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Toric log del Pezzo surfaces with one singularity

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Abstract: This paper focuses on the classification (up to isomorphism) of all toric log Del Pezzo surfaces with exactly one singularity, and on the description of how they are embedded as intersections of finitely many quadrics into suitable projective spaces.

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1 Introduction

A smooth compact complex surface X is called a *del Pezzo surface* if its anticanonical divisor $-K_X$ is ample, i.e. if the rational map $\Phi_{|-mK_X|} : X \dashrightarrow \mathbb{P}(|-mK_X|)$ associated to a base point free linear system $|-mK_X|$ becomes a closed embedding with $\mathcal{O}_X(-mK_X) \cong \Phi^*_{|-mK_X|}(\mathcal{O}_{\mathbb{P}(|-mK_X|)}(1))$, for a suitable positive integer m, where $\mathcal{O}_X(-mK_X)$ is the corresponding invertible sheaf and $\mathcal{O}_{\mathbb{P}(|-mK_X|)}(1)$ the standard twisting sheaf. (Pasquale del Pezzo [16] initiated the study of these surfaces in 1887.) The *degree* $\deg(X)$ of a del Pezzo surface X is defined to be the self-intersection number $(-K_X)^2$. The main classification result about these surfaces can be stated as follows, see [32, Theorem 24.4, pp. 119–121]:

Theorem 1.1. *Let X be a del Pezzo surface of degree $d := \deg(X)$. Then $1 \leq d \leq 9$, and X is classified by d:*

(i) *If $d = 9$, then X is isomorphic to the projective plane $\mathbb{P}^2_{\mathbb{C}}$.*
(ii) *If $d = 8$, then X is isomorphic either to $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ or to the blow-up of the projective plane $\mathbb{P}^2_{\mathbb{C}}$ at one point.*
(iii) *If $1 \leq d \leq 7$, then X is isomorphic to the blow-up of the projective plane $\mathbb{P}^2_{\mathbb{C}}$ at $9 - d$ points.*

For $6 \leq d \leq 9$, such an X is *toric*, i.e. it contains a 2-dimensional algebraic torus $\mathbb{T}$ as a dense open subset, and is equipped with an algebraic action of $\mathbb{T}$ on X which extends the natural action of $\mathbb{T}$ on itself. Taking into account the description of smooth compact toric surfaces by the ($\mathbb{Z}$-weighted) circular graphs (introduced in [37, Chapter I, §8], [38, pp. 42–46] as well as [3, Proposition 6] and [41, Proposition 2.7]), Oda expresses in [38, Proposition 2.21, pp. 88–89] this fact in the language of toric geometry as follows:

Theorem 1.2. *There exist five distinct toric del Pezzo surfaces up to isomorphism. They correspond to the circular graphs* (*with weights* $-1, 0, 1$) *shown in Figure* 1. *They are* (i) $\mathbb{P}^2_{\mathbb{C}}$, (ii) $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ ($\cong \mathbb{F}_0$), (iii) *the Hirzebruch surface* $\mathbb{F}_1$, (iv) *the equivariant blow-up of* $\mathbb{P}^2_{\mathbb{C}}$ *at two of the* $\mathbb{T}$*-fixed points, and* (v) *the equivariant blow-up of* $\mathbb{P}^2_{\mathbb{C}}$ *at the three* $\mathbb{T}$*-fixed points.*

Note 1.3. The *Hirzebruch surfaces*

$$\mathbb{F}_\kappa := \{([z_0 : z_1 : z_2], [t_1 : t_2]) \in \mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \mid z_1 t_1^\kappa = z_2 t_2^\kappa\} \quad \text{with } \kappa \in \mathbb{Z}_{\geq 0}$$

introduced in [26, §2] are toric. Usually $\mathbb{F}_\kappa$ is identified with the total space $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\kappa))$ of the $\mathbb{P}^1_{\mathbb{C}}$-bundle of degree κ over $\mathbb{P}^1_{\mathbb{C}}$. Furthermore, every smooth compact toric surface which has Picard number 2 is necessarily isomorphic to a Hirzebruch surface; cf. [38, Corollary 1.29, p. 45].

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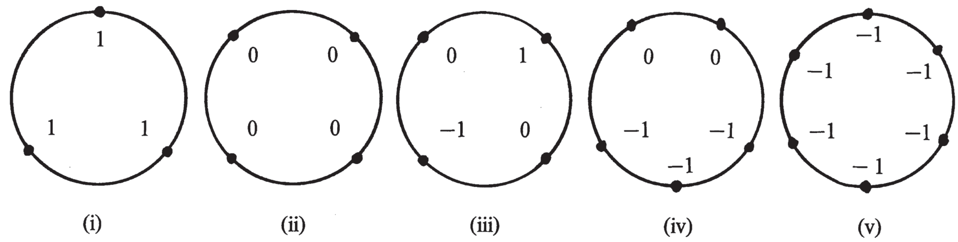

Figure 1

The singular analogues. A normal compact complex surface X with at worst log terminal singularities, i.e. quotient singularities, is called a *log del Pezzo surface* if its anticanonical Weil divisor $-K_X$ is a $\mathbb{Q}$-Cartier ample divisor. The *index* of such an X is defined to be the smallest positive integer ℓ for which $-\ell K_X$ is a Cartier divisor. The family of log del Pezzo surfaces of fixed index ℓ is known to be bounded; see Nikulin [34], [35], [36], and Borisov [6, Theorem 2.1, p. 332]. Consequently, it seems to be rather interesting to classify log del Pezzo surfaces of given index ℓ. This has been done for $\ell = 1$ by Hidaka & Watanabe [25] (by a direct generalization of Theorem 1.1) and Ye [42], and for $\ell = 2$ by Alexeev & Nikulin [1], [2] (in terms of diagrams of exceptional curves with respect to a suitable resolution of singularities). Related results are due to Kojima [31] (whenever the Picard number equals 1) and Nakayama [33] (whose techniques apply even if one replaces $\mathbb{C}$ with an algebraically closed field of arbitrary characteristic). Based on Nakayama's arguments, Fujita & Yasutake [22] succeeded recently to extend the classification to $\ell = 3$. But for $\ell \geq 4$ the situation turns out to be much more complicated, and (apart from some partial results as those in [21], [20]) it is hard to expect a complete characterization of these surfaces in this degree of generality.

On the other hand, if we restrict our study to the subclass of *toric* log del Pezzo surfaces, the classification problem becomes considerably simpler: a) The only singularities which can occur are *cyclic* quotient singularities. b) To classify (not necessarily smooth) compact toric surfaces up to isomorphism it is enough to use the graph-theoretic method proposed in [12, §5] (which generalizes Oda's graphs mentioned above): Two compact toric surfaces are isomorphic to each other if and only if their vertex singly- and edge doubly-weighted circular graphs (WVE^{2}C-*graphs*, for short) are isomorphic; see Theorem 3.3 below. A detailed examination of the number-theoretic properties of the weights of these graphs led to the classification of all toric log del Pezzo surfaces having Picard number 1 and index $\ell \leq 3$ in [12, §6] and [13]. In fact, the purely combinatorial part of the classification problem can be further simplified because it can be reduced to the classification of the so-called *LDP-polygons* (introduced in [15]) *up to unimodular transformation*. For $\ell = 1$ these are the sixteen *reflexive polygons* (which were discovered by Batyrev in the 1980's). More recently, Kasprzyk, Kreuzer & Nill [28, §6] developed a particular algorithm by means of which one creates an LDP-polygon (for given $\ell \geq 2$) by fixing a "special" edge and following a prescribed successive addition of vertices; they produced in this way the long lists of *all* LDP-polygons for $\ell \leq 17$. (Details for each of these 15346 LDP-polygons are available on the webpage [8].)

Restrictions on the singularities. At this point we mention some remarkable results concerning the singularities of log del Pezzo surfaces having Picard number 1: Belousov proved in [4], [5] that each of these surfaces admits at most 4 singularities, Kojima [30] described the nature of the exceptional divisors with respect to the minimal resolution of those possessing exactly one singularity, and Elagin [17] constructed certain (non-toric) surfaces of this kind, realized as hypersurfaces of degree $4n-2$ in $\mathbb{P}^3_{\mathbb{C}}(1, 2, 2n-1, 4n-3)$, and proved the existence of full exceptional sets of coherent sheaves over them.

Obviously, the maximal number of the singularities of a *toric* log del Pezzo surface equals the number of the edges of the corresponding LDP-polygon; for an upper bound of this number see [15, Lemma 3.1]. In the present paper we classify all toric log del Pezzo surfaces *with exactly one singularity* (without imposing a priori any restrictions on the Picard number or on the index) *up to isomorphism*.

