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STRONG McKAY CORRESPONDENCE, STRING-THEORETIC HODGE NUMBERS AND MIRROR SYMMETRY

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WE PROPOSE a new higher dimensional version of the McKay correspondence which enables us to understand the "Hodge numbers" assigned to singular Gorenstein varieties by physicists. Our results lead to the conjecture that string theory indicates the existence of some new cohomology theory $H^*_{st}(X)$ for algebraic varieties with Gorenstein singularities. We give a formal mathematical definition of the Hodge numbers $h^{p,q}_{st}(X)$ inspired from the consideration of strings on orbifolds and from this new (conjectural) version of the McKay correspondence. The numbers $h^{p,q}_{st}(X)$ are expected to give the spectrum of orbifoldized Landau–Ginzburg models and mirror duality relations for higher dimensional Calabi–Yau varieties with Gorenstein toroidal or quotient singularities.

1. INTRODUCTION

Throughout this paper by an *algebraic variety* (or simply *variety*) we mean an integral, separated algebraic scheme over **C**. By a *compact algebraic variety* we mean the representative of a complete variety within the analytic category. The *singular locus* of an algebraic variety X is denoted by Sing X. The words *smooth variety* and *manifold* are used interchangeably. By the word *singularity* we sometimes intimate a singular point and sometimes the underlying space of a neighbourhood or the germ of a singular point, but its meaning will be always clear from the context. Following Danilov [9, Section 13.3], we shall say that an $x \in X$ is a *toroidal singularity* of X, if there is an analytic isomorphism between the germ (X, x) and the germ corresponding to the toric singularity $(\mathbf{A}_\sigma, p_\sigma)$ (see also Section 4).

Our main tool will be certain algebraic varieties with special Gorenstein singularities, primarily having in mind the Calabi–Yau varieties. A *Calabi–Yau variety* is defined to be a normal projective algebraic variety X with trivial canonical sheaf ω_X and $H^i(X, \mathcal{O}_X) = 0$, $0 < i < \dim_{\mathbf{C}} X$, which, in addition, can have at most *canonical Gorenstein singularities*. (For the notion of *canonical singularity* we refer to [38].) If Sing $X = \emptyset$, then X is called, as usual, a *Calabi–Yau manifold*.

In this paper we shall attempt to realize some Hodge-theoretical invariants used by physicists for singular varieties being related to the mirror symmetry phenomenon. The necessity of working with singular varieties becomes unavoidable from the fact that, in many examples of pairs X, X^* of mirror symmetric Calabi–Yau manifolds, at least one of the two manifolds X or X^* is obtained as a crepant desingularization of a singular Calabi–Yau variety [4, 35]. Here, by *a crepant desingularization* of a Gorenstein variety Z, we mean a birational morphism $\pi: Z' \to Z$, such that $\pi^*(\omega_Z) \cong \omega_{Z'}$, where ω_Z and $\omega_{Z'}$ denote the canonical sheaves on Z and Z', respectively. Three-dimensional Gorenstein quotient singularities and their crepant desingularizations have been studied in [6, 24–26, 32, 33, 39, 41–45, 52].

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The most known physical cohomological invariant of singular varieties obtained as quotient-spaces of certain compact manifolds by actions of finite groups is the so-called *physicists Euler number* [12]. It has been investigated by several mathematicians [1, 16, 23, 39, 40, 42].

Let X be a smooth symplectic manifold over $\mathbf{C}$ having an action of a finite group G such that the symplectic volume form ω is G-invariant. For any $g \in G$, we set $X^g := \{x \in X \mid g(x) = x\}$. Physicists have proposed the following formula for computing the *orbifold Euler number* [12]:

$$e(X, G) = \frac{1}{|G|} \sum_{gh = hg} e(X^g \cap X^h). \tag{1}$$

It is expected that $e(X, G)$ coincides with the usual Euler number $e(\widehat{X/G})$ of a crepant desingularization $(\widehat{X/G})$ of the quotient space X/G provided such a desingularization exists. For a volume-invariant linear action on $\mathbf{C}^n$ of a finite group G, the corresponding conjectural local properties of crepant desingularizations were formulated by Reid [39]:

CONJECTURE 1.1. (generalized McKay correspondence). *Let $X = \mathbf{C}^n$, G an arbitrary finite subgroup in $SL(n, \mathbf{C})$. Assume that $Y = X/G$ admits a crepant desingularization $\pi : \hat{Y} \to Y$. Then $H^*(\pi^{-1}(0), \mathbf{C})$ has a basis consisting of classes of algebraic cycles $Z_c \subset \pi^{-1}(0)$ which are in 1-to-1 correspondence with conjugacy classes c of G. In particular, we obtain for the Euler number*

$$e(\hat{Y}) = e(\pi^{-1}(0)) = \#\{\text{conjugacy classes in } G\}.$$

Remark 1.2. For $n = 2$ a one-to-one correspondence between the nontrivial irreducible representations of a subgroup $G \subset SL(2, \mathbf{C})$ and the irreducible components of $\pi^{-1}(0)$ was discovered by McKay [34] and investigated in [14, 29, 46].

Our first purpose is to use some stronger version of Conjecture 1.1 in order to give an analogous interpretation for the *physicists Hodge numbers* $h^{p,q}(X, G)$ of orbifolds considered by Vafa [51] and Zaslow [53]. Let X be a smooth compact Kähler manifold of dimension n over $\mathbf{C}$ being equipped with an action of a finite group G, such that X has a G-invariant volume form. Let $C(g) := \{h \in G \mid hg = gh\}$. Then the action of $C(g)$ on X can be restricted on X^g. For any point $x \in X^g$, the eigenvalues of g in the holomorphic tangent space T_x are roots of unity:

$$e^{2\pi\sqrt{-1}\alpha_1}, \ldots, e^{-2\pi\sqrt{-1}\alpha_d}$$

where $0 \leqslant \alpha_j < 1$ $(j = 1, \ldots, d)$ are locally constant functions on X^g with values in $\mathbf{Q}$. We write $X^g = X_1(g) \cup \cdots \cup X_{r_g}(g)$, where $X_1(g), \ldots, X_{r_g}(g)$ are the smooth connected components of X^g. For each $i \in \{1, \ldots, r_g\}$, the *fermion shift number* $F_i(g)$ is defined to be equal to the value of $\sum_{1 \leqslant j \leqslant n} \alpha_j$ on the connected component $X_i(g)$. We denote by $h^{p,q}_{C(g)}(X_i(g))$ the dimension of the subspace of $C(g)$-invariant elements in $H^{p,q}(X_i(g))$. We set

$$h^{p,q}_g(X, G) := \sum_{i=1}^{r_g} h^{p - F_i(g), q - F_i(g)}_{C(g)}(X_i(g)).$$

The *orbifold Hodge numbers* of X/G are defined by the formula (3.21) in [53]:

$$h^{p,q}(X, G) := \sum_{\{g\}} h^{p,q}_g(X, G) \tag{2}$$

where $\{g\}$ runs over the conjugacy classes of G, so that g represents $\{g\}$. As we shall see in Corollary 6.15, these numbers coincide with the *usual* Hodge numbers of a crepant desingularization of X/G.

One of our next intentions is to convince the reader of the existence of some *new cohomology theory* $H^*_{st}(X)$ of more general algebraic varieties X with mild Gorenstein singularities. Since this cohomology is inspired from the string theory, we call $H^*_{st}(X)$ the *string cohomology of* X. For compact varieties X, we expect that the string cohomology groups $H^*_{st}(X)$ will satisfy the Poincaré duality and will be endowed with a pure Hodge structure. The role of crepant resolutions for the string cohomology $H^*_{st}(X)$ is analogous to the one of small resolutions for the intersection cohomology $IH^*(X)$ with middle perversity. Physicists compute orbifold Hodge numbers without using crepant desingularizations. From the mathematical point of view, however, crepant desingularizations seem to be rather helpful, although they have some disadvantages. Firstly, they might not exist (at least in dimension $\geqslant 4$) and, secondly, even if they exist, they might not be unique. The consistency of the physical approach naturally suggests the formulation of the following conjecture (which is true in dimension 3 in a general setting [30, 31] and can be verified for the toroidal resolutions in arbitrary dimension by Theorem 4.4):

CONJECTURE 1.3. *Hodge numbers of smooth crepant resolutions do not depend on the choice of such a resolution.*

Let us briefly review the rest of the paper. In Section 2, we consider an example showing the importance of the "physical Hodge numbers" in connection with the mirror duality. In Section 3, we recall basic properties of E-polynomials. In Section 4, we study the Hodge structure of the exceptional loci of local crepant toric resolutions. In Section 5, we formulate the conjecture concerning the strong McKay correspondence and we prove that it is true for two- and three-dimensional Gorenstein quotient singularities, as well as for abelian Gorenstein quotient singularities of arbitrary dimension. This correspondence will be used in Section 6 to give the formal definition of the string-theoretic Hodge numbers and to study their main properties. In Section 7, we give some applications relating to the mirror symmetry and formulate the string-theoretic Hodge diamond-mirror conjecture for Calabi–Yau complete intersection in d-dimensional toric Fano varieties. This conjecture will be proved in Section 8 for the case of Δ-regular hypersurfaces in toric Fano varieties $\mathbf{P}_\Delta$ which are defined by d-dimensional reflexive simplices Δ (for arbitrary d); it gives the mirror duality *for all string-theoretic Hodge numbers* $h^{p,q}_{st}$ of abelian quotients of Calabi–Yau Fermat-type hypersurfaces which are embedded in d-dimensional weighted projective spaces. This duality agrees with the mirror construction proposed by Greene and Plesser [19–21] and the polar duality of reflexive polyhedra proposed in [4].

2. HODGE NUMBERS AND MIRROR SYMMETRY

At the beginning we shall state some introductory questions which could be considered also as another motivation for the paper. These questions are related to singular varieties of dimension $\geqslant 4$ which arose as examples of the mirror duality [4, 8, 19, 48–50]. If two d-dimensional Calabi–Yau manifolds X and Y form a mirror pair, then for all $0 \leqslant p, q \leqslant d$ their Hodge numbers must satisfy the relation

$$h^{p,q}(X) = h^{d-p,q}(Y). \tag{3}$$

However, it might happen that a mirror pair consists of two d-dimensional Calabi–Yau varieties X and Y having singularities. In this case, the duality (3) is expected to take place not for X and Y themselves, but for their crepant desingularizations $\hat{X}$ and $\hat{Y}$, if such desingularizations exist. Using the existence of smooth crepant desingularizations of Gorenstein toroidal singularities in dimension $\leqslant 3$, one can check the relations (3) for many examples of three-dimensional mirror pairs [4, 41]. But there are difficulties in proving (3) *for all* p, q and $d \geqslant 4$, even if one heuristically knows a mirror pair of singular Calabi–Yau varieties, for instance, as an orbifold. The main problem in dimension $d \geqslant 4$ is due to the existence of many *terminal* Gorenstein quotient singularities, i.e., to singularities which obviously do not admit any crepant resolution. In [4], the first author constructed the so-called *maximal projective crepant partial desingularizations* (MPCP-desingularizations) of singular Calabi–Yau hypersurfaces in toric varieties. Using MPCP-desingularizations, the relation (3) was proved for $h^{1,1}$ and $h^{d-1,1}$ in [4]. We shall show later that MPCP-desingularizations are sufficient to establish (3) for $q = 1$ and arbitrary p in the case of d-dimensional Calabi–Yau hypersurfaces in toric varieties (see 7.11, 7.12). Although MPCP-desingularizations always exist, it is important to stress that they are not sufficient to prove (3) for all p, q, and $d \geqslant 4$, because of the following two properties which can be easily illustrated by means of various examples:

- In general, a MPCP-desingularization of a Gorenstein toroidal singularity is not a manifold, but a variety with Gorenstein terminal abelian quotient singularities.
- Cohomology and Hodge numbers of different MPCP-desingularizations might be different.

It turns out that, even for three-dimensional Calabi–Yau manifolds, the mirror construction inspired from the superconformal field theory demands consideration of higher dimensional manifolds with singularities [5, 8, 48–50]. In this case, we again meet difficulties if we wish to obtain analogues of the duality in (3). Let us explain them for the example which was discussed in [8].

