{i'}\, e_{i'} + \alpha_{j'}\, e_{j'}\right)$$

$$= \mathbb{Z}\, l\, e_j + \mathbb{Z}\, l\, e_{j'} + \mathbb{Z}\, l\, e_{i'} + \mathbb{Z}\, (-\alpha_i\, e_j + \alpha_{i'}\, e_{i'} + \alpha_{j'}\, e_{j'})\,,$$

the edges $E_{i,j}$ onto $\Phi_{i,j}\,(E_{i,j}) = \operatorname{conv}(\{\mathbf{0}, l\, e_j\})$ and the facets $F_{i'}$ and $F_{j'}$ containing $E_{i,j}$ onto

$$\Phi_{i,j}\,(F_{i'}) = \operatorname{conv}\left(\{\mathbf{0},\, l\, e_j,\, l\,(e_{i'} + e_j)\}\right) \text{ and } \Phi_{i,j}(F_{j'}) = \operatorname{conv}(\{\mathbf{0},\, l\, e_j,\, l(e_j + e_{j'})\}),$$

respectively. Since $\gcd(-\alpha_i, \alpha_{i'}, \alpha_{j'}) = \gcd(\alpha_i, \alpha_{i'}, \alpha_{j'}) = 1$, it is $\det(\widehat{N_{i,j}}) = l^2$. On the other hand,

$$\Phi_{i,j}\,(E_{i,j}) \cap \widehat{N_{i,j}} = \operatorname{conv}\left(\{\mathbf{0}, l\, e_j\}\right) \cap \widehat{N_{i,j}} = \sharp\,(Q_{i'} \cap Q_{j'}) = \gcd\,(\alpha_{i'}, \alpha_{j'}, l)\,,$$

i.e. $\frac{l\ e_j}{\gcd(\alpha_{i'}, \alpha_{j'}, l)}$ is a primitive vector. Setting $\widetilde{N_{i,j}} := \widehat{N_{i,j}}/(\mathbb{Z}\ \frac{l\ e_j}{\gcd(\alpha_{i'}, \alpha_{j'}, l)})$ we get

$$\det(\widetilde{N_{i,j}}) \cdot \frac{l}{\gcd\,(\alpha_{i'}, \alpha_{j'}, l)} = \det(\widehat{N_{i,j}}) \Rightarrow \det(\widetilde{N_{i,j}}) = l \cdot \gcd(\alpha_{i'}, \alpha_{j'}, l).$$

If we denote by $\widetilde{n}_{i'}, \widetilde{n}_{j'}$ the images of $l\,(e_{i'} + e_j)$, $l(e_j + e_{j'})$ under the canonical epimorphism $\widehat{N_{i,j}} \to \widetilde{N_{i,j}}$ (i.e. $\widetilde{n}_{i'} = l e_{i'}$, $\widetilde{n}_{j'} = l e_{j'}$ within $\widetilde{N_{i,j}}$), and by $\widetilde{\widetilde{n}}_{i'}$, $\widetilde{\widetilde{n}}_{j'}$ the primitive vectors of $\operatorname{conv}(\{\mathbf{0}, \widetilde{n}_{i'}\})$ and $\operatorname{conv}(\{\mathbf{0}, \widetilde{n}_{j'}\})$ w.r.t. $\widetilde{N_{i,j}}$, then

$$\left\|\widetilde{\widetilde{n}}_{i'}\right\| \cdot \frac{l}{\gcd\,(\alpha_{i'}, \alpha_{j'}, l)} = \det\left(\operatorname{lin}\,(F_{i'}) \cap \widehat{N_{i,j}}\right)$$

$$= \frac{l^2}{\sharp\left(\operatorname{conv}\left(\{l\, e_j,\, l\,(e_{i'} + e_j)\}\right) \cap \widehat{N_{i,j}}\right)} = \frac{l^2}{\gcd\,(\alpha_{j'}, l)}.$$

Consequently,

$$\widetilde{\widetilde{n}}_{i'} = \left(\frac{l \cdot \gcd(\alpha_{i'}, \alpha_{j'}, l)}{\gcd(\alpha_{j'}, l)}\right) e_{i'} \quad \left(\text{and analogously,} \quad \widetilde{\widetilde{n}}_{j'} = \left(\frac{l \cdot \gcd\left(\alpha_{i'}, \alpha_{j'}, l\right)}{\gcd(\alpha_{i'}, l)}\right) e_{j'}\right).$$

This means that

$$\mathbf{q}_{i,j} = \frac{\left\|\widetilde{\widetilde{n}}_{i'}\right\| \cdot \left\|\widetilde{\widetilde{n}}_{j'}\right\|}{\det\left(\widetilde{N_{i,j}}\right)} = \frac{l \cdot \gcd\,(\alpha_{i'}, \alpha_{j'}, l)}{\gcd\,(\alpha_{i'}, l) \cdot \gcd\,(\alpha_{j'}, l)}\,.$$