Theorem 1.4. *Let X_Q be a toric log del Pezzo surface (associated to an LDP-polygon Q) with exactly one singularity. Then the following hold true:*

(i) *The Picard number $\rho(X_Q)$ of X_Q can take only the values 1, 2 and 3.*

(ii) *Define for every integer $p > 0$ the LDP-polygons*

$$Q_p^{[1]} := \operatorname{conv}(\{\tbinom{1}{-1}, \tbinom{p}{1}, \tbinom{-1}{0}\}),$$
$$Q_p^{[2]} := \operatorname{conv}(\{\tbinom{1}{-1}, \tbinom{p}{1}, \tbinom{p-1}{1}, \tbinom{-1}{0}\}),$$
$$Q_p^{[3]} := \operatorname{conv}(\{\tbinom{1}{-1}, \tbinom{p}{1}, \tbinom{p-1}{1}, \tbinom{-1}{0}, \tbinom{0}{-1}\}),$$

where "conv" denotes the convex hull. Then for $k \in \{1, 2, 3\}$ we have $\rho(X_Q) = k$ if, and only if, there exists an integer $p > 0$ such that $X_Q \cong X_{Q_p^{[k]}}$, and the WVE^{2}C*-graphs $\mathfrak{G}_{\Delta_{Q_p^{[k]}}}$ are those depicted in Figure 2.*

(iii) *$X_{Q_p^{[1]}}$ is isomorphic to the weighted projective plane $\mathbb{P}^2_{\mathbb{C}}(1, 1, p + 1)$ and is obtained by contracting the ∞-section $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(p + 1))$ of $\mathbb{F}_{p+1}$. The surface $X_{Q_p^{[2]}}$ is obtained by blowing up a Hirzebruch surface $\mathbb{F}_p$ at one $\mathbb{T}$-fixed point, and contracting afterwards its ∞-section. $X_{Q_p^{[3]}}$ is obtained by blowing up $X_{Q_p^{[2]}}$ at one non-singular $\mathbb{T}$-fixed point.*

(iv) *If X_Q has index $\ell \geq 1$ and Picard number $\rho(X_Q) = k \in \{1, 2, 3\}$, then for odd $\ell \geq 3$ either $X_Q \cong X_{Q_{\ell-1}^{[k]}}$ or $X_Q \cong X_{Q_{2\ell-1}^{[k]}}$, whereas for $\ell \in \{1\} \cup 2\mathbb{Z}$ we have $X_Q \cong X_{Q_{2\ell-1}^{[k]}}$.*

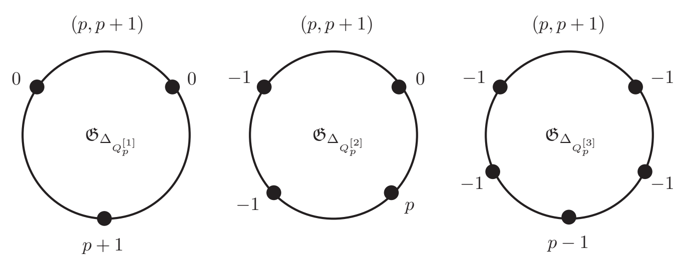

Figure 2

Equations defining closed embeddings. For every del Pezzo surface X of degree d with $3 \leq d \leq 9$ the anticanonical divisor $-K_X$ is already very ample, and $\Phi_{|-K_X|}$ gives rise to a realization of X as a subvariety of projective degree d in $\mathbb{P}^d_{\mathbb{C}}$. (For $d = 1$ and $d = 2$, one has to work with $-3K_X$ and $-2K_X$ instead to obtain realizations of X as a subvariety of degree 9, and of degree 8 in $\mathbb{P}^6_{\mathbb{C}}$, respectively.) Generalizations of these (or similar but more "economic") embeddings of *log* del Pezzo surfaces of index 1 and 2 (in appropriate projective or weighted projective spaces) appear in [25] and [27]. Since every ample divisor on a compact *toric* surface is very ample (cf. [19] or [11, Corollary 2.2.19 (b), p. 71, and Proposition 6.1.10, pp. 269–270]), the map $\Phi_{|-\ell K_{X_Q}|}$ associated to the linear system $|-\ell K_{X_Q}|$ on a toric log del Pezzo surface X_Q of index ℓ becomes a closed embedding. Koelman's Theorem [29] and standard lattice point enumeration techniques enable us to describe $\Phi_{|-\ell K_{X_Q}|}(X_Q)$ for the surfaces X_Q classified in Theorem 1.4 as follows:

Theorem 1.5. *Let X_Q be a toric log del Pezzo surface of index $\ell \geq 1$ with exactly one singularity. Then the image of $X_Q \cong X_{Q_p^{[k]}}$ under the closed embedding*

$$\Phi_{|-\ell K_{X_Q}|} : X_Q \hookrightarrow \mathbb{P}(|-\ell K_{X_Q}|)$$

is isomorphic to a subvariety of $\mathbb{P}_{\mathbb{C}}^{\delta_{Q_p^{[k]}}}$ of projective degree $d_{Q_p^{[k]}}$ which can be expressed as an intersection of finitely many quadrics, where $\delta_{Q_p^{[k]}}$ and $d_{Q_p^{[k]}}$ are given in the following table:

No.	p	k	$d_{Q_p^{[k]}}$	$\delta_{Q_p^{[k]}}$
(i)	odd	1	$\frac{1}{4}(p+1)(p+3)^2$	$\frac{1}{8}(p+3)^3$
(ii)	even	1	$(p+1)(p+3)^2$	$\frac{1}{2}(p+2)(p+3)^2$
(iii)	odd	2	$\frac{1}{4}(p+1)(p^2+5p+8)$	$\frac{1}{8}(p+3)(p^2+5p+8)$
(iv)	even	2	$(p+1)(p^2+5p+8)$	$\frac{1}{2}(p+2)(p^2+5p+8)$
(v)	odd	3	$\frac{1}{4}(p+1)(p^2+4p+7)$	$\frac{1}{8}(p+3)(p^2+4p+7)$
(vi)	even	3	$(p+1)(p^2+4p+7)$	$\frac{1}{2}(p+2)(p^2+4p+7)$

(1.1)

The cardinality $\beta_{Q_p^{[k]}}$ *of any minimal system of quadrics (generating the ideal which determines this subvariety) is given by*

No.	p	k	$\beta_{Q_p^{[k]}}$
(i)	odd	1	$\frac{1}{128}(p+1)(p+3)^2(p^3+11p^2+43p+25)$
(ii)	even	1	$\frac{1}{8}(p+3)^2(p^4+10p^3+37p^2+50p+24)$
(iii)	odd	2	$\frac{1}{128}(p+1)(p^2+5p+8)(p^3+10p^2+37p+16)$
(iv)	even	2	$\frac{1}{8}(p^2+5p+8)(p^4+9p^3+32p^2+42p+20)$
(v)	odd	3	$\frac{1}{128}(p+1)(p^2+4p+7)(p^3+9p^2+31p+7)$
(vi)	even	3	$\frac{1}{8}(p^2+4p+7)(p^4+8p^3+27p^2+34p+16)$

(1.2)

and the sectional genus $g_{Q_p^{[k]}}$ *of* $X_{Q_p^{[k]}}$ *is given by the following table:*

No.	p	k	$g_{Q_p^{[k]}}$
(i)	odd	1	$\frac{1}{8}(p+1)(p^2+4p-1)$
(ii)	even	1	$\frac{1}{2}(p+2)(p^2+4p-1)$
(iii)	odd	2	$\frac{1}{8}p(p+1)(p+3)$
(iv)	even	2	$\frac{1}{2}(p^3+5p^2+8p+2)$
(v)	odd	3	$\frac{1}{8}(p+1)^3$
(vi)	even	3	$\frac{1}{2}(p^3+4p^2+7p+2)$

(1.3)

The paper is organized as follows: In Section 2 we focus on the two non-negative, relatively prime integers $p = p_\sigma$ and $q = q_\sigma$ parametrizing the 2-dimensional, rational, strongly convex polyhedral cones σ, and we explain how they characterize the 2-dimensional toric singularities. In Sections 3 and 4 we recall some auxiliary geometric properties of compact toric surfaces and of those which are log del Pezzo. The proofs of Theorems 1.4 and 1.5 are given in Sections 5 and 6, respectively. We use only tools from discrete and classical toric geometry, adopting the standard terminology from [11], [18], [23], and [38] (and mostly the notation introduced in [12]).

2 Two-dimensional toric singularities

Let $\sigma = \mathbb{R}_{\geq 0}\mathbf{n} + \mathbb{R}_{\geq 0}\mathbf{n}' \subset \mathbb{R}^2$ be a 2-dimensional, rational, strongly convex polyhedral cone. Without loss of generality we may assume that

$$\mathbf{n} = \binom{a}{b}, \quad \mathbf{n}' = \binom{c}{d} \in \mathbb{Z}^2$$

and that both $\mathbf{n}$ and $\mathbf{n}'$ are primitive elements of $\mathbb{Z}^2$, i.e. $\gcd(a, b) = 1$ and $\gcd(c, d) = 1$.

Lemma 2.1. *Consider $\kappa, \lambda \in \mathbb{Z}$ such that $\kappa a - \lambda b = 1$. If $q := |ad - bc|$ and p is the unique integer with $0 \leq p < q$ and $\kappa c - \lambda d \equiv p \pmod{q}$, then $\gcd(p, q) = 1$ and there exists a primitive element $\mathbf{n}'' \in \mathbb{Z}^2$ such that $\mathbf{n}' = p\mathbf{n} + q\mathbf{n}''$ and $\{\mathbf{n}, \mathbf{n}''\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^2$. Moreover, there is a unimodular transformation $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$, $\Psi(\mathbf{x}) := \Xi\mathbf{x}$ with $\Xi \in \mathrm{GL}_2(\mathbb{Z})$, such that*

$$\Psi(\sigma) = \mathbb{R}_{\geq 0}\binom{1}{0} + \mathbb{R}_{\geq 0}\binom{p}{q}.$$

Proof. See [13, Lemma 2.1 and Lemma 2.2]. □

Henceforth, we call σ a (p,q)-*cone*. By $U_\sigma := \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2])$ we denote the affine toric variety associated to σ (by means of the monoid $\sigma^\vee \cap \mathbb{Z}^2$, where $\sigma^\vee$ is the dual of σ) and by $\mathrm{orb}(\sigma)$ the single point being fixed under the usual action of the algebraic torus $\mathbb{T} := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^2, \mathbb{C}^*)$ on U_σ.

Proposition 2.2. *The following conditions are equivalent:*

(i) $\{\mathbf{n}, \mathbf{n}'\}$ *is a* $\mathbb{Z}$*-basis of* $\mathbb{Z}^2$.
(ii) $q = 1$ (*and consequently,* $p = 0$).
(iii) $\mathrm{conv}(\{\mathbf{0}, \mathbf{n}, \mathbf{n}'\}) \cap \mathbb{Z}^2 = \{\mathbf{0}, \mathbf{n}, \mathbf{n}'\}$.
(iv) $U_\sigma \cong \mathbb{C}^2$.