Let E_0 be the unique elliptic curve having an automorphism of order 3 with 3 fixed points $p_0, p_1, p_2 \in E_0$. We consider the natural diagonal action of $G \cong \mathbf{Z}/3\mathbf{Z}$ on $Z = E_0 \times E_0 \times E_0$. The quotient $X = Z/G$ is a singular Calabi–Yau variety whose smooth crepant resolution $\hat{X}$ has Hodge numbers

$$h^{1,1}(\hat{X}) = 36, \qquad h^{2,1}(\hat{X}) = 0.$$

As the mirror partner of X, the seven-dimensional orbifold Y has been proposed which was obtained from the quotient of the Fermat-cubic ($W: z_0^3 + \cdots + z_8^3 = 0$) in $\mathbf{P}^8$ by the order 3 cyclic group action defined by the matrix

$$g = \operatorname{diag}(1, 1, 1, e^{2\pi\sqrt{-1}/3}, e^{2\pi\sqrt{-1}/3}, e^{2\pi\sqrt{-1}/3}, e^{-2\pi\sqrt{-1}/3}, e^{-2\pi\sqrt{-1}/3}, e^{-2\pi\sqrt{-1}/3}).$$

By standard methods, counting G-invariant monomials in the Jacobian ring, one immediately verifies that $h^{4,3}(Y) = 30$. One could expect that a crepant resolution of singularities of Y along the 3 elliptic curves

$$C_0 = \{z_3 = \cdots = z_8 = 0\} \cap Y,$$

$$C_1 = \{z_0 = z_1 = z_2 = z_6 = z_7 = z_8 = 0\} \cap Y,$$

$$C_2 = \{z_0 = \cdots = z_5 = 0\} \cap Y$$

would give the missing 6 dimensions to $h^{4,3}(Y)$ in order to obtain 36 (this would be the analogue of (3)). But also this hope must be given up because of a very simple reason: all singularities along C_0, C_1, C_2 are terminal, i.e., they do not admit any smooth crepant resolution.

Question 2.1. What could be the suitable mathematical reasoning which would give back the missing 6 in the above example?

From the viewpoint of physicists, one should consider Y as an orbifold quotient of W by $G = \{e, g, g^{-1}\}$. By physicists' formula (2),

$$h^{4,3}(W,G) = h_e^{4,3}(W,G) + h_g^{4,3}(W,G) + h_{g^{-1}}^{4,3}(W,G).$$

It is clear that $h_g^{4,3}(W,G) = h_{g^{-1}}^{4,3}(W,G)$ and $h_e^{4,3}(W,G) = h^{4,3}(Y) = 30$. So, it remains to compute $h_g^{4,3}(W,G)$. Notice that $W^g = C_0 \cup C_1 \cup C_2$; i.e., $W_i(g) = C_i$ $(i = 0, 1, 2)$. Moreover, g acts on the tangent space T_w of a point $w \in W^g$ by the matrix

$$\mathrm{diag}(1, e^{2\pi\sqrt{-1}/3}, e^{2\pi\sqrt{-1}/3}, e^{2\pi\sqrt{-1}/3}, e^{-2\pi\sqrt{-1}/3}, e^{-2\pi\sqrt{-1}/3}, e^{-2\pi\sqrt{-1}/3}).$$

Therefore, $F_i(g) = 3$ (for $i = 0, 1, 2$). So $h_g^{4,3}(W,G) = \sum_{i=0}^{2} h^{4-F_i(g), 3-F_i(g)}(C_i) = 3$ and the required 6 is indeed present!

Question 2.2. Is there a local version of the formula (2) for the underlying space of a quotient singularity extending that of 1.1?

We shall answer both questions in Sections 5 and 6.

3. E-POLYNOMIALS OF ALGEBRAIC VARIETIES

In this section we recall some basic properties of the *E-polynomials* of (not necessarily smooth or compact) *algebraic varieties*. E-polynomials are defined by means of the mixed Hodge structure (MHS) of rational cohomology groups with compact supports [10]. As we shall see below, these polynomials obey similar additive and multiplicative laws as those of the *usual* Euler characteristic, which enables us to compute all the Hodge numbers coming into question in a very convenient way.

As Deligne shows in [11], the cohomology groups $H^k(X, \mathbf{Q})$ of a (not necessarily smooth or compact) algebraic variety X carry a natural MHS. By similar methods, one can determine a canonical MHS by considering $H_c^k(X, \mathbf{Q})$, i.e., the cohomology groups *with compact supports*. Compared with $H^k(X, \mathbf{Q})$, the MHS on $H_c^k(X, \mathbf{Q})$ presents some additional technical advantages. One of them is the existence of the following exact sequence:

PROPOSITION 3.1. *Let X be an algebraic variety and $Y \subset X$ a closed subvariety. Then there is an exact sequence*

$$\cdots \to H_c^k(X \setminus Y, \mathbf{Q}) \to H_c^k(X, \mathbf{Q}) \to H_c^k(Y, \mathbf{Q}) \to \cdots$$

consisting of MHS-morphisms.

Definition 3.2. Let X be an algebraic variety over $\mathbf{C}$ which is not necessarily compact or smooth. Denote by $h^{p,q}(H_c^k(X, \mathbf{C}))$ the dimension of the (p,q)-Hodge component of the kth cohomology with compact supports. We define:

$$e^{p,q}(X) := \sum_{k \geq 0} (-1)^k h^{p,q}(H_c^k(X, \mathbf{C})).$$

The polynomial

$$E(X;u,v) := \sum_{p,q} e^{p,q}(X) u^p v^q$$

is called the *E-polynomial* of X.

Remark 3.3. If the Hodge structure of X in 3.2 is *pure*, then the coefficients $e^{p,q}(X)$ of the E-polynomial of X are related to the usual Hodge numbers by $e^{p,q}(X) = (-1)^{p+q} h^{p,q}(X)$. In fact, the E-polynomial (in the general case) can be regarded as a notional refinement of the *virtual Poincaré polynomial* $E(X; -u, -u)$ and, of course, of the *Euler characteristic with compact supports* $e_c(X) := E(X, -1, -1)$. It should be also mentioned, that $e_c(X) = e(X)$, i.e., that $e_c(X)$ is equal to the *usual* Euler characteristic of X (cf. [13, pp. 141–142]).

Using Proposition 3.1, one obtains:

PROPOSITION 3.4. *Let X be a disjoint union of locally closed subvarieties X_i ($i \in I$). Then*

$$E(X;u,v) = \sum_{i \in I} E(X_i;u,v).$$

Definition 3.5. Let X be a disjoint union of locally closed subvarieties X_i ($i \in I$). We shall write $X_{i'} < X_i$, if $X_{i'} \neq X_i$ and $X_{i'}$ is contained in the Zariski closure $\bar{X}_i$ of X_i.

PROPOSITION 3.6. *For any $i_0 \in I$, one has*

$$E(X_{i_0};u,v) = \sum_{k \geq 0} (-1)^k \sum_{X_{i_k} < \cdots < X_{i_1} < X_{i_0}} E(\bar{X}_{i_k};u,v).$$

Proof. By Proposition 3.4, we get

$$E(X_{i_0};u,v) = E(\bar{X}_{i_0};u,v) - E(\bar{X}_{i_0} \backslash X_{i_0};u,v).$$

Moreover,

$$E(\bar{X}_{i_0} \backslash X_{i_0};u,v) = \sum_{X_{i_1} < X_{i_0}} E(X_{i_1};u,v).$$

Repeating the same procedure for $i_1 \in I$, we obtain:

$$E(X_{i_1};u,v) = E(\bar{X}_{i_1};u,v) - E(\bar{X}_{i_1} \backslash X_{i_1};u,v)$$

$$E(\bar{X}_{i_1} \backslash X_{i_1};u,v) = \sum_{X_{i_2} < X_{i_1}} E(X_{i_2};u,v), \quad \text{etc. } \ldots$$

This leads to the claimed formula. □

Applying the Künneth formula, we get:

PROPOSITION 3.7. *Let $\pi : X \to Y$ be a locally trivial fibering in Zariski topology. Denote by F the fiber over a closed point in Y. Then*

$$E(X;u,v) = E(Y;u,v) \cdot E(F;u,v).$$

We shall use Propositions 3.4 and 3.7 in the following situation. Let $\pi : X' \to X$ be a proper birational morphism of algebraic varieties X' and X. Let us further assume that X' is smooth and X has a stratification by locally closed subvarieties X_i ($i \in I$), such that each X_i is smooth and the restriction of π on $\pi^{-1}(X_i)$ is a locally trivial fibering over X_i in Zariski topology. Using Propositions 3.4 and 3.7, we can compute all Hodge numbers of X' as follows:

PROPOSITION 3.8. *Let F_i ($i \in I$) denote the fiber over a closed point of X_i. Then*

$$E(X'; u, v) = \sum_{i \in I} E(X_i; u, v) \cdot E(F_i; u, v).$$

We shall next deal with the case in which $\pi : \tilde{X} \to X$ represents a crepant resolution of singularities of an algebraic variety X having only Gorenstein singularities. The problem of main interest is to characterize the E-polynomials $E(F_i; u, v)$ in terms of singularities of X along the X_i's. This problem will be solved in the case when X has Gorenstein toroidal or quotient singularities.

4. LOCAL CREPANT TORIC RESOLUTIONS

We shall compute here the E-polynomials of the fibers of crepant toric resolution mappings of Gorenstein toric singularities by using their combinatorial description in terms of convex cones. It is assumed that the reader is familiar with the theory of toric varieties as it is presented, for instance, in the expository article of Danilov [9], or in the books of Oda [37] and Fulton [13].

Let M, N be two free abelian groups of rank d, which are dual to each other, and let $M_{\mathbf{R}}$ and $N_{\mathbf{R}}$ be their real scalar extensions. The type of every d-dimensional Gorenstein toroidal singularity can be described combinatorially by a d-dimensional cone $\sigma = \sigma_\Delta \subset N_{\mathbf{R}}$ which supports a $(d-1)$-dimensional lattice polyhedron $\Delta \subset N_{\mathbf{R}}$ [38]. This lattice polyhedron Δ can be defined as $\{x \in \sigma \mid \langle x, m_\sigma \rangle = 1\}$ for some uniquely determined element $m_\sigma \in M$. Let $\check{\sigma} \subset M_{\mathbf{R}}$ be dual to σ and set $\mathbf{A}_\sigma := \operatorname{Spec} \mathbf{C}[\check{\sigma} \cap M]$. Then $\mathbf{A}_\sigma$ is a d-dimensional affine toric variety with only Gorenstein singularities. We denote by $p = p_\sigma$ the unique torus invariant closed point in $\mathbf{A}_\sigma$.

Definition 4.1. A finite collection $\mathcal{T} = \{\theta\}$ of simplices with vertices in $\Delta \cap N$ is called a *triangulation* of Δ if the following properties are satisfied:

(i) if θ' is a face of $\theta \in \mathcal{T}$, then $\theta' \in \mathcal{T}$;
(ii) the intersection of any two simplices $\theta'_1, \theta'_2 \in \mathcal{T}$ is either empty, or a common face of both of them;
(iii) $\Delta = \bigcup_{\theta \in \mathcal{T}} \theta$.

Every triangulation $\mathcal{T}$ of Δ gives rise to a partial crepant toric desingularization $\pi_{\mathcal{T}} : X_{\mathcal{T}} \to \mathbf{A}_\sigma$ of $\mathbf{A}_\sigma$, so that $X_{\mathcal{T}}$ has at most abelian quotient Gorenstein singularities.

Definition 4.2. A simplex $\theta \subset \Delta \subset \{x \in N_{\mathbf{R}} \mid \langle x, m_\sigma \rangle = 1\}$ is called *regular* if its vertices form a part of a $\mathbf{Z}$-basis of N.

It is known (see, for instance, [37, Theorem 1.10, p. 15]) that $X_{\mathcal{T}}$ is smooth if and only if all simplices in $\mathcal{T}$ are regular.

THEOREM 4.3. *Assume that Δ admits a triangulation $\mathcal{T}$ into regular simplices; i.e., that the corresponding toric variety $X_{\mathcal{T}}$ in the crepant resolution*

$$\pi_{\mathcal{T}}: X_{\mathcal{T}} \to \mathbf{A}_\sigma$$

is smooth. Then $F = \pi_{\mathcal{T}}^{-1}(p)$ can be stratified by finitely many affine spaces.