Since

$$\widetilde{N_{i,j}} = \mathbb{Z}\, l\, e_{j'} + \mathbb{Z}\, l\, e_{i'} + \mathbb{Z}\, (\alpha_{i'}\, e_{i'} + \alpha_{j'}\, e_{j'})\,,$$

the vector $\gamma_{i'} \cdot (\alpha_{i'}\, e_{i'})$ +$\gcd(\alpha_{j'}, l)\ e_{j'}$ belongs to $\widetilde{N_{i,j}}$, and from

$$\det(\mathbb{Z}\, \widetilde{\widetilde{n}}_{i'} + \mathbb{Z}\, (\gamma_{i'} \cdot (\alpha_{i'}\, e_{i'}) + \gcd(\alpha_{j'}, l)\, e_{j'})) = l \cdot \gcd(\alpha_{i'}, \alpha_{j'}, l) = \det(\widetilde{N_{i,j}})$$

we conclude that $\{\widetilde{\widetilde{n}}_{i'}, \gamma_{i'} \cdot (\alpha_{i'}\, e_{i'}) + \gcd(\alpha_{j'}, l)\, e_{j'}\}$ is a $\mathbb{Z}$-basis of $\widetilde{N_{i,j}}$ with

$$\widetilde{\widetilde{n}}_{j'} = \left(\tfrac{l \cdot \gcd(\alpha_{i'}, \alpha_{j'}, l)}{\gcd(\alpha_{i'}, l)} \right) e_{j'} = \left(\tfrac{(-\gamma_{i'}) \cdot \alpha_{i'}}{\gcd(\alpha_{i'}, l)} \right) \widetilde{\widetilde{n}}_{i'} + \mathbf{q}_{i,j} \cdot (\gamma_{i'} \cdot (\alpha_{i'}\, e_{i'}) + \gcd(\alpha_{j'}, l)\; e_{j'}).$$

Hence, $\mathbf{p}_{i,j} = \left[\tfrac{(-\gamma_{i'}) \cdot \alpha_{i'}}{\gcd(\alpha_{i'}, l)} \right]_{\mathbf{q}_{i,j}}$, and the proof is completed after the substitution of (D.13), (D.14), (D.15) into (D.1), (D.2). $\square$

REMARK D.6. One can analogously compute the $\mathbf{a}_i(\mathfrak{s}_G)$'s in the case in which the acting group G is abelian (not necessarily cyclic), again by lattice transforming and by Mordell-Pommersheim formula. In these more complicated expressions the greatest common divisors are replaced by denumerants of restricted weighted vectorial partitions.

• **Diaz-Robins formula**. To present *Diaz-Robins formula*, by means of which one computes the coefficients of the Ehrhart polynomial of a lattice simplex of *arbitrary* dimension, let us first recall the notion of *Hermite normal form* which will enable us to choose a convenient coordinate system for the simplex vertices.

THEOREM D.7 ([**95**, II.2 and II.3, pp. 15-18]). *For a given integer (or rational) $(d \times d')$-matrix A of full row rank, there is a unimodular matrix $U \in \mathrm{GL}(d', \mathbb{Z})$, such that AU is lower-triangular with positive diagonal elements. Each off-diagonal element of AU is non-negative and strictly less than the diagonal element in its column. We say that AU is in* Hermite normal form. *If $\det(A) \neq 0$, then U is uniquely determined, and* $\mathbf{HNF}(A) := AU$ *is* the Hermite normal form of A.

Now let $\mathbf{s} \subset \mathbb{R}^d$ be a d-dimensional simplex whose vertices belong to (the standard lattice) $\mathbb{Z}^d$. W.l.o.g., we may assume that $\mathbf{s} = \mathrm{conv}(\{\mathbf{0}, n_1, \ldots, n_{d-1}, n_d\})$. The matrix $(n_1, \ldots, n_{d-1}, n_d)$ is by Theorem D.7 left-equivalent to

$$\mathbf{\Lambda_s} := \mathbf{HNF}((n_1, n_2, \ldots, n_{d-1}, n_d)) = \begin{pmatrix} \lambda_{11} & 0 & \cdots & 0 \\ \lambda_{21} & \lambda_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{d1} & \lambda_{d2} & \cdots & \lambda_{dd} \end{pmatrix}.$$

Consider

$$\widetilde{\mathbf{\Lambda_s}} := \begin{pmatrix} \mathbf{\Lambda_s} & 0 \\ \mathbf{1} & \end{pmatrix}, \qquad \text{with} \quad \mathbf{1} := (\underbrace{1, 1, \ldots, 1, 1}_{(d+1)-\text{times}}).$$