Proof. Let T be the triangle $\mathrm{conv}(\{\mathbf{0}, \mathbf{n}, \mathbf{n}'\})$. The implication (i)⇒(ii) is obvious because $q = |\det(\mathbf{n}, \mathbf{n}')| = 2\mathrm{area}(T)$. By Pick's formula, cf. [23, p. 113], we obtain

$$\frac{q}{2} = \mathrm{area}(T) = \sharp(\mathrm{int}(T) \cap \mathbb{Z}^2) + \frac{1}{2}\sharp(\partial(T) \cap \mathbb{Z}^2) - 1,$$

where "int" and ∂ are abbreviations for interior and boundary, respectively. If $q = 1$, then $\sharp(\partial(T) \cap \mathbb{Z}^2) \geq 3$, hence $\sharp(\mathrm{int}(T) \cap \mathbb{Z}^2) = 0$ and necessarily $\sharp(\partial(T) \cap \mathbb{Z}^2) = 3$. Hence (ii)⇒(iii) is also true. The implication (iii)⇒(i) follows from [24, Theorem 4, p. 20]. For the equivalence of (i) and (iv) see [38, Theorem 1.10, p. 15]. □

If the conditions of Proposition 2.2 are satisfied, then σ is said to be a *basic cone*. On the other hand, whenever $q > 1$ we have the following:

Proposition 2.3. *The point* $\mathrm{orb}(\sigma) \in U_\sigma$ *is a cyclic quotient singularity. In particular,*

$$U_\sigma \cong \mathbb{C}^2/G = \mathrm{Spec}(\mathbb{C}[z_1, z_2]^G),$$

where $G \subset \mathrm{GL}(2, \mathbb{C})$ *denotes the cyclic group of order* q *that is generated by the diagonal matrix* $\mathrm{diag}(\zeta_q^{-p}, \zeta_q)$, *with* $\zeta_q := \exp(2\pi\sqrt{-1}/q)$, *and acts on* $\mathbb{C}^2 = \mathrm{Spec}(\mathbb{C}[z_1, z_2])$ *linearly and effectively.*

Proof. See [11, Proposition 10.1.2, pp. 460–461], [23, §2.2, pp. 32–34] and [38, Proposition 1.24, p.30]. □

By Proposition 2.4 these two numbers $p = p_\sigma$ and $q = q_\sigma$ parametrize uniquely the isomorphism class of the germ$(U_\sigma, \mathrm{orb}(\sigma))$, up to replacement of p by its socius $\widehat{p}$ (which corresponds just to the interchange of the coordinates); the *socius* $\widehat{p}$ of p is defined to be the uniquely determined integer such that $0 \leq \widehat{p} < q$, $\gcd(\widehat{p}, q) = 1$ and $p\widehat{p} \equiv 1 \pmod q$.

Proposition 2.4. *Let* $\sigma, \tau \subset \mathbb{R}^2$ *be two 2-dimensional, rational, strongly convex polyhedral cones. Then the following conditions are equivalent:*

(i) *There is a* $\mathbb{T}$*-equivariant isomorphism* $U_\sigma \cong U_\tau$ *mapping* $\mathrm{orb}(\sigma)$ *onto* $\mathrm{orb}(\tau)$.
(ii) *There is a unimodular transformation* $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$, $\Psi(\mathbf{x}) := \Xi\mathbf{x}$ *with* $\Xi \in \mathrm{GL}_2(\mathbb{Z})$, *such that* $\Psi(\sigma) = \tau$.
(iii) *The numbers* $p_\sigma, p_\tau, q_\sigma, q_\tau$ *associated to* σ, τ (*by Lemma* 2.1) *satisfy* $q_\tau = q_\sigma$ *and either* $p_\tau = p_\sigma$ *or* $p_\tau = \widehat{p}_\sigma$.

Proof. See [13, Proposition 2.4]. □

3 Compact toric surfaces

Every compact toric surface is a 2-dimensional toric variety X_Δ associated to a *complete* fan Δ in $\mathbb{R}^2$, i.e. a fan having 2-dimensional cones as maximal cones and whose support $|\Delta|$ is the entire $\mathbb{R}^2$; see [38, Theorem 1.11, p. 16]. Consider a complete fan Δ in $\mathbb{R}^2$ and suppose that

$$\sigma_i = \mathbb{R}_{\geq 0}\mathbf{n}_i + \mathbb{R}_{\geq 0}\mathbf{n}_{i+1}, \quad i \in \{1, \ldots, \nu\}, \tag{3.1}$$

are its 2-dimensional cones (with $\nu \geq 3$ and $\mathbf{n}_i \in \mathbb{Z}^2$ primitive for all $i \in \{1, \ldots, \nu\}$), enumerated in such a way that $\mathbf{n}_1, \ldots, \mathbf{n}_\nu$ go *anticlockwise* around the origin exactly once in this order (under the usual convention: $\mathbf{n}_{\nu+1} := \mathbf{n}_1, \mathbf{n}_0 := \mathbf{n}_\nu$). The variety X_Δ is obtained by gluing the affine charts U_{σ_i} along the open subsets defined by the rays $\sigma_i \cap \sigma_{i+1}$ with $i \in \{1, \ldots, \nu\}$; cf. [38, Theorem 1.4, p. 7]. Since Δ is simplicial, the Picard number $\rho(X_\Delta)$ of X_Δ, i.e. the rank of its Picard group $\mathrm{Pic}(X_\Delta)$ equals

$$\rho(X_\Delta) = \nu - 2, \tag{3.2}$$

see [23, p. 65]. Now suppose that σ_i is a (p_i, q_i)-cone for all $i \in \{1, \ldots, \nu\}$ and introduce the notation

$$I_\Delta := \{i \in \{1, \ldots, \nu\} \mid q_i > 1\}, \qquad J_\Delta := \{i \in \{1, \ldots, \nu\} \mid q_i = 1\}, \tag{3.3}$$

to separate the indices corresponding to non-basic cones from those corresponding to basic cones. By Propositions 2.2 and 2.3 the singular locus of X_Δ equals

$$\mathrm{Sing}(X_\Delta) = \{\mathrm{orb}(\sigma_i) \mid i \in I_\Delta\}.$$

For all $i \in I_\Delta$ consider the negative-regular continued fraction expansion of

$$\frac{q_i}{q_i - p_i} = b_1^{(i)} - \cfrac{1}{b_2^{(i)} - \cfrac{1}{\ddots \; b_{s_i-1}^{(i)} - \cfrac{1}{b_{s_i}^{(i)}}}}$$

and define $\mathbf{u}_0^{(i)} := \mathbf{n}_i$, $\mathbf{u}_1^{(i)} := \frac{1}{q_i}((q_i - p_i)\mathbf{n}_i + \mathbf{n}_{i+1})$, and

$$\mathbf{u}_{j+1}^{(i)} := b_j^{(i)}\mathbf{u}_j^{(i)} - \mathbf{u}_{j-1}^{(i)}, \quad \text{for all } j \in \{1, \ldots, s_i\}.$$

It is easy to see that $\mathbf{u}_{s_i+1}^{(i)} = \mathbf{n}_{i+1}$ and that the $b_j^{(i)}$ are integers ≥ 2, for all indices $j \in \{1, \ldots, s_i\}$. According to [12, Proposition 4.9, p. 99], the self-intersection number of the canonical divisor K_{X_Δ} of X_Δ equals

$$K_{X_\Delta}^2 = 12 - \nu + \sum_{i \in I_\Delta} \left(\tfrac{q_i - p_i + 1}{q_i} + \tfrac{q_i - \widehat{p}_i + 1}{q_i} - 2 + \sum_{j=1}^{s_i} (b_j^{(i)} - 3) \right). \tag{3.4}$$

By construction, the birational morphism $f : X_{\widetilde{\Delta}} \longrightarrow X_\Delta$ induced by the refinement $\widetilde{\Delta}$ of Δ consisting of the cones $\{\sigma_i \mid i \in J_\Delta\}$ and $\{\mathbb{R}_{\geq 0}\mathbf{u}_j^{(i)} + \mathbb{R}_{\geq 0}\mathbf{u}_{j+1}^{(i)} \mid i \in I_\Delta, j \in \{0, 1, \ldots, s_i\}\}$ together with their faces is the *minimal desingularization* of X_Δ. The *exceptional divisor*

$$E^{(i)} := \sum_{j=1}^{s_i} E_j^{(i)}, \quad i \in I_\Delta,$$

replacing $\mathrm{orb}(\sigma_i)$ via f has the closures

$$E_j^{(i)} := \overline{\mathrm{orb}_{\widetilde{\Delta}}(\mathbb{R}_{\geq 0}\mathbf{u}_j^{(i)})} \quad (\cong \mathbb{P}^1_{\mathbb{C}}) \quad \text{with } j \in \{1, 2, \ldots, s_i\}$$

(i.e. the closures of the orbits of the new rays with respect to $\widetilde{\Delta}$) as its components, and the self-intersection number $(E_j^{(i)})^2 = -b_j^{(i)}$. Moreover,

$$\overline{C}_i := \overline{\mathrm{orb}_{\widetilde{\Delta}}(\mathbb{R}_{\geq 0}\mathbf{n}_i)}$$

is the *strict transform* of $C_i := \overline{\mathrm{orb}_\Delta(\mathbb{R}_{\geq 0}\mathbf{n}_i)}$ with respect to f for every $i \in \{1, 2, \ldots, \nu\}$.