Proof. Let θ_0 be an arbitrary $(d-1)$-dimensional simplex in $\mathcal{T}$ with vertices $e_1, \dots, e_d$. Choose a 1-parameter multiplicative subgroup $G_\omega \subset (\mathbf{C}^*)^d$ whose action on $\mathbf{A}_\sigma$ is defined by a weight-vector $\omega \in \sigma \cap N$, so that $\omega = \omega_1 e_1 + \cdots + \omega_d e_d$, where $\omega_1, \dots, \omega_d$ are positive integers. The action of G_ω on $\mathbf{A}_\sigma$ extends naturally to an action on $X_{\mathcal{T}}$. If $\{\theta_0, \theta_1, \dots, \theta_s\}$ denotes the set of all $(d-1)$-dimensional simplices in $\mathcal{T}$, then $\sigma = \bigcup_{i=0}^s \sigma_{\theta_i}$, and $X_{\mathcal{T}}$ is canonically covered by the corresponding G_ω-invariant open subsets $U_0, \dots, U_s$, so that $U_i \cong \mathbf{C}^d$. Denote by p_i $(i = 0, 1, \dots, s)$ the unique torus invariant point in U_i. We assume that ω has been already chosen in such a way that p_i is the unique G_ω-invariant point in U_i (this property holds if ω is not contained in some $(d-1)$-dimensional subspace generated by some integral vectors in Δ). We consider a multiplicative parameter t on G_ω for which the action of G_ω on U_0 is defined as follows:

$$t \cdot (x_1, \dots, x_d) := (t^{\omega_1} x_1, \dots, t^{\omega_d} x_d).$$

Furthermore, we set:

$$X_i := \{x \in U_i \mid \lim_{t \to \infty} t(x) = p_i\}.$$

Let $e_1^{(i)}, \dots, e_d^{(i)}$ be vertices of θ_i and $\omega = \omega_1^{(i)} e_1^{(i)} + \cdots + \omega_d^{(i)} e_d^{(i)}$. Without loss of generality, we can assume that $\omega_1^{(i)}, \dots, \omega_k^{(i)}$ are positive and $\omega_{k+1}^{(i)}, \dots, \omega_d^{(i)}$ are negative. Let $x_1^{(i)}, \dots, x_d^{(i)}$ be torus coordinates on U_i. G_ω acts by

$$t \cdot (x_1^{(i)}, \dots, x_k^{(i)}, x_{k+1}^{(i)}, \dots, x_d^{(i)}) = (t^{\omega_1^{(i)}} x_1^{(i)}, \dots, t^{\omega_k^{(i)}} x_k^{(i)}, t^{\omega_{k+1}^{(i)}} x_{k+1}^{(i)}, \dots, t^{\omega_d^{(i)}} x_d^{(i)}).$$

Hence X_i is defined by the equations $x_1^{(i)} = \cdots = x_k^{(i)} = 0$. Therefore, X_i is isomorphic to an affine space. (In particular, $X_0 = \{p_0\}$). Since $\pi_{\mathcal{T}}(p_i) = p$ and $\{x \in \mathbf{A}_\sigma \mid \lim_{t \to \infty} t(x) = p\} = \{p\}$, we have $X_i \subset F$. By compactness of F, for every point $x \in F$, there exists $\lim_{t \to \infty} t(x)$ which is a G_ω-invariant point; i.e., $\lim_{t \to \infty} t(x) = p_i$ for some i $(0 \leqslant i \leqslant s)$. So $\bigcup_{i=0}^s X_i = F$ with $X_i \cap X_j = \emptyset$ for $i \neq j$. □

Let $l(k\Delta)$ denote the number of lattice points of $k\Delta$. Then the *Ehrhart power series*

$$P_\Delta(t) := \sum_{k \geqslant 0} l(k\Delta) t^k$$

can be considered as a numerical characteristic of the toric singularity at p_σ. It is well known (see, for instance, [4, Theorem 2.11, p. 356]) that $P_\Delta(t)$ can always be written in the form:

$$P_\Delta(t) = \frac{\psi_0(\Delta) + \psi_1(\Delta)t + \cdots + \psi_{d-1}(\Delta)t^{d-1}}{(1-t)^d}$$

where $\psi_0(\Delta) = 1$ and $\psi_1(\Delta), \dots, \psi_{d-1}(\Delta)$ are certain nonnegative integers.

THEOREM 4.4. *Let Δ be as in* 4.3, *and* $F = \pi_{\mathcal{T}}^{-1}(p)$. *Then the cohomology groups* $H^{2i}(F, \mathbf{C})$, $i = 0, \ldots, d-1$ *are generated by the* (i,i)*-classes of algebraic cycles, and* $H_c^j(F, \mathbf{C}) = 0$ *for odd values of* j. *Moreover,* $h^{i,i}(F) = \psi_i(\Delta)$, $i = 0, \ldots, d-1$. *In particular, the dimensions* $h^{i,i}(F) = \dim H^{2i}(F, \mathbf{C})$ $(0 \leqslant i \leqslant d-1)$ *do not depend on the choice of the triangulation* $\mathcal{T}$.

Proof. The first statement follows immediately from Theorem 4.3. Since F is compact, we have $H^i(F, \mathbf{C}) = H_c^i(F, \mathbf{C})$. Therefore, it is sufficient to compute the E-polynomial

$$E(F; u, v) = \sum_{p,q} e^{p,q}(F) u^p v^q.$$

Since $X_{\mathcal{T}}$ is a toric variety, it admits a natural stratification by strata which are isomorphic to algebraic tori T_θ corresponding to regular subsimplices $\theta \in \mathcal{T}$, such that

$$\dim T_\theta + \dim \theta = d - 1.$$

The natural stratification of $X_{\mathcal{T}}$ induces a stratification of F. Notice that $\pi_{\mathcal{T}}(T_\theta) = p_\sigma$ (i.e., $T_\theta \in F$) if and only if θ does not belong to the boundary of Δ. If a_i denotes the number of i-dimensional regular simplices of $\mathcal{T}$ which do not belong to the boundary of Δ, then a_i can be identified with the number of $(d-1-i)$-dimensional tori in the natural stratification of $\pi_{\mathcal{T}}^{-1}(p)$. By 3.4, we get:

$$E(F; u, v) = \sum_{\pi_{\mathcal{T}}(T_\theta) = p} E(T_\theta; u, v).$$

Since $E((\mathbf{C}^*)^k; u, v) = (uv - 1)^k$, we obtain

$$E(F; u, v) = a_0(uv - 1)^{d-1} + a_1(uv - 1)^{d-2} + \cdots + a_{d-1}.$$

Now we compute $P_\Delta(t)$ by using the numbers a_i. If $\theta \in \mathcal{T}$ is an i-dimensional regular simplex, then

$$l(k\theta) = \binom{k+i}{k}; \quad \text{i.e.,} \quad P_\theta(t) = \frac{1}{(1-t)^{i+1}}.$$

Applying the usual inclusion–exclusion principle for the counting of lattice points of $k\Delta$, we obtain:

$$l(k\Delta) = \sum_{i=0}^{d-1} \sum_{\dim\theta = d-1-i} (-1)^i l(k\theta)$$

where θ runs over all regular simplices on $\mathcal{T}$ which do not belong to the boundary of Δ. Thus,

$$P_\Delta(t) = \frac{a_{d-1}}{(1-t)^d} - \frac{a_{d-2}}{(1-t)^{d-1}} + \cdots + (-1)^{d-1} \frac{a_0}{(1-t)}$$

and the polynomial

$$\psi_0(\Delta) + \psi_1(\Delta)t + \cdots + \psi_{d-1}(\Delta)t^{d-1} = P_\Delta(t)(1-t)^d$$

is equal to

$$a_{d-1} + a_{d-2}(t-1) + \cdots + a_0(t-1)^{d-1}.$$

The latter coincides with the E-polynomial $E(F; u, v)$ after making the substitution $t = uv$. Hence, $\psi_i(\Delta) = e^{i,i}(F)$ $(0 \leqslant i \leqslant d-1)$. □

Definition 4.5. Let Δ be a $(d-1)$-dimensional lattice polyhedron defining a d-dimensional Gorenstein toric singularity $p \in \mathbf{A}_\sigma$. Then

$$S(\Delta; uv) := \psi_0(\Delta) + \psi_1(\Delta)uv + \cdots + \psi_{d-1}(\Delta)(uv)^{d-1}$$

will be called the *S-polynomial* of the Gorenstein toric singularity at p.

COROLLARY 4.6. *The Euler number $e(F)$ equals $S(\Delta, 1) = (d-1)!\,\mathrm{vol}(\Delta)$.*

Proof. By definition of $P_\Delta(t)$,

$$\psi_0(\Delta) + \psi_1(\Delta) + \cdots + \psi_{d-1}(\Delta) = (d-1)!\,\mathrm{vol}(\Delta).$$

Obviously, the left hand side equals $e(F)$. □

Remark 4.7. It is known that the coefficient $\psi_{d-1}(\Delta)$ equals $l^*(\Delta)$, i.e., the number of rational points in the interior of Δ (see [10, pp. 292–293]).

5. GORENSTEIN QUOTIENT SINGULARITIES

Let G be a finite subgroup of $SL(d, \mathbf{C})$. We shall use the fact that any element $g \in G$ is obviously conjugate to a diagonal matrix.

Definition 5.1. If an element $g \in G$ is conjugate to

$$\mathrm{diag}(e^{2\pi\sqrt{-1}\alpha_1}, \ldots, e^{2\pi\sqrt{-1}\alpha_d})$$

with $\alpha_i \in \mathbf{Q} \cap [0, 1)$, then the sum

$$wt(g) := \alpha_1 + \cdots + \alpha_d$$

will be called the *weight* of the element $g \in G$. The number $ht(g) := rk(g - e)$ will be called the *height* of g.

PROPOSITION 5.2. *For any $g \in G$, one has*

$$wt(g) + wt(g^{-1}) = ht(g) = ht(g^{-1}).$$

Proof. Let $g = \mathrm{diag}(e^{2\pi\sqrt{-1}\alpha_1}, \ldots, e^{2\pi\sqrt{-1}\alpha_d})$, $g^{-1} = \mathrm{diag}(e^{2\pi\sqrt{-1}\beta_1}, \ldots, e^{2\pi\sqrt{-1}\beta_d})$. Then $ht(g)$ equals the number of nonzero elements in $\{\alpha_1, \ldots, \alpha_d\}$. On the other hand, $\alpha_i + \beta_i = 1$ if $\alpha_i \neq 0$, and $\alpha_i + \beta_i = 0$ otherwise. Hence $\sum_{i=1}^{d} (\alpha_i + \beta_i) = ht(g)$. □

CONJECTURE 5.3 (strong McKay correspondence). *Let $G \subset SL(d, \mathbf{C})$ be a finite group. Assume that $X = C^d/G$ admits a smooth crepant desingularization $\pi: \hat{X} \to X$ and $F := \pi^{-1}(0)$. Then $H^*(F, \mathbf{C})$ has a basis consisting of classes of algebraic cycles $Z_{\{g\}} \subset F$ which are in 1-to-1 correspondence with the conjugacy classes $\{g\}$ of G, so that*

$$\dim H^{2i}(F, \mathbf{C}) = \#\{\text{conjugacy classes } \{g\} \subset G, \text{ such that } wt(g) = i\}.$$

Now we give several evidences in support of Conjecture 5.3.

THEOREM 5.4. *Let $G \subset SL(d, \mathbf{C})$ be a finite abelian group. Suppose that $X = \mathbf{C}^d/G$ admits a smooth crepant toric desingularization $\pi: \hat{X} \to X$ and $F := \pi^{-1}(0)$. Then $H^*(F, \mathbf{C})$ has* *a basis consisting of classes of algebraic cycles $Z_g \subset F$ which are in 1-to-1 correspondence with the elements g of G, so that*

$$\dim H^{2i}(F,\mathbf{C}) = \#\{\text{elements } g \in G, \text{ such that } wt(g) = i\}.$$

In particular, the Euler number of F equals $|G|$.