Define

$$\varpi_j := \prod_{1 \le i < j} \lambda_{ii}, \quad \forall j, \ \ 1 \le j \le d, \quad \text{and} \quad \varpi_{d+1} := \varpi_d,$$

$$\mathfrak{G}_\mathbf{s} := (\mathbb{Z} / \varpi_1 \mathbb{Z}) \times (\mathbb{Z} / \varpi_2 \mathbb{Z}) \times \cdots \times (\mathbb{Z} / \varpi_d \mathbb{Z}).$$

Moreover, for each group element $\mathfrak{g} = (\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_d) \in \mathfrak{G}_\mathbf{s}$, set:

$$\boldsymbol{\varepsilon}_j(\mathfrak{g}) := \left\langle (0, \mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_d), (j\text{-th column of } \widetilde{\mathbf{\Lambda_s}}) \right\rangle.$$

THEOREM D.8 (Diaz-Robins formula; cf. [**35, 36**] and [**17**, §5]). *The Ehrhart polynomial*

$$\mathbf{Ehr}_N(\mathbf{s}, \nu) = \sum_{i=0}^{d} \mathbf{a}_i(\mathbf{s})\ \nu^i$$

of $\mathbf{s}$ *has the following exponential generating function*:

$$\sum_{\nu=0}^{\infty} \mathbf{Ehr}_N(\mathbf{s},\nu)\, e^{-2\pi x\nu} = \tfrac{1}{2^{d+1}\ |\mathfrak{G}_{\mathbf{s}}|} \sum_{\mathfrak{g}\in\mathfrak{G}_{\mathbf{s}}} \prod_{j=1}^{d+1} (1+\coth(\tfrac{\pi}{\varpi_j}(x+\sqrt{-1}\boldsymbol{\varepsilon}_j(\mathfrak{g})))) \tag{D.16}$$

and $\mathbf{a}_i(\mathbf{s})$ *equals the coefficient of* $\frac{1}{x^{i+1}}$ *in the Laurent expansion at* $x=0$ *of*

$$\frac{\pi^{i+1}}{i!\ 2^{d-i}\ \ |\mathfrak{G}_{\mathbf{s}}|} \sum_{\mathfrak{g}\in\mathfrak{G}_{\mathbf{s}}} \left[\prod_{j=1}^{d+1} (1+\coth(\frac{\pi}{\varpi_j}(x+\sqrt{-1}\ \boldsymbol{\varepsilon}_j(\mathfrak{g}))))\right].$$

Let us now describe how can one apply Diaz-Robins formula to the case of the simplex we are interested in. Let $(\mathbb{C}^r/G,[\mathbf{0}])$, $r\geq 4$, be a Gorenstein AQS of type (3.6), and $\mathfrak{s}_G = \mathrm{conv}(\{e_1,e_2,\ldots,e_r\})$ the corresponding junior simplex. As our lattice N_G is "skew", to count

$$\sharp(\mathfrak{s}_G\cap N_G) = \sharp\left\{ (j_1,..,j_\kappa)\in\prod_{\mu=1}^{\kappa}(\{0,1,..,q_\mu\}) \,\middle|\, \sum_{i=1}^{r}\delta_i(j_1,..,j_\kappa) = \exp(G)\right\} + r$$

we have to transform $\mathfrak{s}_G$ onto another appropriate simplex.

Corollary D.9. *There exists a lattice simplex* $\bar{\mathfrak{s}}''_G$ *w.r.t.* $\mathbb{Z}^{r-1}$ *such that*

$$\sharp\left(\bar{\mathfrak{s}}''_G\cap\mathbb{Z}^{r-1}\right) = \ \sharp(\mathfrak{s}_G\cap N_G), \tag{D.17}$$

and therefore $\sharp(\mathfrak{s}_G\cap N_G)$ *can be computed by applying formula* (D.16) *for* $\bar{\mathfrak{s}}''_G$.

Proof. This will be done in three steps.

$\triangleright$ First step. We first perform a translation in order to insert the zero point as a vertex, and define $\bar{\mathfrak{s}}_G := \mathfrak{s}_G - e_1 = \mathrm{conv}(\{\mathbf{0}, e_2-e_1,\ldots,e_r-e_1\})$ with vertex set belonging to the lattice $\overline{N}_G := \sum_{i=2}^{r}\mathbb{Z}\,(e_i-e_1) + \sum_{\mu=1}^{\kappa}\mathbb{Z}\,\frac{1}{q_\mu}(\alpha_\mu - \vartheta_\mu\, e_1)$, where $\alpha_\mu := (\alpha_{\mu,1},\ldots,\alpha_{\mu,r})^{\intercal}$ and $\vartheta_\mu := \sum_{i=1}^{r}\alpha_{\mu,i}$. Obviously,

$$\sharp(\mathfrak{s}_G\cap N_G) = \sharp\left(\bar{\mathfrak{s}}_G\cap\overline{N}_G\right).$$