Definition 3.1. For $i \in \{1, \ldots, \nu\}$ we introduce integers r_i *uniquely determined* by the conditions:

$$r_i\mathbf{n}_i = \begin{cases} \mathbf{u}_{s_{i-1}}^{(i-1)} + \mathbf{u}_1^{(i)}, & \text{if } i \in I'_\Delta, \\ \mathbf{n}_{i-1} + \mathbf{u}_1^{(i)}, & \text{if } i \in I''_\Delta, \\ \mathbf{u}_{s_{i-1}}^{(i-1)} + \mathbf{n}_{i+1}, & \text{if } i \in J'_\Delta, \\ \mathbf{n}_{i-1} + \mathbf{n}_{i+1}, & \text{if } i \in J''_\Delta, \end{cases} \tag{3.5}$$

where

$$I'_\Delta := \{i \in I_\Delta \mid q_{i-1} > 1\}, \qquad I''_\Delta := \{i \in I_\Delta \mid q_{i-1} = 1\},$$

and

$$J'_\Delta := \{i \in J_\Delta \mid q_{i-1} > 1\}, \qquad J''_\Delta := \{i \in J_\Delta \mid q_{i-1} = 1\},$$

with I_Δ, J_Δ as in (3.3). By [12, Lemma 4.3], $-r_i$ is the self-intersection number $\overline{C}_i^2$ of $\overline{C}_i$ for $i \in \{1, \dots, \nu\}$. The triples (p_i, q_i, r_i) with $i \in \{1, 2, \dots, \nu\}$ are used to define the WVE^2 C-graph $\mathfrak{G}_\Delta$.

Definition 3.2. A *circular graph* is a plane graph whose vertices are points on a circle and whose edges are the corresponding arcs (on this circle, each of which connects two consecutive vertices). We say that a circular graph $\mathfrak{G}$ is *$\mathbb{Z}$-weighted at its vertices* and *double $\mathbb{Z}$-weighted at its edges* (and call it WVE^{2}C-*graph*, for short) if it is accompanied by two maps

$$\{\text{Vertices of } \mathfrak{G}\} \longmapsto \mathbb{Z}, \quad \{\text{Edges of } \mathfrak{G}\} \longmapsto \mathbb{Z}^2,$$

assigning to each vertex an integer and to each edge a pair of integers, respectively. To every complete fan Δ in $\mathbb{R}^2$ (as described above) we associate an anticlockwise directed WVE^{2}C-graph $\mathfrak{G}_\Delta$ with

$$\{\text{Vertices of } \mathfrak{G}_\Delta\} = \{\mathbf{v}_1, \dots, \mathbf{v}_\nu\} \quad \text{and} \quad \{\text{Edges of } \mathfrak{G}_\Delta\} = \{\overline{\mathbf{v}_1\mathbf{v}_2}, \dots, \overline{\mathbf{v}_\nu\mathbf{v}_1}\},$$

$(\mathbf{v}_{\nu+1} := \mathbf{v}_1)$, by defining its "weights" as follows:

$$\mathbf{v}_i \longmapsto -r_i, \quad \overline{\mathbf{v}_i\mathbf{v}_{i+1}} \longmapsto (p_i, q_i), \quad \text{for } i \in \{1, \dots, \nu\}.$$

The *reverse graph* $\mathfrak{G}_\Delta^{\text{rev}}$ of $\mathfrak{G}_\Delta$ is the directed WVE^{2}C-graph which is obtained by changing the double weight (p_i, q_i) of the edge $\overline{\mathbf{v}_i\mathbf{v}_{i+1}}$ into $(\widehat{p}_i, q_i)$ and reversing the initial anticlockwise direction of $\mathfrak{G}_\Delta$ into clockwise direction; see Figure 3.

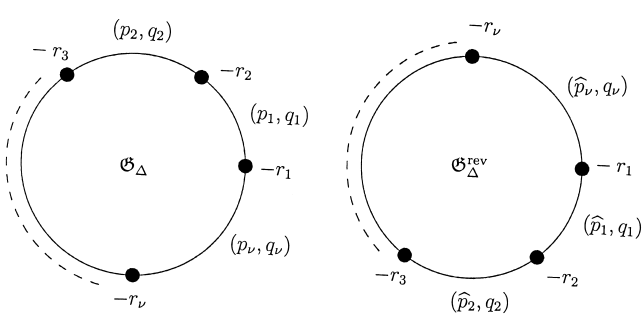

Figure 3

Theorem 3.3. *Let Δ and Δ' be two complete fans in $\mathbb{R}^2$. Then the following conditions are equivalent:*

(i) *The compact toric surfaces X_Δ and $X_{\Delta'}$ are isomorphic.*

(ii) *Either $\mathfrak{G}_{\Delta'} \overset{\text{gr}}{\cong} \mathfrak{G}_\Delta$ or $\mathfrak{G}_{\Delta'} \overset{\text{gr}}{\cong} \mathfrak{G}_\Delta^{\text{rev}}$.*

Here $\overset{\text{gr}}{\cong}$ indicates graph-theoretic isomorphism (i.e. a bijection between the sets of vertices which preserves the corresponding weights). For further details and for the proof of Theorem 3.3 (which can be viewed as an appropriate generalization of Proposition 2.4 for complete fans in $\mathbb{R}^2$) the reader is referred to [12, §5]. [Convention: To be absolutely compatible with Oda's circular graphs we omit the weights of the edges which are equal to $(0, 1)$, i.e. those corresponding to basic cones, whenever we draw a WVE^{2}C-graph.]

4 Toric log del Pezzo surfaces and LDP-polygons

Definition 4.1. Let $Q \subset \mathbb{R}^2$ be a convex polygon. Denote by $\mathcal{V}(Q)$ and $\mathcal{F}(Q)$ the set of its vertices and the set of its facets (edges), respectively. Q is called an *LDP-polygon* if it contains the origin in its interior, and its vertices belong to $\mathbb{Z}^2$ and are primitive. (Obviously, the image of an LDP-polygon under a unimodular transformation is again an LDP-polygon.)

If Q is an LDP-polygon, we denote by X_Q the compact toric surface X_{Δ_Q} constructed by means of the fan

$$\Delta_Q := \{\text{the cones } \sigma_F \text{ together with their faces } \mid F \in \mathcal{F}(Q)\},$$

where $\sigma_F := \{\lambda \mathbf{x} \mid \mathbf{x} \in F \text{ and } \lambda \in \mathbb{R}_{\geq 0}\}$ for all $F \in \mathcal{F}(Q)$.

Proposition 4.2. (i) *A compact toric surface is a log del Pezzo surface if and only if it is isomorphic to X_Q for some LDP-polygon Q.*

(ii) *There is a one-to-one correspondence $[Q] \mapsto [X_Q]$ between the lattice-equivalence classes of LDP-polytopes Q and the isomorphism classes $[X_Q]$ of toric log del Pezzo surfaces.*

Proof. (i) This follows from [12, Remark 6.7, p. 107].

(ii) If Q is an LDP-polygon, if $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$, $\Psi(\mathbf{x}) := \Xi\mathbf{x}$ with $\Xi \in \mathrm{GL}_2(\mathbb{Z})$, is a unimodular transformation, and if $Q' := \Psi(Q)$, then

$$\mathfrak{G}_{\Delta_{Q'}} \overset{\text{gr}}{\cong} \mathfrak{G}_{\Delta_Q} \quad \text{whenever } \det(\Xi) = 1, \quad \text{and } \mathfrak{G}_{\Delta_{Q'}} \overset{\text{gr}}{\cong} \mathfrak{G}^{\text{rev}}_{\Delta_Q} \quad \text{whenever } \det(\Xi) = -1.$$

By Theorem 3.3, X_Q and $X_{Q'}$ are isomorphic. And conversely, if X_Q and $X_{Q'}$ are isomorphic for some LDP-polygons Q, Q', then

$$\text{either } \mathfrak{G}_{\Delta_{Q'}} \overset{\text{gr}}{\cong} \mathfrak{G}_{\Delta_Q} \quad \text{or} \quad \mathfrak{G}_{\Delta_{Q'}} \overset{\text{gr}}{\cong} \mathfrak{G}^{\text{rev}}_{\Delta_Q}. \tag{4.1}$$

Thus by (4.1) there exists an automorphism ϖ of the lattice $\mathbb{Z}^2 = \mathbb{Z}\binom{1}{0} \oplus \mathbb{Z}\binom{0}{1}$ with

$$\det(\varpi) = \begin{cases} 1, & \text{in the first case,} \\ -1, & \text{in the second case,} \end{cases}$$

such that $\varpi_{\mathbb{R}}(\Delta_Q) = \Delta_{Q'}$ (preserving/reversing the ordering of the cones), where

$$\varpi_{\mathbb{R}} := \varpi \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{R}} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

denotes its scalar extension. Obviously, $\varpi_{\mathbb{R}}(Q) = Q'$. □

Note 4.3. Let Q be an arbitrary LDP-polygon. For each $F \in \mathcal{F}(Q)$ assume that σ_F is a (p_F, q_F)-cone. Then from [12, Lemma 6.8] one concludes that the index ℓ of X_Q equals

$$\ell = \mathrm{lcm}\{l_F \mid F \in \mathcal{F}(Q)\} \quad \text{with} \quad l_F := \frac{q_F}{\gcd(q_F, p_F - 1)}. \tag{4.2}$$

We denote by $\mathring{Q} := \{\mathbf{y} \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R}) \mid \langle \mathbf{y}, \mathbf{x} \rangle \geq -1 \text{ for all } \mathbf{x} \in Q\}$ the *polar polygon* of Q, where $\langle \cdot, \cdot \rangle$: $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}$ is the usual inner product. Then $\mathring{Q}$ contains the origin in its interior, and the index ℓ of X_Q equals

$$\ell = \min\{\kappa \in \mathbb{Z}_{>0} \mid \mathcal{V}(\kappa\mathring{Q}) \subset \mathbb{Z}^2\}, \quad \text{where } \kappa\mathring{Q} := \{\kappa\mathbf{y} \mid \mathbf{y} \in \mathring{Q}\}.$$

Moreover, if $F \in \mathcal{F}(Q)$, denoting by $\boldsymbol{\eta}_F$ the unique primitive $\boldsymbol{\eta}_F \in \mathbb{Z}^2$ satisfying $\langle \boldsymbol{\eta}_F, \mathbf{x} \rangle = l_F$ for every $\mathbf{x} \in F$, we have

$$\mathcal{V}(\mathring{Q}) = \left\{ \frac{-1}{l_F} \boldsymbol{\eta}_F \,\middle|\, F \in \mathcal{F}(Q) \right\}. \tag{4.3}$$