Proof. Let $N \subset \mathbf{R}^d$ be the free abelian group generated by $\mathbf{Z}^d \subset \mathbf{R}^d$ and all vectors $(\alpha_1, \dots, \alpha_d)$ where $g = \mathrm{diag}(e^{2\pi\sqrt{-1}\alpha_1}, \dots, e^{2\pi\sqrt{-1}\alpha_d})$ runs over all the elements of G. Then N is a full sublattice of $\mathbf{R}^d = N_{\mathbf{R}}$, $\mathbf{Z}^d$ is a subgroup of finite index in N, and $N/\mathbf{Z}^d$ is canonically isomorphic to G. Let $M = \mathrm{Hom}(N,\mathbf{Z})$. We identify $\mathbf{Z}^d$ with $\mathrm{Hom}(\mathbf{Z}^d,\mathbf{Z})$ by using the dual basis. M is a canonical sublattice of $\mathbf{Z}^d$ and can be identified with the set of all Laurent monomials in variables $t_1, \dots, t_d$ which are G-invariant. Therefore, the cone σ defining the affine toric variety $X = \mathbf{A}_\sigma$ is the positive d-dimensional octant $\mathbf{R}^d_{\geqslant 0} \subset \mathbf{R}^d = N_{\mathbf{R}}$. Furthermore, the element $m_\sigma \in M$, which was mentioned at the beginning of the previous section, equals $(1, \dots, 1) \in \mathbf{Z}^d$. Now if $S := \mathbf{C}[\sigma \cap N]$ and if for any $x \in \sigma \cap N$, we define a *degree* $\deg x := \langle x, m_\sigma \rangle$, S becomes a graded ring, so that

$$n_1 := (1,0, \dots, 0), \dots, \quad n_d := (0, \dots, 0, 1)$$

form a regular sequence of elements of degree 1 in S. This means that $S/(n_1, \dots, n_d)$ has a monomial basis corresponding to those elements of N which are not in $\mathbf{Z}^d$. The element $(\alpha_1, \dots, \alpha_d) \in N$ corresponds precisely to the element $g = \mathrm{diag}(e^{2\pi\sqrt{-1}\alpha_1}, \dots, e^{2\pi\sqrt{-1}\alpha_d}) \in G$. Moreover,

$$\langle (\alpha_1, \dots, \alpha_d), m_\sigma \rangle = wt(g).$$

Thus, the Poincaré series of the quotient ring $S/(n_1, \dots, n_d)$ equals

$$\psi_0(\Delta) + \psi_1(\Delta)t + \cdots + \psi_{d-1}(\Delta)t^{d-1}$$

with coefficients

$$\psi_i(\Delta) = \#\{\text{elements } g \in G \text{ such that } wt(g) = i\}$$

and $\sigma = \sigma_\Delta$ as in Section 4. The proof is completed after making use of Theorem 4.4 and Corollary 4.6. □

Example 5.5. For an abelian finite group $G \subset SL(3,\mathbf{C})$, the quotient $X = \mathbf{C}^3/G$ admits always smooth crepant toric desingularizations coming from the full triangulations of the corresponding triangle Δ which is determined by n_1, n_2, n_3. All these triangulations contain only regular simplices and each of them differs from the other ones by finitely many elementary transformations (cf. [37, Proposition 1.30 (ii)]). In particular, if G is a cyclic group generated by

$$\mathrm{diag}(e^{\frac{2\pi\sqrt{-1}\lambda_1}{|G|}}, e^{\frac{2\pi\sqrt{-1}\lambda_2}{|G|}}, e^{\frac{2\pi\sqrt{-1}\lambda_3}{|G|}})$$

with

$$0 < \lambda_1, \lambda_2, \lambda_3 < |G|, \qquad \lambda_1 + \lambda_2 + \lambda_3 = |G|, \qquad gcd(\lambda_1, \lambda_2, \lambda_3) = 1,$$

then:

$$\dim H^0(F,\mathbf{C}) = 1, \qquad \dim H^1(F,\mathbf{C}) = \dim H^3(F,\mathbf{C}) = \dim H^5(F,\mathbf{C}) = 0,$$

$$\dim H^2(F,\mathbf{C}) = \frac{1}{2}\left(|G| + \sum_{i=1}^{3} gcd(\lambda_i, |G|)\right) - 2$$

and

$$\dim H^4(F,\mathbf{C}) = \frac{1}{2}\left(|G| - \sum_{i=1}^{3} gcd(\lambda_i, |G|) \right) + 1.$$

PROPOSITION 5.6. *The Conjecture* 5.3 *is true for* $d \leqslant 3$.

Proof. If $d = 2$, then $wt(g) = 1$ unless $g = e$. The number of the conjugacy classes with weight 1 is equal to the number of nontrivial irreducible representations of G. Since the exceptional locus F of a crepant resolution is a tree of rational curves, $\dim H^0(F,\mathbf{C}) = 1$, and $\dim H^2(F,\mathbf{C})$ is the number of irreducible components of F. By the classical McKay correspondence [14, 29, 34], we obtain the statement 5.3.

If $d = 3$, we use the result of Roan [44] about the existence of crepant resolutions and the Euler number of the exceptional locus. Let F be the exceptional locus over 0 of a crepant resolution $\pi: \hat{X} \to X$. Then F is a strong deformation retract of $\hat{X}$; i.e., $H^i(F,\mathbf{C}) = H^i(\hat{X},\mathbf{C})$. On the other hand, $H^4(\hat{X},\mathbf{C})$ is Poincaré dual to $H_c^2(\hat{X},\mathbf{C})$. Note that $\dim H^4(F,\mathbf{C})$ is nothing but the number of irreducible two-dimensional components of F. Since $H^2(\hat{X},\mathbf{Z})$ is isomorphic to the Picard group of $\hat{X}$, $\dim H^2(\hat{X},\mathbf{C})$ is equal to the number of π-exceptional divisors. Moreover, the subspace $H_c^2(\hat{X},\mathbf{C}) \subset H^2(\hat{X},\mathbf{C})$ is spanned exactly by the classes of those exceptional divisors whose image under π is 0. Therefore,

$$\dim H^2(\hat{X},\mathbf{C}) - \dim H^4(\hat{X},\mathbf{C}) = \#\{\text{exceptional divisors } E \subset \hat{X}, \text{ such that } \pi(E) \text{ is a curve on } X\}.$$

By the classical McKay correspondence in dimension 2,

$$\dim H^2(\hat{X},\mathbf{C}) - \dim H^4(\hat{X},\mathbf{C}) = \#\{\text{conjugacy classes } \{g\} \subset G, \text{ such that } wt(g) = 1 \text{ and } ht(g) = 2\}.$$

By [44],

$$1 + \dim H^2(\hat{X},\mathbf{C}) + \dim H^4(\hat{X},\mathbf{C}) = \#\{\text{all conjugacy classes } \{g\} \subset G\}. \tag{4}$$

By 5.2,

$$\#\{\text{conjugacy classes } \{g\} \subset G, \text{ with } wt(g) = 1 \text{ and } ht(g) = 3\}$$
$$= \#\{\text{conjugacy classes } \{g\} \subset G, \text{ with } wt(g) = 2 \text{ and } ht(g) = 3\}.$$

Hence,

$$\#\{\text{conjugacy classes } \{g\} \subset G, \text{ with } wt(g) = 2 \text{ and } ht(g) = 3\} = \dim H^4(\hat{X},\mathbf{C}).$$

Notice that if $wt(g) = 2$, then the height of g must be equal to 3. Thus,

$$\dim H^4(F,\mathbf{C}) = \#\{\text{conjugacy classes } \{g\} \subset G, \text{ such that } wt(g) = 2\}.$$

Finally,

$$\dim H^2(F,\mathbf{C}) = \#\{\text{conjugacy classes } \{g\} \subset G, \text{ such that } wt(g) = 1\}$$

follows immediately from (4). □

Definition 5.7. Let G be a finite subgroup of $SL(d,\mathbf{C})$ and $0 \in \mathbf{C}^d/G$ the corresponding d-dimensional Gorenstein toric singularity. If we denote by $\psi_i(G)$ the number of the conjugacy classes of G having the weight i, then

$$S(G; uv) := \psi_0(G) + \psi_1(G)uv + \cdots + \psi_{d-1}(G)(uv)^{d-1}$$

will be called the *S-polynomial* of the regarded Gorenstein quotient singularity at 0.

Definition 5.8. Let G be a finite subgroup of $SL(d, \mathbf{C})$ and $0 \in \mathbf{C}^d/G$ the corresponding d-dimensional Gorenstein toric singularity. If we denote by $\tilde{\psi}_i(G)$ the number of the conjugacy classes of G having the weight i, and the height d, then

$$\tilde{S}(G; uv) := \tilde{\psi}_0(G) + \tilde{\psi}_1(G)uv + \cdots + \tilde{\psi}_{d-1}(G)(uv)^{d-1}$$

will be called the *$\tilde{S}$-polynomial* of the Gorenstein quotient singularity at 0.

By 5.2, we easily obtain:

PROPOSITION 5.9. *The $\tilde{S}$-polynomial satisfies the following reciprocity relation*:

$$\tilde{S}(G; uv) = (uv)^d \tilde{S}(G; (uv)^{-1}).$$

6. STRING-THEORETIC HODGE NUMBERS

Let X be a compact d-dimensional Gorenstein variety with Sing X consisting of at most toroidal or quotient singularities.

Definition 6.1. Let $x \in Sing\ X$. We say that the d-dimensional singularity at x has the *splitting codimension* k, if k is the maximal number for which the analytic germ at x is locally isomorphic to the product of $\mathbf{C}^{d-k}$ and a k-dimensional toric singularity defined by a $(k-1)$-dimensional lattice polyhedron Δ' or, correspondingly, to the product of $\mathbf{C}^{d-k}$ and the underlying space $\mathbf{C}^k/G'$ of a k-dimensional quotient singularity defined by a finite subgroup $G' \subset SL(k, \mathbf{C})$. For simplicity, we also say that the singularity at x is defined by Δ', or by G'.

Using standard arguments, we can easily show that X is always stratified by locally closed subvarieties X_i $(i \in I)$, such that the germs of the singularities of X along X_i are analytically isomorphic to those of a Gorenstein toric singularity defined by means of a $(k-1)$-dimensional lattice polytope Δ_i or to those of a quotient singularity defined by means of a finite subgroup G_i of $SL(k, \mathbf{C})$, respectively, where k denotes the splitting codimension of singularities on X_i.

Definition 6.2. We denote by $S(X_i; uv)$ the S-polynomial $S(\Delta_i; uv)$ or $S(G_i; uv)$. Analogously, $\tilde{S}(X_i; uv)$ will denote the $\tilde{S}$-polynomial $\tilde{S}(G_i; uv)$ if X_i has only Gorenstein quotient singularities.

Definition 6.3. Suppose that X has at most quotient Gorenstein singularities. A stratification $X = \bigcup_{i \in I} X_i$, as above, is called *canonical*, if for every $i \in I$ and every $x \in X_i$, there exists an open subset $U \cong \mathbf{C}^d/G_i$ in X and an element $g \in G_i$, such that $\overline{X_i} \cap U = (\mathbf{C}^d)^g/C(g)$, where $(\mathbf{C}^d)^g$ is the set of g-invariant points of $\mathbf{C}^d$.

Remark 6.4. An algebraic variety is called *V-variety* if it has at most quotient singularities. A *Gorenstein V-variety* (abbreviated *GV-variety*) is a V-variety having at most Gorenstein quotient singularities. The notion of V-variety (or V-manifold) was first introduced by Satake [47]. The existence and the uniqueness of the canonical stratification for a V-variety was proved by Kawasaki [27]. (Note that our *canonical* stratification in Definition 6.3 is not the first, but the second stratification of X defined by Kawasaki in [27, p. 77].)

PROPOSITION 6.5. *Suppose that X is a GV-variety and $X = \bigcup_{i\in I} X_i$ is its canonical stratification. Then for any $i_0 \in I$, one has:*

$$S(X_{i_0}; uv) = \tilde{S}(X_{i_0}; uv) + \sum_{X_{i_0} < X_{i_1}} \tilde{S}(X_{i_1}; uv).$$

Proof. It is sufficient to prove the corresponding local statement; i.e., we can assume, without loss of generality, that $X_{i_0} = \mathbf{C}^k/G_{i_0}$. For simplicity, we set $Y = \mathbf{C}^k$, $Z = X_{i_0}$. Denote by π the natural mapping $Y \to Z$. For $g \in G_{i_0}$, the image $Z(g) := \pi(Y^g) \subset Z$ depends only on the conjugacy class of g. Since $ht(g)$ equals the codimension of $Z(g)$ in Z, we obtain

$$S(X_{i_0}; uv) = \tilde{S}(X_{i_0}; uv) + \sum_{X_{i_0} < X_{i_1}} \tilde{S}(X_{i_1}; uv). \qquad \square$$

COROLLARY 6.6. *Suppose that X is a GV-variety and $X = \bigcup_{i\in I} X_i$ is its canonical stratification. Then for any $i_0 \in I$ one has:*

$$\tilde{S}(X_{i_0}; uv) = \sum_{k \geq 0} (-1)^k \sum_{X_{i_0} < \cdots < X_{i_k}} S(X_{i_k}; uv).$$

Proof. By Propositon 6.5, we have

$$\tilde{S}(X_{i_0}; uv) = S(X_{i_0}; uv) - \sum_{X_{i_0} < X_{i_1}} \tilde{S}(X_{i_1}; uv).$$

After that we apply Proposition 6.5 to X_{i_1}:

$$\tilde{S}(X_{i_1}; uv) = S(X_{i_1}; uv) - \sum_{X_{i_1} < X_{i_2}} \tilde{S}(X_{i_2}; uv), \text{ etc.}$$

The repetition of this procedure completes the proof of the assertion. $\square$

Definition 6.7. Let X be a stratified variety with at most Gorenstein toroidal or quotient singularities. We shall call the polynomial

$$E_{\text{st}}(X; u, v) := \sum_{i\in I} E(X_i; u, v) \cdot S(X_i; uv)$$

the *string-theoretic E-polynomial of X*. Let us write $E_{\text{st}}(X; u, v)$ in the following expanded form:

$$E_{\text{st}}(X; u, v) = \sum_{p,q} a_{p,q} u^p v^q.$$

The numbers $h_{\text{st}}^{p,q}(X) := (-1)^{p+q} a_{p,q}$ will be called the *string-theoretic Hodge numbers of X*. Correspondingly,

$$e_{\text{st}}(X) := E_{\text{st}}(X; -1, -1) = \sum_{p,q} (-1)^{p+q} h_{\text{st}}^{p,q}(X)$$

will be called the *string-theoretic Euler number of X*.