$\triangleright$ Second step. We define a unimodular transformation $\Phi:\mathbb{R}^r\longrightarrow\mathbb{R}^r$ by

$$\Phi(e_1) := e_1 - e_2,\ \ \Phi(e_2) := e_1,\ \ \Phi(e_j) := e_1+e_j,\ \ \forall j,\ \ 3\leq j\leq r,$$

and linear extension. Φ maps $\bar{\mathfrak{s}}_G$ onto

$$\bar{\mathfrak{s}}'_G := \Phi(\bar{\mathfrak{s}}_G) = \mathrm{conv}(\{\mathbf{0},e_2,e_2+e_3,e_2+e_4,\ldots,e_2+e_r\})$$

and $\overline{N}_G$ onto

$$\overline{N}'_G := \Phi\left(\overline{N}_G\right) = \mathbb{Z}\ e_2 + \mathbb{Z}\ \ (e_2+e_3) + \cdots + \mathbb{Z}\ \ (e_2+e_r) + \sum_{\mu=1}^{\kappa}\mathbb{Z}\,\frac{\alpha'_\mu}{q_\mu},$$

with

$$\alpha'_\mu := \Phi(\alpha_\mu - \vartheta_\mu\, e_1) = \Phi(\alpha_\mu) - \vartheta_\mu\ \Phi(e_1)$$

$$= (\vartheta_\mu - \alpha_{\mu,1})\, e_2 + \sum_{j=3}^{r}\ \alpha_{\mu,j}\ e_j\ .$$

Thus,

$$\overline{N}'_G = \mathbb{Z}\ e_2 + \mathbb{Z}\ e_3 + \cdots + \mathbb{Z}\ e_r + \sum_{\mu=1}^{\kappa}\mathbb{Z}\,\frac{\alpha'_\mu}{q_\mu}$$

and we can consider both $\overline{\mathfrak{s}}_G'$ and $\overline{N}_G'$ within $\mathbb{R}^{r-1}$ by identifying it with the set $\{(x_1,\ldots,x_r)\in\mathbb{R}^r \mid x_1=0\}$. Moreover, $\sharp\left(\overline{\mathfrak{s}}_G'\cap\overline{N}_G'\right)=\sharp\left(\overline{\mathfrak{s}}_G\cap\overline{N}_G\right)$.

$\triangleright$ Third step. Define A to be the rational $(r-1)\times(r-1+\kappa)$-matrix formed by the column vectors which generate $\overline{N}_G'$:

$$A := (e_2, e_3, \ldots, e_r, \tfrac{1}{q_1}\ \alpha_1', \ldots, \tfrac{1}{q_\kappa}\ \alpha_\kappa').$$

According to Theorem D.7 (with $d=r-1$, $d'=r-1+\kappa$), we can determine a unimodular matrix U such that $AU=(R\ \ 0)$, where R is a rational, non-singular $(r-1)\times(r-1)$-matrix being in Hermite normal form. Hence, if we set

$$\begin{aligned}\overline{\mathfrak{s}}_G'' &:= R^{-1}\overline{\mathfrak{s}}_G'\\ &= \mathrm{conv}\left(\left\{\mathbf{0},\ R^{-1}\ e_2, R^{-1}\ \left(e_2+e_3\right), R^{-1}\ \left(e_2+e_4\right), \ldots, R^{-1}\ \left(e_2+e_r\right)\right\}\right),\end{aligned}$$

the lattice $\overline{N}_G'$ is transformed onto $\overline{N}_G'' := R^{-1}\ \overline{N}_G' = \mathbb{Z}^{r-1}$, i.e., onto the *standard* lattice of rank $r-1$. But then we are done because $\sharp\left(\overline{\mathfrak{s}}_G''\cap\mathbb{Z}^{r-1}\right)=\ \sharp(\overline{\mathfrak{s}}_G'\cap\overline{N}_G')$, and we can apply Theorem D.8 for the simplex $\overline{\mathfrak{s}}_G''\subset\mathbb{Z}^{r-1}$ (with $d=r-1$). $\square$

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University of Crete, Department of Mathematics, Division Algebra and Geometry, Knossos Avenue, P.O. Box 2208, GR-71409, Heraklion, Crete, Greece
E-mail address: `ddais@math.uoc.gr`

Technical University Otto von Guericke, Institut für Algebra und Geometrie, PSF 4120, D-39016 Magdeburg, Germany
E-mail address: `henk@math.uni-magdeburg.de`

TU-Berlin, Institut für Mathematik, MA 6.2, Strasse des 17 Juni 136, D-10623 Berlin, Germany
E-mail address: `ziegler@math.tu-berlin.de`