5 Proof of the Classification Theorem 1.4

Let Q be an LDP-polygon with vertex set $\mathcal{V}(Q) = \{\mathbf{n}_1, \dots, \mathbf{n}_\nu\}$, $\nu \geq 3$. Assume that σ_i, $i \in \{1, \dots, \nu\}$, are the 2-dimensional cones of Δ_Q, defined and ordered (anticlockwise) as in (3.1), and that only one of these cones, say σ_1, is a non-basic (p, q)-cone (i.e. $q > 1$). By Lemma 2.1 there is a unimodular transformation $\Psi_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\Psi_1(\mathbf{x}) := \Xi\mathbf{x}$ with $\Xi \in \mathrm{GL}_2(\mathbb{Z})$, such that

$$\Psi_1(\sigma_1) = \mathbb{R}_{\geq 0}\tbinom{1}{0} + \mathbb{R}_{\geq 0}\tbinom{p}{q}.$$

Without loss of generality we may assume that $\det(\Xi) = 1$ (because otherwise the proof of Theorem 1.4 which follows can be performed similarly if one works with the vertices ordered clockwise). This means that $\Psi_1(\mathbf{n}_1) = \binom{1}{0}$ and $\Psi_1(\mathbf{n}_2) = \binom{p}{q}$. We set $\mathbf{w}_i := \Psi_1(\mathbf{n}_i)$ for all $i \in \{1, \dots, \nu\}$ (and $\mathbf{w}_{\nu+1} := \mathbf{w}_1$). Since all cones of $\Delta_{\Psi_1(Q)}$ are strongly convex and $|\Delta_{\Psi_1(Q)}| = \mathbb{R}^2$, there exists an index $\mu \in \{3, \dots, \nu\}$ such that

$$\mathbf{w}_\mu = \tbinom{a}{b} \in \left\{\tbinom{x}{y} \in \mathbb{R}^2 \mid x < 0\right\} \cap \mathbb{Z}^2. \tag{5.1}$$

Lemma 5.1. (i) *The cones* $\mathbb{R}_{\geq 0}\mathbf{w}_\mu + \mathbb{R}_{\geq 0}\mathbf{w}_1$ *and* $\mathbb{R}_{\geq 0}\mathbf{w}_2 + \mathbb{R}_{\geq 0}\mathbf{w}_\mu$ *are basic.*
(ii) $q = p + 1$ (*and consequently,* $\widehat{p} = p$ *and* $\mathbf{w}_\mu = \binom{-1}{-1}$).

Proof. (i) Using Proposition 2.2 it suffices to prove that

$$\mathrm{conv}(\{\mathbf{0}, \mathbf{w}_\mu, \mathbf{w}_1\}) \cap \mathbb{Z}^2 = \{\mathbf{0}, \mathbf{w}_\mu, \mathbf{w}_1\} \quad \text{and} \quad \mathrm{conv}(\{\mathbf{0}, \mathbf{w}_2, \mathbf{w}_\mu\}) \cap \mathbb{Z}^2 = \{\mathbf{0}, \mathbf{w}_2, \mathbf{w}_\mu\}. \tag{5.2}$$

Obviously, $\mathcal{V}(\Psi_1(Q)) \setminus \{\mathbf{w}_\mu, \mathbf{w}_1, \mathbf{w}_2\}$ is either empty or a subset of $(\mathcal{U}_1 \cup \mathcal{U}_2) \cap \mathbb{Z}^2$, where

$$\mathcal{U}_1 := \left\{\tbinom{x}{y} \in \mathbb{R}^2 \mid y < 0, y < x, \text{ and } qx - (p-1)y < q\right\},$$

and

$$\mathcal{U}_2 := \left\{\tbinom{x}{y} \in \mathbb{R}^2 \mid qx < py, y > x, \text{ and } qx - (p-1)y < q\right\}.$$

The set $\{\binom{x}{y} \in \mathbb{R}^2 \mid qx - (p-1)y = q\}$ is the supporting line of the edge $\mathrm{conv}(\{\mathbf{w}_1, \mathbf{w}_2\})$ of $\Psi_1(Q)$. If $\mathrm{conv}(\{\mathbf{w}_\mu, \mathbf{w}_1\}) \in \mathcal{F}(\Psi_1(Q))$, i.e. if $\mu = \nu$, the first equality in (5.2) is obvious (because $\Psi_1(\sigma_\nu)$ is basic by definition). If $\mathrm{conv}(\{\mathbf{w}_\mu, \mathbf{w}_1\}) \notin \mathcal{F}(\Psi_1(Q))$, then $\mathcal{V}(\Psi_1(Q)) \cap \mathcal{U}_1 \neq \varnothing$, and the existence of an element

$$\mathbf{m} \in (\mathrm{conv}(\{\mathbf{0}, \mathbf{w}_\mu, \mathbf{w}_1\}) \cap \mathbb{Z}^2) \setminus \{\mathbf{0}, \mathbf{w}_\mu, \mathbf{w}_1\}$$

would imply that

$$\mathbf{m} \in (\mathrm{conv}(\{\mathbf{0}, \mathbf{w}_{\xi-1}, \mathbf{w}_\xi\}) \cap \mathbb{Z}^2) \setminus \{\mathbf{0}, \mathbf{w}_{\xi-1}, \mathbf{w}_\xi\} \quad \text{for some } \xi \in \{\mu+1, \mu+2, \dots, \nu, \nu+1\},$$

which leads to a contradiction (because $\Psi_1(\sigma_{\xi-1})$ is basic by definition). Similar arguments (using $\mathcal{U}_2$ instead of $\mathcal{U}_1$) show that the second equality in (5.2) is also true.

(ii) By (i) we have $|\det(\mathbf{w}_\mu, \mathbf{w}_1)| = |\det(\mathbf{w}_2, \mathbf{w}_\mu)| = 1$, i.e. $b \in \{\pm 1\}$, and one of the following conditions is satisfied:

$$b = 1 \quad \text{and} \quad aq - p = 1, \tag{5.3}$$
$$b = 1 \quad \text{and} \quad aq - p = -1, \tag{5.4}$$
$$b = -1 \quad \text{and} \quad aq + p = 1, \tag{5.5}$$
$$b = -1 \quad \text{and} \quad aq + p = -1. \tag{5.6}$$

Condition (5.3) gives $a = \frac{1+p}{q} > 0$ which is not true, because $a < 0$ by (5.1). Similarly, (5.4) is not true, as we would have $a = \frac{p-1}{q} \geq 0$. In Case (5.5) we have $a = \frac{-(p-1)}{q}$, hence q divides $p - 1$; this yields $p < q \leq p - 1$, which is absurd.

Therefore, (5.6) is necessarily true and $a = -\frac{p+1}{q}$. Now since $q \mid p + 1$ and $p < q$, we have $q = p + 1$, hence $a = -1$ and $b = -1$. This implies that $\mathbf{w}_\mu = \binom{-1}{-1}$. From $p + 1 \mid (p^2 - 1)$ we infer that $\widehat{p} = p$. ☐

Lemma 5.2. *There exists a unimodular transformation $\Psi_2 : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$\Psi_2(\Psi_1(\sigma_1)) = \mathbb{R}_{\geq 0}\tbinom{1}{-1} + \mathbb{R}_{\geq 0}\tbinom{p}{1},$$

with $\Psi_2\binom{1}{0} = \binom{1}{-1}$, $\Psi_2\binom{p}{p+1} = \binom{p}{1}$ *and* $\Psi_2\binom{-1}{-1} = \binom{-1}{0}$.

Proof. It is enough to define $\Psi_2(\mathbf{x}) := \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$. □

Next, we set $\Upsilon := \Psi_2 \circ \Psi_1$, $\mathbf{v}_i := \Upsilon(\mathbf{n}_i)$ for every $i \in \{1, \dots, \nu\}$ (and $\mathbf{v}_{\nu+1} := \mathbf{v}_1$). Starting with the minimal generators $\mathbf{v}_1 = \binom{1}{-1}$, $\mathbf{v}_2 = \binom{p}{1}$ of the unique non-basic cone $\Upsilon(\sigma_1)$ of $\Delta_{\Upsilon(Q)}$, and with $\mathbf{v}_\mu = \binom{-1}{0} \in \mathcal{V}(\Upsilon(Q))$, we study in detail the restrictions on the location of the remaining vertices of $\Upsilon(Q)$.

Lemma 5.3. *There is no convex polygon having three collinear vertices.*

Proof. This is due to the fact that the vertices of a convex polygon are its extreme points; see e.g. [7, p. 30 and p. 45]. □

Lemma 5.4. *The LDP-polygon $\Upsilon(Q)$, with $\mathcal{V}(\Upsilon(Q)) = \{\mathbf{v}_1, \dots, \mathbf{v}_\nu\}$, has the following properties:*

(i) *Setting $k := \nu - 2$, we have $k \in \{1, 2, 3\}$. Moreover, $\Upsilon(Q) = Q_p^{[k]}$ for $k \in \{1, 3\}$, and either $\Upsilon(Q) = Q_p^{[2]}$ or $\Upsilon(Q) = \check{Q}_p^{[2]}$ for $k = 2$, where $Q_p^{[1]}$, $Q_p^{[2]}$, $Q_p^{[3]}$ are the polygons defined in Theorem* 1.4(ii) *and*

$$\check{Q}_p^{[2]} := \mathrm{conv}(\{\tbinom{1}{-1}, \tbinom{p}{1}, \tbinom{-1}{0}, \tbinom{0}{-1}\}).$$