Remark 6.8. If X admits a smooth crepant toroidal desingularization $\pi:\hat{X} \to X$, then, by Proposition 3.8 and Theorem 4.4, the E-polynomial of $\hat{X}$ equals

$$E(\hat{X};u,v) = \sum_{i\in I} E(X_i;u,v)\cdot E(F_i;u,v)$$

where F_i denotes a the special fiber $\pi^{-1}(x)$ over a point $x \in X_i$.

By Remark 6.8, we obtain:

THEOREM 6.9. *If X admits, a smooth crepant toroidal desingularization $\hat{X}$, then the string-theoretic Hodge numbers $h^{p,q}_{\rm st}(X)$ coincide with the ordinary Hodge numbers $h^{p,q}(\hat{X})$. In particular, the numbers $h^{p,q}_{\rm st}(X)$ are nonnegative and satisfy the Poincaré duality $h^{p,q}_{\rm st}(X) = h^{d-p,d-q}_{\rm st}(X)$.*

The next theorem will play an important role in the forthcoming statements:

THEOREM 6.10. *Suppose that X is a GV-variety and $X = \bigcup_{i\in I} X_i$ denotes its canonical stratification. Then*

$$E_{\rm st}(X;u,v) = \sum_{i\in I} E(\bar{X}_i;u,v)\cdot \tilde{S}(X_i;uv).$$

Proof. By Proposition 3.6, we get

$$E(X_{i_0};u,v) = \sum_{k\geqslant 0} (-1)^k \sum_{X_{i_k}<\cdots<X_{i_1}<X_{i_0}} E(\bar{X}_{i_k};u,v).$$

Therefore,

$$\begin{aligned} E_{\rm st}(X;u,v) &= \sum_{i_0\in I}\left(\sum_{k\geqslant 0}(-1)^k \sum_{X_{i_k}<\cdots<X_{i_1}<X_{i_0}} E(\bar{X}_{i_k};u,v)\right)\cdot S(X_{i_0};uv)\\ &= \sum_{i_k\in I} E(\bar{X}_{i_k};u,v)\cdot\left(\sum_{k\geqslant 0}(-1)^k \sum_{X_{i_k}<\cdots<X_{i_1}<X_{i_0}} S(X_{i_0};uv)\right). \end{aligned}$$

By Corollary 6.6, we have

$$\tilde{S}(X_{i_k};uv) = \sum_{k\geqslant 0}(-1)^k\left(\sum_{X_{i_k}<\cdots<X_{i_1}<X_{i_0}} S(X_{i_0};uv)\right).$$

This implies the required formula. □

COROLLARY 6.11. *Suppose that X is a GV-variety. Then the numbers $h^{p,q}_{\rm st}(X)$ are nonnegative and satisfy the Poincaré duality $h^{p,q}_{\rm st}(X) = h^{d-p,d-q}_{\rm st}(X)$.*

Proof. Since $\bar{X}_i$ itself is a V-variety, one has $h^{p,q}(\bar{X}_i) \geqslant 0$, as well as the Poincaré duality

$$E(\bar{X}_i;u,v) = (uv)^{\dim \bar{X}_i} E(\bar{X}_i;u^{-1},v^{-1}).$$

On the other hand, by Proposition 5.9, we obtain

$$\tilde{S}(X_i;uv) = (uv)^{\dim \bar{X}_i}\tilde{S}(X_i;(uv)^{-1}).$$

This implies

$$E_{st}(X;u,v) = (uv)^{\dim X} E_{st}(X;u^{-1},v^{-1}).$$

Since $\tilde{S}(X_i;uv)$ is a polynomial of uv with nonnegative coefficients, we conclude that $h_{st}^{p,q}(X) \geq 0$. □

THEOREM 6.12. *Suppose that X has at most toroidal Gorenstein singularities. Let $\pi:\hat{X} \to X$ be a MPCP-desingularization of X. Then*

$$E_{st}(X;u,v) = E_{st}(\hat{X};u,v).$$

Moreover,

$$h_{st}^{p,1}(X) = h^{p,1}(\hat{X}), \quad \textit{for all } p.$$

Proof. Let $X = \bigcup_{i \in I} X_i$ be a stratification of X, such that

$$E_{st}(X;u,v) = \sum_{i \in I} E(X_i;u,v) \cdot S(\Delta_i;uv)$$

and $\pi:\hat{X} \to X$ be a MPCP-desingularization of X. We set $\hat{X}_i := \pi^{-1}(X_i)$. Then $\hat{X}_i$ has the natural stratification by products $X_i \times (\mathbf{C}^*)^{\operatorname{codim}\theta}$ induced by the triangulation

$$\Delta_i = \bigcup_{\theta \in \mathcal{T}_i} \theta.$$

Thus,

$$E_{st}(\hat{X};u,v) = \sum_{i \in I} \left(\sum_{\theta \in \mathcal{T}_i} (uv-1)^{\operatorname{codim}\theta} E(X_i;u,v) \cdot S(\theta;uv) \right).$$

By counting lattice points in $k\Delta_i$, we obtain

$$S(\Delta_i;uv) = \sum_{\theta \in \mathcal{T}_i} (uv-1)^{\operatorname{codim}\theta} S(\theta;uv).$$

Hence,

$$E_{st}(X;u,v) = E_{st}(\hat{X};u,v).$$

Since $\hat{X}$ has only terminal $\mathbf{Q}$-factorial singularities, for any $\theta \in \mathcal{T}_i$ we obtain

$$\psi_1(\theta) = 0; \quad \text{i.e.,} \quad S(\theta;uv) = 1 + \psi_2(\theta)(uv)^2 + \cdots.$$

Therefore, the coefficient of $u^p v$ in $E_{st}(\hat{X};u,v)$ coincides with the coefficient of $u^p v$ in the usual E-polynomial $E(\hat{X};u,v)$. As $\hat{X}$ is a V-variety, the Hodge structure in $H^*(\hat{X},\mathbf{C})$ is pure, and

$$h_{st}^{p,1}(X) = h^{p,1}(\hat{X}), \quad \text{for all } p. \qquad \square$$

COROLLARY 6.13. *Suppose that X has at most toroidal Gorenstein singularities. Then the numbers $h_{st}^{p,q}(X)$ are nonnegative and satisfy the Poincaré duality $h_{st}^{p,q}(X) = h_{st}^{d-p,d-q}(X)$.*

Proof. By Theorem 6.12, it is sufficient to prove the statement for an *MPCP*-desingularization $\hat{X}$ of X. The latter follows from Corollary 6.11. □

THEOREM 6.14. *Let X be a smooth compact Kähler manifold of dimension n over $\mathbf{C}$ being equipped with an action of a finite group G, such that X has a G-invariant volume form. Then* *the orbifold Hodge numbers $h^{p,q}(X, G)$ which were defined in the introduction coincide with the string-theoretic Hodge numbers $h_{\rm st}^{p,q}(X/G)$. Moreover,*

$$e(X, G) = e_{\rm st}(X/G).$$

Proof. We use the canonical stratification of $Y = X/G$:

$$Y = \bigcup_{i \in I} Y_i.$$

For every stratum Y_i, there exists an element $g_i \in G$, such that $\bar{Y}_i = X^{g_i}/C(g_i)$. We note that

$$E(\bar{Y}_i; u, v) = \sum_{p,q} (-1)^{p+q} \dim H^{p,q}(X^{g_i})^{C(g_i)} u^p v^q.$$

Now the equality

$$h_{\rm st}^{p,q}(X/G) = h^{p,q}(X, G)$$

follows from Theorem 6.10.

In order to get $e(X, G) = e_{\rm st}(X/G)$, it remains to prove the equality

$$e(X, G) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X, G).$$

We shall make use of the notation which was introduced in Section 1. Since $\{g\}$ expresses a system of representatives for $G/C(g)$ and the number of conjugacy classes of G equals

$$\frac{1}{|G|} \sum_{g \in G} |C(g)|$$

one can rewrite the physicists Euler number (1) as

$$e(X, G) = \frac{1}{|G|} \sum_{g} |C(g)| \cdot e(X^g/C(g)) = \sum_{\{g\}} e(X^g/C(g))$$

where $\{g\}$ runs over all conjugacy classes of G with g representing $\{g\}$. We show that

$$\sum_{p,q} (-1)^{p+q} h_g^{p,q}(X, G) = e(X^g/C(g)).$$

This follows from the equalities

$$\begin{aligned} \sum_{p,q} (-1)^{p+q} h_g^{p,q}(X, G) &= \sum_{i=1}^{r_g} \sum_{p,q} (-1)^{p+q-2F_i(g)} h_{C(g)}^{p-F_i(g), q-F_i(g)}(X_i(g)) \\ &= \sum_{p,q} (-1)^{p+q} h_{C(g)}^{p,q}(X^g) = e(X^g/C(g)). \end{aligned}$$

□

COROLLARY 6.15. *Suppose that X/G has a crepant desingularization $\widehat{X/G}$ and that the strong McKay correspondence (Conjecture 5.3) holds true for the singularities occurring along every stratum of X/G. Then*

$$h^{p,q}(\widehat{X/G}) = h_{\rm st}^{p,q}(X/G).$$

Example 6.16. Let us first give a three-dimensional example of an orbit space (with a simple acting group) containing both abelian and non-abelian quotient singularities, and which was proposed by F. Hirzebruch. We consider the Fermat quintic

$X = \{[z_1, \dots, z_5] \in \mathbf{P}^4 \mid \sum_{i=1}^5 z_i^5 = 0\}$ and let the alternating group $\mathscr{A}_5$ act on it coordinatewise. The group $\mathscr{A}_5$ has five conjugacy classes: the trivial, the one consisting of all 20 3-cycles, one consisting of the 15 products of disjoint transpositions, and two more conjugacy classes of 5-cycles, each of which has 12-elements. Note that the action of the elements of these last two conjugacy classes is fixed point free. Each of the 20 3-cycles fixes a plane quintic and two additional points. Correspondingly, each of the 15 products of disjoint transpositions fixes a plane quintic and a projective line (without common points). As $X/\mathscr{A}_5$ is a Calabi–Yau variety, the generic points of the one-dimensional components of Sing $X/\mathscr{A}_5$ are compound Du Val points [38]. Up to the above-mentioned 40 additional points coming from the 3-cycles and having isotropy groups $\cong \mathbf{Z}/3\mathbf{Z}$, there exist 175 more fixed points on X creating (after appropriate group identifications) *dissident* points on $X/\mathscr{A}_5$ (we follow here the terminology of M. Reid). Namely, the 25 points of the intersection locus of the 20 plane quintics (with isotropy groups $\cong \mathscr{A}_4$), a further 125 points lying in the intersection locus of the 15 plane quintics (with isotropy groups $\cong \mathscr{S}_3$), as well as $15 + 10 = 25$ points coming from the intersection of the projective lines (with isotropy groups isomorphic to the Kleinian four-group and to $\mathscr{S}_3$, respectively). Using Ito's results [24, 25], we can construct global crepant desingularizations $\pi: \widehat{X/\mathscr{A}_5} \to X/\mathscr{A}_5$. By Corollary 6.15, $h^{p,q}(\widehat{X/\mathscr{A}_5}) = h^{p,q}_{\mathrm{st}}(X/\mathscr{A}_5)$. Thus, for the computation of $h^{p,q}(\widehat{X/\mathscr{A}_5})$, we just need to choose two representatives, say (123) and (12) (34), of the two nonfreely acting conjugacy classes. We have:

- $h^{p,q}(X/\mathscr{A}_5) = h^{p,q}_{\{1\}}(X, \mathscr{A}_5)$ equals $\delta_{p,q}$ (=Kronecker symbol) for $p + q \neq 3$, $h^{p,q}_{\{1\}}(X, \mathscr{A}_5) = 1$ for $(p,q) \in \{(3,0), (0,3)\}$ and $h^{p,q}_{\{1\}}(X, \mathscr{A}_5) = 5$ for $(p,q) \in \{(2,1), (1,2)\}$;
- $h^{p,q}_{\{(123)\}}(X, \mathscr{A}_5)$ equals 2 for $(p,q) \in \{(1,1), (2,2)\}$, $h^{p,q}_{\{(123)\}}(X, \mathscr{A}_5) = 6$ for $(p,q) \in \{(2,1), (1,2)\}$, $h^{p,q}_{\{(123)\}}(X, \mathscr{A}_5) = 0$ otherwise;
- $h^{p,q}_{\{(12)(34)\}}(X, \mathscr{A}_5)$ equals 2 for $1 \leqslant p, q \leqslant 2$ and 0 otherwise.