(ii) *The polygons $Q_p^{[2]}$ and $\check{Q}_p^{[2]}$ are lattice-equivalent.*

Proof. (i) If $\mathcal{U}_1' := \{\binom{x}{y} \in \Psi_2(\mathcal{U}_1 \mid y \leq -2\}$, we claim that $\mathcal{U}_1' \cap \mathcal{V}(\Upsilon(Q)) = \varnothing$. If $\mathbf{v}_{\mu+1} \in \mathcal{U}_1' \cap \mathcal{V}(\Upsilon(Q))$, then we would have $|\det(\mathbf{v}_\mu, \mathbf{v}_{\mu+1})| = 2$, which is a contradiction to the basicness of the cone $\Upsilon(\sigma_\mu)$. If

$$\mathbf{v}_{\mu+1} \in \{\tbinom{x}{y} \in \mathbb{Z}^2 \mid x \leq 0, y = -1\} \quad \text{and} \quad \mathbf{v}_{\mu+2} \in \mathcal{U}_1' \cap \mathcal{V}(\Upsilon(Q)),$$

then we would have $|\det(\mathbf{v}_{\mu+1}, \mathbf{v}_{\mu+2})| \geq 2$, which is a contradiction to the basicness of the cone $\Upsilon(\sigma_{\mu+1})$. Repeating successively this procedure (until we arrive at $\mathbf{v}_\nu$) we bear out our assertion, as well as the implication

$$\mu \leq \nu - 1 \Rightarrow \{\mathbf{v}_\xi \mid \mu + 1 \leq \xi \leq \nu\} \subset \{\tbinom{x}{y} \in \mathbb{Z}^2 \mid x \leq 0, y = -1\}.$$

For $\mathcal{U}_2' := \{\binom{x}{y} \in \Psi_2(\mathcal{U}_2) \mid y \geq 2\}$ we show, analogously, that $\mathcal{U}_2' \cap \mathcal{V}(\Upsilon(Q)) = \varnothing$ and and that

$$\mu \geq 4 \Rightarrow \{\mathbf{v}_\xi \mid 3 \leq \xi \leq \mu - 1\} \subset \{\tbinom{x}{y} \in \mathbb{Z}^2 \mid x \leq p - 1, y = 1\}.$$

Hence $\mathcal{V}(\Upsilon(Q)) \setminus \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_\mu\}$ is either empty or a subset of

$$\{\tbinom{x}{y} \in \mathbb{Z}^2 \mid x \leq 0 \text{ and } y = -1\} \cup \{\tbinom{x}{y} \in \mathbb{Z}^2 \mid x \leq p - 1 \text{ and } y = 1\}.$$

Taking into account Lemma 5.3 we conclude that

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_\mu\} \subseteq \mathcal{V}(\Upsilon(Q)) \subseteq \{\mathbf{v}_1, \mathbf{v}_2, \tbinom{p-1}{1}, \mathbf{v}_\mu, \tbinom{0}{-1}\}.$$

Therefore, $k \in \{1, 2, 3\}$ and there are only the following possibilities:

- If $k = 1$, then $\nu = \mu = 3$ and $\Upsilon(Q) = \mathrm{conv}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) = Q_p^{[1]}$.
- If $k = 2$, then $\nu = 4$ and *either* $\Upsilon(Q) = Q_p^{[2]}$, $\mu = 4$, *or* $\Upsilon(Q) = \check{Q}_p^{[2]}$, $\mu = 3$.
- If $k = 3$, then $\nu = 5$, $\mu = 4$ and $\Upsilon(Q) = Q_p^{[3]}$.

(ii) The polygon $Q_p^{[2]}$ is mapped onto $\check{Q}_p^{[2]}$ under the unimodular transformation

$$\mathfrak{Y} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \mathfrak{Y}(\mathbf{x}) := \begin{pmatrix} 1 & 1-p \\ 0 & -1 \end{pmatrix} \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^2,$$

and $\mathfrak{Y}(\mathbf{v}_1) = \mathbf{v}_2$, $\mathfrak{Y}(\mathbf{v}_2) = \mathbf{v}_1$, $\mathfrak{Y}\binom{-1}{0} = \binom{-1}{0}$, $\mathfrak{Y}\binom{p-1}{1} = \binom{0}{-1}$. □

Note 5.5. The set $\mathrm{conv}(\{\mathbf{v}_1, \mathbf{v}_2\}) \cap \mathbb{Z}^2$ is empty if p is even, and it consists of the single lattice point $\binom{\frac{1}{2}(p+1)}{0}$ if p is odd. Thus, the number of the lattice points belonging to the boundary of $Q_p^{[k]}$, $k \in \{1, 2, 3\}$, equals $k + 2$ whenever p is even and $k + 3$ whenever p is odd. Since $\mathrm{area}(Q_p^{[k]}) = \frac{p+k}{2} + 1$, Pick's formula gives

$$\sharp(\mathrm{int}(Q_p^{[k]}) \cap \mathbb{Z}^2) = \begin{cases} \frac{p}{2} + 1, & \text{if } p \text{ is even,} \\ \frac{p-1}{2} + 1, & \text{if } p \text{ is odd.} \end{cases}$$

Proof of Theorem 1.4. (i)–(ii) Up to isomorphism, every toric log del Pezzo surface with exactly one singularity is of the form X_Q with Q as above. By (3.2), Lemma 5.4 and Proposition 4.2 we infer that the Picard number $\rho(X_Q)$ of X_Q can take only the values 1, 2 and 3, and that for $k \in \{1, 2, 3\}$ we have $\rho(X_Q) = k$ if and only if $X_Q \cong X_{Q_p^{[k]}}$ for some $p \in \mathbb{Z}_{>0}$. Note that for $k = 2$, $\mathfrak{Y}$ induces a graph-theoretic isomorphism $\mathfrak{G}_{\Delta_{\check{Q}_p^{[2]}}} \overset{\mathrm{gr}}{\cong} \mathfrak{G}^{\mathrm{rev}}_{\Delta_{Q_p^{[2]}}}$, meaning that $X_{Q_p^{[2]}} \cong X_{\check{Q}_p^{[2]}}$.

The fan $\widetilde{\Delta}_{Q_p^{[k]}}$ which is used to construct the minimal desingularization of $X_{Q_p^{[k]}}$ (as explained in Section 3) contains just one additional ray (compared with $\Delta_{Q_p^{[k]}}$), namely $\mathbb{R}_{\geq 0}\binom{1}{0}$. The closure of its orbit constitutes the single exceptional divisor, say E, with respect to this desingularization, with $E^2 = -(p+1)$. Setting $\mathbf{u}_E := \binom{1}{0}$ we compute the integers r_i, $i \in \{1, \ldots, k+2\}$, defined in (3.5) in the three different cases:

- Case (a): If $k = 1$, then $\mathbf{v}_1 = \binom{1}{-1}$, $\mathbf{v}_2 = \binom{p}{1}$, $\mathbf{v}_3 = \binom{-1}{0}$, and
$$[\mathbf{v}_3 + \mathbf{u}_E = \mathbf{0}, \mathbf{v}_2 + \mathbf{v}_1 = -(p+1)\mathbf{v}_3] \Rightarrow r_1 = r_2 = 0, r_3 = -(p+1).$$
- Case (b): If $k = 2$, then $\mathbf{v}_1 = \binom{1}{-1}$, $\mathbf{v}_2 = \binom{p}{1}$, $\mathbf{v}_3 = \binom{p-1}{1}$, $\mathbf{v}_4 = \binom{-1}{0}$, and
$$\left.\begin{array}{r} \mathbf{v}_4 + \mathbf{u}_E = \mathbf{0}, \mathbf{u}_E + \mathbf{v}_3 = \mathbf{v}_2 \\ \mathbf{v}_2 + \mathbf{v}_4 = \mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_1 = -p\mathbf{v}_4 \end{array}\right\} \Rightarrow r_1 = 0, r_2 = r_3 = 1, r_4 = -p.$$
- Case (c): If $k = 3$, then $\mathbf{v}_1 = \binom{1}{-1}$, $\mathbf{v}_2 = \binom{p}{1}$, $\mathbf{v}_3 = \binom{p-1}{1}$, $\mathbf{v}_4 = \binom{-1}{0}$, $\mathbf{v}_5 = \binom{0}{-1}$, and
$$\left.\begin{array}{r} \mathbf{v}_5 + \mathbf{u}_E = \mathbf{v}_1, \mathbf{u}_E + \mathbf{v}_3 = \mathbf{v}_2, \\ \mathbf{v}_2 + \mathbf{v}_4 = \mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_5 = -(p+1)\mathbf{v}_4, \mathbf{v}_4 + \mathbf{v}_1 = \mathbf{v}_5 \end{array}\right\} \Rightarrow r_1 = r_2 = r_3 = r_5 = 1, r_4 = -(p-1).$$

Hence, the WVE^{2}C-graphs $\mathfrak{G}_{\Delta_{Q_p^{[k]}}}$ are indeed those depicted in Figure 2.