Thus, we get

$$h^{2,1}_{\mathrm{st}}(X/\mathscr{A}_5) = h^{1,2}_{\mathrm{st}}(X/\mathscr{A}_5) = 13,$$

$$h^{1,1}_{\mathrm{st}}(X/\mathscr{A}_5) = h^{2,2}_{\mathrm{st}}(X/\mathscr{A}_5) = 5.$$

In particular, $e(\widehat{X/\mathscr{A}_5}) = e_{\mathrm{st}}(X/\mathscr{A}_5) = -16$, in agreement with the calculations of physicists (cf. [28, p. 57]).

Example 6.17. Let $X^{(n)} := X^n/\mathscr{S}_n$ be the nth symmetric power of a smooth projective surface X. As it is known (see, for instance, [16, p. 54] or [23, p. 258]), $X^{(n)}$ is endowed with a canonical crepant desingularization $X^{[n]} := \mathit{Hilb}^n(X) \to X^{(n)}$ given by the Hilbert scheme of finite subschemes of length n. In [15, 16], Göttsche computed the Poincaré polynomial of $X^{[n]}$. In particular, his formula for the Euler number gives:

$$\sum_{n=0}^{\infty} e(X^{[n]}) t^n = \prod_{k=1}^{\infty} (1 - t^k)^{-e(X)}.$$

Using power series comparison and the above formula, Hirzebruch and Höfer gave in [23] a formal proof of the equality $e(X^{[n]}) = e(X^{(n)}, \mathscr{S}_n)$. In fact, for the proof of the validity of *orbifold Euler formulae* of this kind, it is enough to check locally that the Conjecture 1.1 of M. Reid is true (cf. [44, Lemma 1]). Our results Theorem 6.14 and Corollary 6.15 say more: in order to obtain the equality $h^{p,q}(X^{[n]}) = h^{p,q}(X^{(n)}, G)$ it is sufficient to verify locally our "strong" McKay correspondence. The latter has been checked by Göttsche in [17]. The numbers $h^{p,q}(X^{[n]})$ can be computed by means of the Hodge polynomial $h(X^{[n]}; u, v) := E(X^{[n]}; -u, -v)$.

If $\Pi(n)$ denotes the set of all finite series $(\alpha) = (\alpha_1, \alpha_2, \dots)$ of nonnegative integers with $\sum_i i\alpha_i = n$, then the conjugacy class of a permutation $\sigma \in \mathscr{S}_n$ is determined by its type $(\alpha) = (\alpha_1, \alpha_2, \dots) \in \Pi(n)$, where α_i expresses the number of cycles of length i in σ. Göttsche and Soergel [16, 18] proved that

$$h(X^{[n]}; u, v) = \sum_{(\alpha) \in \Pi(n)} (uv)^{n - |\alpha|} \prod_{k=1}^{\infty} h(X^{(\alpha_k)}; u, v),$$

where $|\alpha| := \alpha_1 + \alpha_2 + \cdots$ denotes the sum of the members of $(\alpha) \in \Pi(n)$.

(Similar formulae can be obtained for the even-dimensional Kummer varieties of higher order, cf. [16–18].)

7. APPLICATIONS TO QUANTUM COHOMOLOGY AND MIRROR SYMMETRY

From now on, and throughout this section, we use the notion of *reflexive polyhedron* being introduced in [4].

PROPOSITION 7.1. *Let Δ be a reflexive polyhedron of dimension d. Then*

$$S(\Delta, t) = (t-1)^d + \sum_{\substack{0 \leqslant \dim \theta \leqslant d-1 \\ \theta \subset \Delta}} S(\theta, t) \cdot (t-1)^{\dim \theta^*}.$$

Proof. Denote by $\partial\Delta$ the $(d-1)$-dimensional boundary of Δ which is homeomorphic to the $(d-1)$-dimensional sphere. Let $l(k \cdot \partial\Delta)$ be the number of lattice points belonging to the boundary of $k\Delta$. The reflexivity of Δ implies:

$$l(k \cdot \partial\Delta) = \sum_{0 \leqslant \dim \theta \leqslant d-1} (-1)^{d-1-\dim \theta} l(k\theta), \quad \text{for } k > 0.$$

Since the Euler number of a $(d-1)$-dimensional sphere is $1 + (-1)^{d-1}$, we obtain

$$(-1)^{d-1} + (1-t) P_\Delta(t) = (-1)^{d-1} \sum_{0 \leqslant \dim \theta \leqslant d-1} (-1)^{\dim \theta} P_\theta(t),$$

i.e.,

$$(-1)^{d-1} + \frac{S(\Delta; t)}{(1-t)^d} = (-1)^{d-1} \sum_{0 \leqslant \dim \theta \leqslant d-1} (-1)^{\dim \theta} \frac{S(\theta; t)}{(1-t)^{\dim \theta + 1}}.$$

This implies the required equality. □

We prove the following relation between the polar duality of lattice polyhedra and string-theoretic cohomology:

THEOREM 7.2. *Let $\mathbf{P}_\Delta$ be a d-dimensional Gorenstein toric Fano variety corresponding to a d-dimensional reflexive polyhedron Δ. Then*

$$E_{\rm st}(\mathbf{P}_\Delta; u, v) = (1 - uv)^{d+1} P_{\Delta^*}(uv)$$

where Δ^* is the dual reflexive polyhedron.

Proof. $\mathbf{P}_\Delta$ has a natural stratification being defined by the strata T_θ, where θ runs over all the faces of Δ. On the other hand, the Gorenstein singularities along T_θ are determined by the dual face θ^* of the dual polyhedron Δ^* (cf. [4, 4.2.4]). We set $S(\theta^*, uv) = 1$ if $\theta = \Delta$.

Then

$$E_{\rm st}(\mathbf{P}_\Delta; u, v) = \sum_{\theta \subset \Delta} E(T_\theta; u, v) \cdot S(\theta^*; uv).$$

Note that

$$E(T_\theta; u, v) = (uv - 1)^{\dim \theta}$$

and that, for $\dim \theta < d$, one has by definition:

$$S(\theta^*; uv) = (1 - uv)^{\dim \theta^* + 1} P_{\theta^*}(uv).$$

If we apply Proposition 7.1 to the dual reflexive polyhedron Δ^*, then, using $\dim \theta + \dim \theta^* = d - 1$, we obtain the desired formula for $E_{\rm st}(\mathbf{P}_\Delta; u, v)$. □

COROLLARY 7.3. *The string-theoretic Euler number of* $\mathbf{P}_\Delta$ *is equal to* $d!(\mathrm{vol}\, \Delta^*)$.

Remark 7.4. The quantum cohomology ring of a smooth toric variety was described in [3]. It was proved that the usual cohomology of a smooth toric manifold can be obtained as a limit of the quantum cohomology ring. On the other hand, one can immediately extend the description of the quantum cohomology ring to an arbitrary (possibly singular) toric variety (cf. [3, 5.1]). In particular, one can easily show that $\dim QH^*_\varphi(\mathbf{P}_\Delta, \mathbf{C}) = d!(\mathrm{vol}\, \Delta^*)$, for any d-dimensional reflexive polyhedron. Comparing dimensions, we see that, for singular toric Fano varieties $\mathbf{P}_\Delta$, the limit of the quantum cohomology ring is not the usual cohomology ring, but rather the cohomology of a smooth crepant desingularization, if such a desingularization exists (cf. [3, 6.5]). By our general philosophy, we should consider the string-theoretic Hodge numbers $h^{p,p}_{\rm st}(\mathbf{P}_\Delta)$ as the Betti numbers of a limit of the quantum cohomology ring $QH^*_\varphi(\mathbf{P}_\Delta, \mathbf{C})$.

Let $\bar{Z}_f := \bar{Z}_{f_1} \cap \cdots \cap \bar{Z}_{f_r}$ be a generic $(d - r)$-dimensional Calabi–Yau complete intersection variety, which is embedded in a Gorenstein toric Fano variety $\mathbf{P}_\Delta$ corresponding to a d-dimensional reflexive polyhedron $\Delta = \Delta_1 + \cdots + \Delta_r$, where Δ_i is the Newton polyhedron of f_i $(i = 1, \ldots, r)$. Assume that the lattice polyhedra $\Delta_1, \ldots, \Delta_r$ are defined by a *nef-partition* of vertices of the dual reflexive polyhedron $\Delta^* = \mathrm{Conv}\{\nabla_1, \ldots, \nabla_r\}$. (For definitions and notations the reader is referred to [5, 7].) Denote by $\bar{Z}_g := \bar{Z}_{g_1} \cap \cdots \cap \bar{Z}_{g_r}$, a generic Calabi–Yau complete intersection variety in the Gorenstein toric Fano variety $\mathbf{P}_{\nabla^*}$, which is defined by the reflexive polyhedron $\nabla^* = \mathrm{Conv}\{\Delta_1, \ldots, \Delta_r\}$, where ∇_i is the Newton polyhedron of g_i $(i = 1, \ldots, r)$.

CONJECTURE 7.5 (Mirror duality of string-theoretic Hodge numbers). *The string-theoretic E-polynomials of* $\bar{Z}_f$ *and* $\bar{Z}_g$ *obey the following reciprocity law:*

$$E_{\rm st}(\bar{Z}_f; u, v) = (-u)^{d-r} E_{\rm st}(\bar{Z}_g; u^{-1}, v).$$

Equivalently, the string-theoretic Hodge numbers of $\bar{Z}_f$ *and* $\bar{Z}_g$ *are related to each other by:*

$$h^{p,q}_{\rm st}(\bar{Z}_f) = h^{d-r-p,q}_{\rm st}(\bar{Z}_g), \quad \text{for all } p, q.$$

We want to show some evidence in support of Conjecture 7.5 for Calabi–Yau hypersurfaces $(r = 1)$.

THEOREM 7.6. *Let $\bar{Z}_f$ be a Δ-regular Calabi–Yau hypersurface in $\mathbf{P}_\Delta$. Then*

$$E_{st}(\bar{Z}_f; 1, v) = \frac{S(\Delta^*; v)}{v} + (-1)^{d-1}\frac{S(\Delta; v)}{v} + \sum_{\substack{1 \leqslant \dim\theta \leqslant d-2 \\ \theta \subset \Delta}} \frac{(-1)^{\dim\theta - 1}}{v}(S(\theta; v)\cdot S(\theta^*; v))$$

$$- \sum_{\substack{\dim\theta = d-1 \\ \theta \subset \Delta}} (-1)^{d-1}\frac{S(\theta, v)}{v} - \sum_{\substack{\dim\theta^* = d-1 \\ \theta^* \subset \Delta^*}} \frac{S(\theta^*, v)}{v}.$$

COROLLARY 7.7.

$$E_{st}(\bar{Z}_f; 1, v) = (-1)^{d-1}E_{st}(\bar{Z}_g; 1, v).$$

At first we need the following formula:

PROPOSITION 7.8. *Let θ be a face of Δ and* $\dim\theta \geqslant 1$. *Then*

$$E(Z_{f,\theta}; 1, v) = \frac{(v-1)^{\dim\theta}}{v} + (-1)^{\dim\theta - 1}\frac{S(\theta, v)}{v}.$$

Proof. It follows from the formula of Danilov and Khovanskiĭ ([10, Remark 4.6]):

$$(-1)^{\dim\theta - 1}\sum_p e^{p,q}(Z_{f,\theta}) = (-1)^q\binom{n}{q+1} + \psi_{q+1}(\theta).$$ □

Proof of Theorem 7.6. By definition,

$$E_{st}(\bar{Z}_f; 1, v) = E(Z_{f,\Delta}; 1, v) + \sum_{\substack{\dim\theta = d-1 \\ \theta \subset \Delta}} E(Z_{f,\theta}; 1, v)$$

$$+ \sum_{\substack{1 \leqslant \dim\theta \leqslant d-2 \\ \theta \subset \Delta}} E(Z_{f,\theta}; 1, v)\cdot S(\theta^*; v).$$

Substituting the expressions which follow from Proposition 7.8, we get:

$$E(\bar{Z}_{f,\Delta}; 1, v) = \frac{(v-1)^d}{v} + (-1)^{d-1}\frac{S(\Delta, v)}{v} + \sum_{\substack{\dim\theta = d-1 \\ \theta \subset \Delta}} \left((-1)^{d-2}\frac{S(\theta, v)}{v} + \frac{(v-1)^{d-1}}{v}\right)$$

$$+ \sum_{\substack{1 \leqslant \dim\theta \leqslant d-2 \\ \theta \subset \Delta}} \left((-1)^{\dim\theta - 1}\frac{S(\theta, v)}{v} + \frac{(v-1)^{\dim\theta}}{v}\right)\cdot S(\theta^*; uv).$$

It remains to use Propositions 7.8 and 7.1 □

Definition 7.9. For a face θ of Δ, we denote by $\mathbf{v}(\theta)$ the normalized volume of θ: $(\dim\theta)!\,\mathrm{vol}(\theta)$.