(iii) For every integer $p > 0$, let $\mathfrak{D}_p$ be the complete fan consisting of the four cones $\mathbb{R}_{\geq 0}\binom{1}{-1} + \mathbb{R}_{\geq 0}\binom{1}{0}$, $\mathbb{R}_{\geq 0}\binom{1}{0} + \mathbb{R}_{\geq 0}\binom{p}{1}$, $\mathbb{R}_{\geq 0}\binom{p}{1} + \mathbb{R}_{\geq 0}\binom{-1}{0}$ and $\mathbb{R}_{\geq 0}\binom{-1}{0} + \mathbb{R}_{\geq 0}\binom{1}{-1}$ together with their faces. We see that $X_{\mathfrak{D}_p} \cong \mathbb{F}_{p+1}$, having $\mathrm{orb}_{\mathfrak{D}_p}(\mathbb{R}_{\geq 0}\binom{1}{0})$ as its ∞-section. The surfaces $X_{Q_p^{[k]}}$ are characterized as follows:

- Case (a): If $k = 1$, then $X_{Q_p^{[1]}} \cong \mathbb{P}^2_{\mathbb{C}}(1, 1, p+1)$, see [13, Lemma 6.1], and it is obtained by contracting the ∞-section of $X_{\mathfrak{D}_p}$. In fact, since $X_{\mathfrak{D}_p} = X_{\widetilde{\Delta}_{Q_p^{[1]}}}$ is the minimal desingularization of $X_{Q_p^{[1]}}$, the surface $X_{Q_p^{[1]}}$ is nothing but the *anticanonical model* of $X_{\mathfrak{D}_p}$ in the sense of Sakai [39].
- Case (b): If $k = 2$, then the star subdivision of $\mathfrak{D}_{p-1}$ with respect to the cone $\mathbb{R}_{\geq 0}\binom{1}{0} + \mathbb{R}_{\geq 0}\binom{p-1}{1}$ induces the equivariant blow-up $X_{\widetilde{\Delta}_{Q_p^{[2]}}} \longrightarrow X_{\mathfrak{D}_{p-1}}$ with the orbit of this cone as centre; cf. [11, Proposition 3.3.15, p. 130], [37, Corollary 7.5, p. 45] or [18, Theorem VI.7.2, pp. 249–250]. Thus the surface $X_{Q_p^{[2]}}$ is obtained by contracting the strict transform of the ∞-section of $X_{\mathfrak{D}_{p-1}}$ on the surface $X_{\widetilde{\Delta}_{Q_p^{[2]}}}$.
- Case (c): If $k = 3$, we construct the surface $X_{Q_p^{[3]}}$ from $X_{Q_p^{[2]}}$ by using the equivariant birational morphism induced by the star subdivision of $\mathfrak{D}_{p-1}$ with respect to the cone $\mathbb{R}_{\geq 0}\binom{-1}{0} + \mathbb{R}_{\geq 0}\binom{1}{-1}$, i.e. by blowing up its orbit (which is a non-singular $\mathbb{T}$-fixed point of $X_{Q_p^{[2]}}$).

Taking into account that we pass from $X_{\mathfrak{D}_{p-1}}$ to $X_{\mathfrak{D}_p}$ (and vice versa) by an elementary transformation, cf. [12, Remark 6.3, pp. 105–106], we illustrate in Figure 4 how the equivariant birational morphisms connecting all the above mentioned compact toric surfaces affect their WVE^{2}C-graphs.

(iv) Since $q = p + 1$ and $\gcd(p+1, p-1) = \gcd(p+1, 2) \in \{1, 2\}$, Formula (4.2) shows that the index ℓ of $X_Q \cong X_{Q_p^{[k]}}$ equals $\frac{p+1}{2}$ whenever p is odd and $p+1$ whenever p is even. This bears out our assertion about ℓ. □

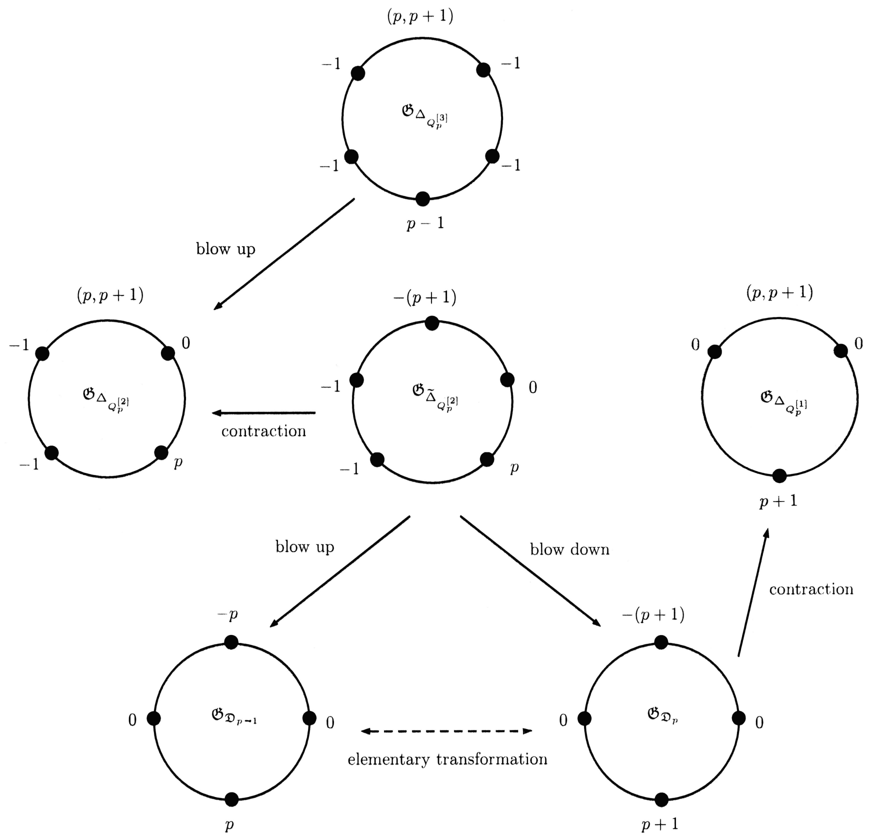

Figure 4

Remark 5.6. Among the LDP-polygons $Q_p^{[k]}$, only $Q_1^{[1]}$, $Q_1^{[2]}$, $Q_1^{[3]}$ are *reflexive* (with index $\ell = 1$ and a unique *Gorenstein* singularity).

6 Defining equations

Let Q be an arbitrary LDP-polygon. Since the Cartier divisor $-\ell K_{X_Q}$ on X_Q is very ample, setting

$$\delta_Q := \sharp((\ell\mathring{Q}) \cap \mathbb{Z}^2) - 1,$$

the complete linear system $|-\ell K_{X_Q}|$ induces the closed embedding $\Phi_{|-\ell K_{X_Q}|}$,

$$\begin{tikzcd} \mathbb{T} \arrow[r, hook, "\iota"] \arrow[rr, bend right] & X_Q \arrow[r, hook, "\Phi_{|-\ell K_{X_Q}|}"] & \mathbb{P}_{\mathbb{C}}^{\delta_Q} \end{tikzcd}$$

with

$$\mathbb{T} \ni t \longmapsto (\Phi_{|-\ell K_{X_Q}|} \circ \iota)(t) := [\dots : z_{(i,j)} : \dots]_{(i,j)\in(\ell\mathring{Q})\cap\mathbb{Z}^2} \in \mathbb{P}_{\mathbb{C}}^{\delta_Q}, \quad z_{(i,j)} := \chi^{(i,j)}(t),$$

where $\mathbb{T}$ denotes the algebraic torus $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^2, \mathbb{C}^*)$ and $\chi^{(i,j)} : \mathbb{T} \to \mathbb{C}^*$ is the character associated to the lattice point $(i,j) \in (\ell\mathring{Q}) \cap \mathbb{Z}^2$. The image $\Phi_{|-\ell K_{X_Q}|}(X_Q)$ of the surface X_Q under $\Phi_{|-\ell K_{X_Q}|}$ is the Zariski closure of $\mathrm{Im}(\Phi_{|-\ell K_{X_Q}|} \circ \iota)$ in $\mathbb{P}_{\mathbb{C}}^{\delta_Q}$ and can be viewed as the projective variety $\mathrm{Proj}(S_{\ell\mathring{Q}})$, where

$$S_{\ell\mathring{Q}} := \mathbb{C}[C(\ell\mathring{Q}) \cap \mathbb{Z}^3] = \bigoplus_{\kappa=0}^{\infty} \Big(\bigoplus_{(i,j)\in(\kappa(\ell\mathring{Q}))\cap\mathbb{Z}^2} \mathbb{C}{\cdot}\chi^{(i,j)} s^{\kappa} \Big),$$

with $C(\ell\mathring{Q}) := \{(\lambda y_1, \lambda y_2, \lambda) \mid \lambda \in \mathbb{R}_{\geq 0} \text{ and } (y_1, y_2) \in \ell\mathring{Q}\}$, is the semigroup algebra which is naturally graded by setting $\deg(\chi^{(i,j)} s^{\kappa}) := \kappa$; for a detailed exposition see [11, Theorem 2.3.1, p. 75; Proposition 5.4.7, pp. 237–238; Theorem 5.4.8, pp. 239–240, and Theorem 7.1.13, pp. 325–326]. Equivalently, it can be viewed as the zero set $\mathbb{V}(I_{\mathcal{A}_Q}) \subset \mathbb{P}_{\mathbb{C}}^{\delta_Q}$ of the homogeneous ideal $I_{\mathcal{A}_Q} := \operatorname{Ker}(\pi_Q)$, where

$$\mathcal{A}_Q := \{(i, j, 1) \mid (i, j) \in (\ell\mathring{Q}) \cap \mathbb{Z}^2\} \subset \mathbb{Z}^2 \times \{1\} \subset \mathbb{Z}^3$$

and π_Q is the $\mathbb{C}$-algebra homomorphism

$$\mathbb{C}[\ldots : z_{(i,j)} : \ldots]_{(i,j)\in(\ell\mathring{Q})\cap\mathbb{Z}^2} \xrightarrow{\pi_Q} \mathbb{C}[\ldots, \chi^{(i,j,1)}, \ldots]_{(i,j,1)\in\mathcal{A}_Q}, \quad z_{(i,j)} \longmapsto \chi^{(i,j,1)}.$$

Furthermore, the *projective degree* $d_Q := \deg(\mathbb{V}(I_{\mathcal{A}_Q}))$ of $\mathbb{V}(I_{\mathcal{A}_Q})$, i.e. the double of the leading coefficient of the Hilbert polynomial of the homogeneous coordinate ring $\mathbb{C}[\ldots : z_{(i,j)} : \ldots]_{(i,j)\in(\ell\mathring{Q})\cap\mathbb{Z}^2}/I_{\mathcal{A}_Q}$, is given by

$$d_Q = 2\operatorname{area}(\ell\mathring{Q}); \tag{6.1}$$

see Sturmfels [40, Theorem 4.16, pp. 36–37, and p. 131] and [11, Proposition 9.4.3, pp. 432–433].