COROLLARY 7.10. *Let Δ be a d-dimensional reflexive polyhedron. Then*

$$e_{st}(\bar{Z}_f) = \sum_{i=1}^{d-2} \sum_{\dim\theta = i} (-1)^i \mathbf{v}(\theta)\cdot\mathbf{v}(\theta^*).$$

In particular,

$$e_{st}(\bar{Z}_f) = (-1)^{d-1}e_{st}(\bar{Z}_g).$$

We remark that Conjecture 7.5 is evident if $q = 0$, because $h_{\rm st}^{p,0}(\bar{Z}_f) = 1$, for $q = 0, d-1$ and $h_{\rm st}^{p,0}(\bar{Z}_f) = 0$ otherwise. For $q = 1$ ($r = 1$), and $p \in \{1, d-2\}$, Conjecture 7.5 is proved by Theorem 6.12 combined with Theorem 4.4.3 from [4]. We generalize this for arbitrary values of p.

THEOREM 7.11. *For a face θ of Δ, we denote by $l^*(\theta)$ the number of lattice points in the relative interior of θ. Assume that $d \geqslant 5$.* Then for $2 \leqslant p \leqslant d-3$ *one has*

$$h_{\rm st}^{p,1}(\bar{Z}_f) = \sum_{\operatorname{codim}\theta = p} l^*(\theta)\cdot l^*(\theta^*).$$

By the duality among faces, one has

$$h_{\rm st}^{p,1}(\bar{Z}_f) = h_{\rm st}^{d-1-p,1}(\bar{Z}_g).$$

Proof. By the Poincaré duality, it is enough to compute $h_{\rm st}^{d-1-p,d-2}(\bar{Z}_f) = h_{\rm st}^{p,1}(\bar{Z}_f)$. We use

$$E_{\rm st}(\bar{Z}_f; u, v) = \sum_{\theta \subset \Delta} E(Z_{f,\theta}; u, v)\cdot S(\theta^*; uv).$$

By Remark 4.7,

$$S(\theta^*; uv) = l^*(\theta^*)(uv)^{\dim\theta^*} + \{\text{lower order terms in } uv\}.$$

On the other hand, by [10, Proposition 3.9],

$$e^{p,q}(Z_{f,\theta}) = 0 \quad \text{if } p + q > \dim\theta - 1 = \dim Z_{f,\theta} \text{ and } p \neq q.$$

Hence, the only possible case in which we can meet the monomial of type $u^{d-1-p}v^{d-2}$ within the product $E(Z_{f,\theta}; u, v)\cdot S(\theta^*; u, v)$ is that occurring by consideration of the product of the term $l^*(\theta^*)(uv)^{\dim\theta^*}$ from $S(\theta^*; uv)$ and the term

$$e^{0,\dim\theta-1}(Z_{f,\theta})v^{\dim\theta-1}$$

where $\dim\theta^* = d-1-p$. As it is known (cf. [10, Proposition 5.8]):

$$e^{0,\dim\theta-1}(Z_{f,\theta}) = (-1)^{\dim\theta-1} l^*(\theta).$$

Therefore,

$$h_{\rm st}^{d-1-p,d-2}(\bar{Z}_f) = l^*(\theta)\cdot l^*(\theta^*). \qquad \square$$

COROLLARY 7.12. *Let $\hat{Z}_f$ be an MPCP-desingularization of $\bar{Z}_f$. Assume that $d \geqslant 5$.* Then, for $2 \leqslant p \leqslant d-3$, *one has*

$$h^{p,1}(\hat{Z}_f) = \sum_{\operatorname{codim}\theta = p} l^*(\theta)\cdot l^*(\theta^*).$$

Proof. It follows from Theorems 7.11 and 6.12. $\square$

8. DUALITY OF STRING-THEORETIC HODGE NUMBERS FOR THE GREENE–PLESSER CONSTRUCTION

In [20, 21] Greene and Plesser proposed an explicit construction of mirror pairs of Calabi–Yau orbifolds which are obtained as abelian quotients of Fermat hypersurfaces in weighted projective spaces. As it was shown in [4, 5.5], the Greene–Plesser construction can be interpreted in terms of the polar duality of *reflexive simplices*. The main purpose of this section is to verify the mirror duality of all string-theoretic Hodge numbers for this construction.

From now on, we assume that Δ and Δ^* are d-dimensional reflexive simplices. We shall prove Conjecture 7.5 for Δ-regular Calabi–Yau hypersurfaces in $\mathbf{P}_\Delta$ and $\mathbf{P}_{\Delta^*}$. (We recall that, for this kind of hypersurface and for $d = 4$, Conjecture 7.5 was proved in [4, 41].)

Definition 8.1. Let Θ be a k-dimensional lattice simplex. We denote by $\tilde{S}(\Theta; uv)$ the $\tilde{S}$-polynomial of the $(k+1)$-dimensional abelian quotient singularity defined by Θ. We denote the corresponding finite abelian subgroup of $SL(k+1, \mathbf{C})$ by G_Θ (in the sense of Sections 4 and 5).

Our main statement is an immediate consequence of the following:

THEOREM 8.2. *Let $\bar{Z}_f$ be a Δ-regular Calabi–Yau hypersurface in $\mathbf{P}_\Delta$. Then*

$$E_{\rm st}(\bar{Z}_f; u, v) = \frac{1}{uv}\tilde{S}(\Delta^*; uv) + (-1)^{d-1}\frac{u^d}{v}\tilde{S}(\Delta; u^{-1}v) + \sum_{\substack{1 \leqslant \dim\theta \leqslant d-2 \\ \theta \subset \Delta}} (-1)^{\dim\theta - 1}\left(\frac{u^{\dim\theta}}{v}\tilde{S}(\theta; u^{-1}v)\cdot\tilde{S}(\theta^*; uv)\right).$$

Indeed, if we apply Theorem 8.2 to the dual polyhedron Δ^*, then we get

$$E_{\rm st}(\bar{Z}_g; u, v) = \frac{1}{uv}\tilde{S}(\Delta; uv) + (-1)^{d-1}\frac{u^d}{v}\tilde{S}(\Delta^*; u^{-1}v) + \sum_{\substack{1 \leqslant \dim\theta^* \leqslant d-2 \\ \theta^* \subset \Delta^*}} (-1)^{\dim\theta^* - 1}\left(\frac{u^{\dim\theta^*}}{v}\tilde{S}(\theta^*; u^{-1}v)\cdot\tilde{S}(\theta; uv)\right).$$

Now the required equality

$$E_{\rm st}(\bar{Z}_f; u, v) = (-u)^{d-1}E_{\rm st}(\bar{Z}_g; u^{-1}, v)$$

follows evidently from the 1-to-1 correspondence $\theta \leftrightarrow \theta^*$ ($1 \leqslant \dim\theta$, $\dim\theta^* \leqslant d-1$) and from the property: $\dim\theta + \dim\theta^* = d - 1$.

For the proof of Theorem 8.2, we need some preliminary facts.

PROPOSITION 8.3. *Let θ be a face of Δ and* $\dim\theta \geqslant 1$. *Then*

$$E(Z_{f,\theta}; u, v) = \frac{(uv-1)^{\dim\theta} - (-1)^{\dim\theta}}{uv} + (-1)^{\dim\theta - 1}\left(\sum_{\substack{\dim\tau \geqslant 1 \\ \tau \subset \theta}} \frac{u^{\dim\tau}}{v}\tilde{S}(\tau; u^{-1}v)\right).$$

Proof. By [10, Proposition 3.9], the natural mapping

$$H_c^i(Z_{f,\theta}) \to H_c^{i+1}(T_\theta)$$

is an isomorphism if $i > \dim\theta - 1$ and surjective if $i = \dim\theta - 1$. Moreover, $H_c^i(Z_{f,\theta}) = 0$ if $i < \dim\theta - 1$. In order to compute the mixed Hodge structure in $H_c^{\dim\theta - 1}(Z_{f,\theta})$, we use the explicit description of the weight filtration in $H_c^{\dim\theta-1}(Z_{f,\theta})$ (see [2]). Note that if we choose a θ-regular Laurent polynomial f containing only $\dim\theta+1$ monomials associated with vertices of θ (such a polynomial f defines a Fermat-type hypersurface $\bar{Z}_f$ in $\mathbf{P}_\theta$), then the corresponding Jacobian ring R_f has a monomial basis. Thus, the weight filtration on R_f can be described in terms of the partition of monomials in R_f which is defined by the faces $\tau\subset\theta$. To get the claimed formula, it suffices to identify the partition of monomials in R_f with the height-partition of elements of the finite abelian group $G_\theta\subset SL(\dim\theta+1,\mathbf{C})$ and its subgroups $G_\tau\subset G_\theta$.

Another way to obtain the same result is to use the formulae of Danilov and Khovanskiĭ (cf. [10, Section 5.6, 5.7]) which are valid for an arbitrary simple polyhedron Δ. □

PROPOSITION 8.4. *Let θ be a face of Δ and* $\dim\theta\geqslant 1$. *Then*

$$S(\theta;t)=1+\sum_{\substack{\dim\eta\geqslant 1\\ \eta\subset\theta}}\tilde{S}(\eta;t).$$

Proof. It is similar to that of Proposition 6.5. □

PROPOSITION 8.5. *We fix a face $\tau\subset\Delta$ and a face $\eta\subset\Delta^*$, such that τ is a face of η^*. Then*

$$\sum_{0,\tau\subset\theta\subset\eta^*}(-1)^{\dim\theta}=(-1)^{\dim\tau}\quad \text{if } \tau=\eta^*$$

and

$$\sum_{0,\tau\subset\theta\subset\eta^*}(-1)^{\dim\theta}=0\quad \text{if } \tau\neq\eta^*.$$

Proof. If $\eta^*=\tau$, this is obvious. For $\dim\eta^*>\dim\tau$, the number of faces $\theta\subset\Delta$ of fixed dimension, for which $\tau\subset\theta\subset\eta^*$ is equal to $\binom{\dim\eta^*-\dim\tau}{\dim\theta-\dim\tau}$. It remains to use the equality

$$\sum_{0,\tau\subset\theta\subset\eta^*}(-1)^{\dim\theta}=(-1)^{\dim\tau}\left(\sum_{i=0}^{\dim\eta^*-\dim\tau}(-1)^i\binom{\dim\eta^*-\dim\tau}{i}\right)=0.\quad\square$$

PROPOSITION 8.6.

$$\frac{1}{uv}\tilde{S}(\Delta;uv)=\frac{(uv)^d-1}{uv-1}+\sum_{\substack{1\leqslant\dim\tau\leqslant d-2\\ \tau\subset\Delta}}\left(\frac{(uv)^{\dim\tau^*}-1}{uv-1}\right)\cdot\tilde{S}(\eta;uv).$$

Proof. By Proposition 7.1, we have

$$(-1)^{d-1}+\frac{S(\Delta;t)}{(1-t)^d}=(-1)^{d-1}\sum_{0\leqslant\dim\theta\leqslant d-1}(-1)^{\dim\theta}\frac{S(\theta;t)}{(1-t)^{\dim\theta+1}}.$$

Applying Proposition 8.4 to both sides of this equality, we get

$$(-1)^{d-1}+\frac{1}{(1-t)^d}+\sum_{\substack{\dim\tau\geqslant 1\\ \tau\subset\Delta}}\frac{\tilde{S}(\tau;t)}{(1-t)^d}$$

$$=(-1)^{d-1}\sum_{0\leqslant\dim\theta\leqslant d-1}\frac{(-1)^{\dim\theta}}{(1-t)^{\dim\theta+1}}+(-1)^{d-1}\sum_{0\leqslant\dim\theta\leqslant d-1}(-1)^{\dim\theta}\sum_{\dim\tau\geqslant 1}\frac{\tilde{S}(\tau;t)}{(1-t)^{\dim\theta+1}}.$$

As the number of k-dimensional faces of Δ equals $\binom{d+1}{k+1}$, we have

$$-(-1)^{d-1} - \frac{1}{(1-t)^d} + (-1)^d \sum_{0 \leqslant \theta \leqslant d-1} \frac{(-1)^{\dim \theta}}{(1-t)^{\dim \theta + 1}}$$

$$= -(-1)^{d-1} - \frac{1}{(1-t)^d} + \sum_{k=0}^{d-1} \frac{(-1)^k}{(1-t)^{k+1}} \binom{d+1}{k+1} = (-1)^d \frac{t^{d+1}-t}{(t-1)^{d+1}}$$

and we can deduce that:

$$\frac{\tilde{S}(\Delta,t)}{(1-t)^d} + \sum_{\dim \tau = d-1} \frac{\tilde{S}(\tau,t)}{(1-t)^d} + \sum_{1 \leqslant \dim \tau \leqslant d-2} \frac{\tilde{S}(\tau,t)}{(1-t)^d}$$

$$= (-1)^d \frac{t^{d+1}-t}{(t-1)^{d+1}} + \sum_{\dim \tau = d-1} \frac{\tilde{S}(\tau,t)}{(1-t)^d} + \sum_{\dim \theta = d-1} \sum_{\substack{1 \leqslant \dim \tau \leqslant d-2 \\ \tau \subset \theta}} \frac{\tilde{S}(\tau,t)}{(1-t)^d}$$

$$+ (-1)^{d-1} \sum_{1 \leqslant \dim \theta \leqslant d-2} (-1)^{\dim \theta} \sum_{\substack{\dim \tau \geqslant 1 \\ \tau \subset \theta}} \frac{\tilde{S}(\tau,t)}{(1-t)^{\dim \theta + 1}}.$$

The terms containing $\tilde{S}(\tau,t)$, with $\dim \tau = d-1$, have the same contribution to the right- and left-hand sides. The coefficient of $\tilde{S}(\tau,t)$ ($1 \leqslant \dim \tau \leqslant d-2$) in the right-hand side of the last equality is

$$(-1)^{d-1} \sum_{\substack{\dim \theta \leqslant d-2 \\ \tau \subset \theta}} (-1)^{\dim \theta} \frac{1}{(1-t)^{\dim \theta + 1}} = \frac{(-1)^d}{(t-1)^{d+1}} (t^{d - \dim \tau} - 1 - (d - \dim \tau)(t-1)).$$

Correspondingly, the coefficient of $\tilde{S}(\tau,t)$ ($1 \leqslant \dim \tau \leqslant d-2$) in the left-hand side equals

$$\frac{d-1-\dim \tau}{(1-t)^d}.$$

Finally, using $\dim \tau + \dim \tau^* = d-1$, we obtain:

$$\frac{\tilde{S}(\Delta,t)}{(1-t)^d} = (-1)^d \frac{t^{d+1}-t}{(t-1)^{d+1}} + (-1)^d \sum_{1 \leqslant \tau \leqslant d-2} \tilde{S}(\tau,t) \frac{(t^{\dim \tau^* + 1} - t)}{(t-1)^{d+1}}. \qquad \square$$

Proof of Theorem 8.2. By definition,

$$E_{\text{st}}(\bar{Z}_f; u, v) = E(Z_{f,\Delta}; u, v) + \sum_{\substack{\dim \theta = d-1 \\ \theta \subset \Delta}} E(Z_{f,\theta}; u, v) + \sum_{\substack{1 \leqslant \dim \theta \leqslant d-2 \\ \theta \subset \Delta}} E(Z_{f,\theta}; u, v) \cdot S(\theta^*; uv).$$

Substituting the expressions from Proposition 8.3 for the E-polynomials of the above three summands, we get:

$$E(Z_{f,\Delta}; u, v) = \frac{(uv-1)^d - (-1)^d}{uv} + (-1)^{d-1} \left(\sum_{\substack{\dim \tau \geqslant 1 \\ \tau \subset \Delta}} \frac{u^{\dim \tau}}{v} \hat{S}(\tau; u^{-1}v) \right)$$

$$\sum_{\substack{\dim \theta = d-1 \\ \theta \subset \Delta}} E(Z_{f,\theta}; u, v) = \sum_{\substack{\dim \theta = d-1 \\ \theta \subset \Delta}} \frac{(uv-1)^{\dim \theta} - (-1)^{\dim \theta}}{uv}$$

$$+ \sum_{\substack{\dim \theta = d-1 \\ \theta \subset \Delta}} (-1)^{\dim \theta - 1} \left(\sum_{\substack{\dim \tau \geqslant 1 \\ \tau \subset \theta}} \frac{u^{\dim \tau}}{v} S(\tau; u^{-1}v) \right)$$

and

$$\sum_{\substack{1 \leq \dim \theta \leq d-2 \\ \theta \subset \Delta}} E(Z_{f,\theta}; u, v) \cdot S(\theta^*; uv) = \sum_{\substack{1 \leq \dim \theta \leq d-2 \\ \theta \subset \Delta}} \frac{(uv-1)^{\dim \theta} - (-1)^{\dim \theta}}{uv}$$

$$+ \sum_{\substack{1 \leq \dim \theta \leq d-2 \\ \theta \subset \Delta}} (-1)^{\dim \theta - 1} \left(\sum_{\substack{\dim \tau \geq 1 \\ \tau \subset \theta}} \frac{u^{\dim \tau}}{v} \tilde{S}(\tau; u^{-1}v) \right)$$

$$\left(1 + \sum_{\substack{\dim \eta \geq 1 \\ \eta \subset \theta^*}} \tilde{S}(\eta; uv) \right).$$

Hene, $E_{st}(\bar{Z}_f; u, v)$ can be written as the sum of the following four terms E_i ($i = 1, 2, 3, 4$):

$$E_1 = \sum_{\substack{1 \leq \dim \theta \\ \theta \subset \Delta}} \frac{(uv-1)^{\dim \theta} - (-1)^{\dim \theta}}{uv}$$

$$E_2 = \sum_{\substack{1 \leq \dim \theta \\ \theta \subset \Delta}} (-1)^{\dim \theta - 1} \left(\sum_{\substack{\dim \tau \geq 1 \\ \tau \subset \theta}} \frac{u^{\dim \tau}}{v} \tilde{S}(\tau; u^{-1}v) \right)$$

$$E_3 = \sum_{\substack{1 \leq \dim \theta \leq d-2 \\ \theta \subset \Delta}} \left(\frac{(uv-1)^{\dim \theta} - (-1)^{\dim \theta}}{uv} \right) \cdot \left(\sum_{\substack{\dim \eta \geq 1 \\ \eta \subset \theta^*}} \tilde{S}(\eta; uv) \right)$$

and

$$E_4 = \sum_{\substack{1 \leq \dim \theta \leq d-2 \\ \theta \subset \Delta}} (-1)^{\dim \theta - 1} \left(\sum_{\substack{\dim \tau \geq 1 \\ \tau \subset \theta}} \frac{u^{\dim \tau}}{v} \tilde{S}(\tau; u^{-1}v) \right) \cdot \left(\sum_{\substack{\dim \eta \geq 1 \\ \eta \subset \theta^*}} \tilde{S}(\eta; uv) \right).$$

By Proposition 8.5, we can simplify the the multiple summation into a single sum:

$$E_4 = \sum_{\substack{1 \leq \dim \theta \leq d-2 \\ \theta \subset \Delta}} (-1)^{\dim \theta - 1} \left(\frac{u^{\dim \theta}}{v} \tilde{S}(\theta; u^{-1}v) \cdot \tilde{S}(\theta^*; uv) \right).$$

If we make use of the combinatorial identity

$$\sum_{\substack{1 \leq \dim \theta \\ \theta \subset \Delta}} a^{\dim \theta} = \sum_{k=2}^{d+1} \binom{d+1}{k} a^{k-1} = a^{-1-}((a+1)^{d+1} - 1 - (d+1)a)$$

we obtain:

$$E_1 = \sum_{\substack{1 \leq \dim \theta \\ \theta \subset \Delta}} \frac{(uv-1)^{\dim \theta} - (-1)^{\dim \theta}}{uv}$$

$$= [uv(uv-1)]^{-1}((uv)^{d+1} - 1 - (d+1)(uv-1)) + d(uv)^{-1} = \frac{(uv)^d - 1}{uv - 1}.$$

By Proposition 8.5, we get

$$E_2 = \sum_{\substack{1 \leq \dim \theta \\ \theta \subset \Delta}} (-1)^{\dim \theta - 1} \sum_{\substack{\dim \tau \geq 1 \\ \tau \subset \theta}} \frac{u^{\dim \tau}}{v} \tilde{S}(\tau; u^{-1}v) = (-1)^{d-1} \frac{u^d}{v} \tilde{S}(\Delta; u^{-1}v).$$

It remains to compute E_3. As above for E_1, we have

$$\sum_{\substack{1 \leq \dim \theta \\ \theta \subset \eta^*}} \frac{(uv-1)^{\dim \theta} - (-1)^{\dim \theta}}{uv} = \frac{(uv)^{\dim \eta^*} - 1}{uv - 1}.$$

Hence, by Proposition 8.6,

$$E_3 = \sum_{\substack{1 \leq \dim \eta \leq d-2 \\ \eta \subset \Delta^*}} \left(\frac{(uv)^{\dim \eta^*} - 1}{uv - 1}\right) \cdot \tilde{S}(\eta; uv) = \frac{1}{uv} \tilde{S}(\Delta^*; uv) - \frac{(uv)^d - 1}{uv - 1}.$$

Finally, we get altogether

$$E_{\mathrm{st}}(\bar{Z}_f; u, v) = \frac{1}{uv} \tilde{S}(\Delta^*; uv) + (-1)^{d-1} \frac{u^d}{v} \tilde{S}(\Delta; u^{-1}v)$$
$$+ \sum_{\substack{1 \leq \dim \theta \leq d-2 \\ \theta \subset \Delta}} (-1)^{\dim \theta - 1} \left(\frac{u^{\dim \theta}}{v} \tilde{S}(\theta; u^{-1}v) \cdot \tilde{S}(\theta^*; uv)\right). \qquad \square$$

Example 8.7. The polar duality between reflexive simplices shows (cf. [4, Theorem 5.1.1]) that the family of all smooth Calabi–Yau hypersurfaces X_{d+1} of degree $d+1$ in $\mathbf{P}^d$ has its mirror partner the one-parameter family $\{Q_{d+1}(\lambda)/G_{d+1}\}$, where

$$Q_{d+1}(\lambda) := \left\{[z_0, \ldots, z_d] \in \mathbf{P}^d \mid \sum_{i=0}^{d} z_i^{d+1} - (d+1)\lambda \prod_{i=0}^{d} z_i = 0\right\}$$

denotes the so called *Dwork pencil* and G_{d+1} the acting finite abelian group

$$G_{d+1} := \left\{(\alpha_0, \ldots, \alpha_d) \in (\mathbf{Z}/(d+1)\mathbf{Z})^{d+1} \mid \prod_{i=0}^{d} \alpha_i = 1\right\} \Big/ \{\textit{scalars}\},$$

which is abstractly isomorphic to $(\mathbf{Z}/(d+1)\mathbf{Z})^{d-1}$. The moduli space $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ of $\{Q_{d+1}(\lambda)/G_{d+1}\}_\lambda$ can be described by means of the parameter λ^{d+1} (cf. [19, Section 3.1], [35, Section 5], and [36, Section 11]).

Since Conjecture 7.5 is true for the case under consideration, the quotient $Q_{d+1}(\lambda)/G_{d+1}$ has the following string-theoretic Hodge numbers:

$$h_{\mathrm{st}}^{p,q}(Q_{d+1}(\lambda)/G_{d+1}) = h^{p,q}(Q_{d+1}(\lambda), G_{d+1}) = h^{d-1-p,q}(X_{d+1}) = \delta_{d-1-p,q} \quad \text{for } p \neq q;$$

$$h_{\mathrm{st}}^{p,p}(Q_{d+1}(\lambda)/G_{d+1}) = h^{p,p}(Q_{d+1}(\lambda), G_{d+1}) = h^{d-1-p,p}(X_{d+1})$$
$$= \sum_{i=0}^{p} (-1)^i \binom{d+1}{i} \binom{(p+1-i)d+p}{d} + \delta_{2p,d-1}.$$

In particular, the string-theoretic Euler number is given by:

$$e_{\mathrm{st}}(Q_{d+1}(\lambda)/G_{d+1}) = e(Q_{d+1}(\lambda), G_{d+1}) = -e(X_{d+1})$$
$$= \frac{1}{d+1}((-1)^{d+2} \cdot d^{d+1} + 1) - d - 1.$$

The first two equalities follow from Lefschetz hyperplane section theorem and from the "four-term formula" (cf. [22, Section 2.2]). The third one can be obtained directly by computing the $(d-1)$-th Chern class of X_{d+1}.

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