Theorem 6.1 (Koelman [29]). *If $\sharp(\partial(\ell\mathring{Q}) \cap \mathbb{Z}^2) \geq 4$, then $I_{\mathcal{A}_Q}$ is generated by all possible quadratic binomials, i.e. we have*

$$I_{\mathcal{A}_Q} = \left\langle \left\{ z_{(i_1,j_1)} z_{(i_2,j_2)} - z_{(i_1',j_1')} z_{(i_2',j_2')} \;\middle|\; \begin{array}{l} (i_1, j_1), (i_2, j_2), (i_1', j_1'), (i_2', j_2') \in (\ell\mathring{Q}) \cap \mathbb{Z}^2 \\ \text{with } (i_1, j_1) + (i_2, j_2) = (i_1', j_1') + (i_2', j_2') \end{array} \right\} \right\rangle.$$

Corollary 6.2 (Castryck & Cools [9, Section 2]). *If $\sharp(\partial(\ell\mathring{Q}) \cap \mathbb{Z}^2) \geq 4$ and if we denote by β_Q the cardinality of any minimal system of quadrics generating the ideal $I_{\mathcal{A}_Q}$, then*

$$\beta_Q = \binom{\delta_Q + 2}{2} - \sharp(2(\ell\mathring{Q}) \cap \mathbb{Z}^2). \tag{6.2}$$

Proof. Let $\mathrm{HP}_2(\mathbb{P}_{\mathbb{C}}^{\delta_Q})$ be the set of all homogeneous polynomials in $\delta_Q + 1$ variables of degree 2. Then the $\mathbb{C}$-vector space homomorphism

$$f : \mathrm{HP}_2(\mathbb{P}_{\mathbb{C}}^{\delta_Q}) \longrightarrow \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \ \text{ mapping } z_{(i_1,j_1)} z_{(i_2,j_2)} \text{ to } x^{i_1+i_2} y^{j_1+j_2}$$

has the $\mathbb{C}$-vector space of homogeneous polynomials of degree 2 belonging to $I_{\mathcal{A}_Q}$ as kernel $\operatorname{Ker}(f)$, and the linear span of $\{x^i y^j \mid (i, j) \in 2(\ell\mathring{Q}) \cap \mathbb{Z}^2\}$ is the image $\operatorname{Im}(f)$, because every lattice point in $2(\ell\mathring{Q})$ is the sum of two lattice points of $\ell\mathring{Q}$, cf. [11, Theorem 2.2.12, pp. 68–69]. Taking into account Koelman's Theorem 6.1, [40, Lemma 4.1, p. 31], and the fact that $\mathbb{V}(I_{\mathcal{A}_Q})$ is not contained in any hyperplane of $\mathbb{P}_{\mathbb{C}}^{\delta_Q}$, the equality

$$\dim_{\mathbb{C}}(\operatorname{Ker}(f)) = \dim_{\mathbb{C}}(\mathrm{HP}_2(\mathbb{P}_{\mathbb{C}}^{\delta_Q})) - \dim_{\mathbb{C}}(\operatorname{Im}(f))$$

gives (6.2). □

Back to toric log del Pezzo surfaces with one singularity. Let Q be an LDP-polygon such that X_Q has exactly one singularity. By Theorem 1.4 there exist $p \in \mathbb{Z}_{>0}$ and $k \in \{1, 2, 3\}$ such that $X_Q \cong X_{Q_p^{[k]}}$ with index $\ell = \frac{p+1}{2}$ if p is odd and $\ell = p + 1$ if p is even. For this reason, to apply Corollary 6.2 and to prove Theorem 1.5 we shall take a closer look at the dilated polars $\ell\mathring{Q}_p^{[k]}$ of the polygons $Q_p^{[k]}$ defined in Theorem 1.4(ii).

Lemma 6.3. *The vertex sets of the polygons $\ell\mathring{Q}_p^{[k]}$, $k \in \{1, 2, 3\}$, are the following:*

$$\mathcal{V}(\ell\mathring{Q}_p^{[1]}) = \begin{cases} \left\{ \binom{-1}{(p-1)/2}, \binom{(p+1)/2}{-(p+1)^2/2}, \binom{(p+1)/2}{p+1} \right\} & \text{if } p \text{ is odd,} \\ \left\{ \binom{-2}{p-1}, \binom{p+1}{-(p+1)^2}, \binom{p+1}{2(p+1)} \right\} & \text{if } p \text{ is even,} \end{cases}$$

$$\mathcal{V}(\ell\mathring{Q}_p^{[2]}) = \begin{cases} \left\{ \binom{-1}{(p-1)/2}, \binom{0}{-(p+1)/2}, \binom{(p+1)/2}{-p(p+1)/2}, \binom{(p+1)/2}{p+1} \right\} & \text{if } p \text{ is odd,} \\ \left\{ \binom{-2}{p-1}, \binom{0}{-(p+1)}, \binom{p+1}{-p(p+1)}, \binom{p+1}{2(p+1)} \right\} & \text{if } p \text{ is even,} \end{cases}$$

$$\mathcal{V}(\ell\mathring{Q}_p^{[3]}) = \begin{cases} \left\{ \binom{-1}{(p-1)/2}, \binom{0}{-(p+1)/2}, \binom{(p+1)/2}{-p(p+1)/2}, \binom{(p+1)/2}{(p+1)/2}, \binom{0}{(p+1)/2} \right\} & \text{if } p \text{ is odd,} \\ \left\{ \binom{-2}{p-1}, \binom{0}{-(p+1)}, \binom{p+1}{-p(p+1)}, \binom{p+1}{p+1}, \binom{0}{p+1} \right\} & \text{if } p \text{ is even.} \end{cases}$$

Proof. Since $Q_p^{[1]} = \operatorname{conv}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$ with $\mathbf{v}_1 = \binom{1}{-1}$, $\mathbf{v}_2 = \binom{p}{1}$, $\mathbf{v}_3 = \binom{-1}{0}$, we have

$$\boldsymbol{\eta}_{\operatorname{conv}(\{\mathbf{v}_1,\mathbf{v}_2\})} = \binom{x}{y} \text{ with } \left\langle \binom{x}{y}, \binom{1}{-1} \right\rangle = \ell = \left\langle \binom{x}{y}, \binom{p}{1} \right\rangle,$$

hence $x = \frac{2\ell}{p+1}$ and $y = -\frac{(p-1)\ell}{p+1}$ (and similarly with the others). Thus

$$\boldsymbol{\eta}_{\operatorname{conv}(\{\mathbf{v}_1,\mathbf{v}_2\})} = \begin{pmatrix} \frac{2\ell}{p+1} \\ -\frac{(p-1)\ell}{p+1} \end{pmatrix}, \quad \boldsymbol{\eta}_{\operatorname{conv}(\{\mathbf{v}_2,\mathbf{v}_3\})} = \begin{pmatrix} -1 \\ p+1 \end{pmatrix}, \quad \boldsymbol{\eta}_{\operatorname{conv}(\{\mathbf{v}_3,\mathbf{v}_1\})} = \begin{pmatrix} -1 \\ -2 \end{pmatrix},$$

where $l_{\operatorname{conv}(\{\mathbf{v}_1,\mathbf{v}_2\})} = \ell$, $l_{\operatorname{conv}(\{\mathbf{v}_2,\mathbf{v}_3\})} = l_{\operatorname{conv}(\{\mathbf{v}_3,\mathbf{v}_1\})} = 1$, and (4.3) gives

$$\mathcal{V}(\mathring{Q}_p^{[1]}) = \left\{ \begin{pmatrix} -\frac{2}{p+1} \\ \frac{p-1}{p+1} \end{pmatrix}, \begin{pmatrix} 1 \\ -(p+1) \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

Analogously, we conclude that

$$\mathcal{V}(\mathring{Q}_p^{[2]}) = \left\{ \begin{pmatrix} -\frac{2}{p+1} \\ \frac{p-1}{p+1} \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -p \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}, \qquad \mathcal{V}(\mathring{Q}_p^{[3]}) = \left\{ \begin{pmatrix} -\frac{2}{p+1} \\ \frac{p-1}{p+1} \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -p \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

After multiplication with the index ℓ we get $\mathcal{V}(\ell\mathring{Q}_p^{[k]})$ for $k \in \{1, 2, 3\}$. □

Lemma 6.4. *The number of lattice points on $\partial(\ell\mathring{Q}_p^{[k]})$ is given in the following table:*

No.	p	k	$\sharp(\partial(\ell\mathring{Q}_p^{[k]}) \cap \mathbb{Z}^2)$	No.	p	k	$\sharp(\partial(\ell\mathring{Q}_p^{[k]}) \cap \mathbb{Z}^2)$
(i)	odd	1	$\frac{1}{2}(p+3)^2$	(iv)	even	2	$p^2 + 5p + 8$
(ii)	even	1	$(p+3)^2$	(v)	odd	3	$\frac{1}{2}(p^2 + 4p + 7)$
(iii)	odd	2	$\frac{1}{2}(p^2 + 5p + 8)$	(vi)	even	3	$p^2 + 4p + 7$

Proof. Since the number of lattice points lying on the boundary of a lattice-polygon (with respect to $\mathbb{Z}^2$) is computed by the sum of the greatest common divisors of the differences of the vertex-coordinates of its edges, the above table is produced directly by using Lemma 6.3. □

Remark 6.5. Since $\sharp(\partial(\ell\mathring{Q}_p^{[k]}) \cap \mathbb{Z}^2) \geq 6$ for all $p \in \mathbb{Z}_{>0}$ and all $k \in \{1, 2, 3\}$, Theorem 6.1 and Corollary 6.2 can be applied for the LDP-polygons $Q_p^{[k]}$.

Lemma 6.6. *The projective degree $d_{Q_p^{[k]}}$ of $\mathbb{V}(I_{\mathcal{A}_{Q_p^{[k]}}})$ is given in the following table:*

No.	p	k	$d_{Q_p^{[k]}}$	No.	p	k	$d_{Q_p^{[k]}}$
(i)	odd	1	$\frac{1}{4}(p+1)(p+3)^2$	(iv)	even	2	$(p+1)(p^2 + 5p + 8)$
(ii)	even	1	$(p+1)(p+3)^2$	(v)	odd	3	$\frac{1}{4}(p+1)(p^2 + 4p + 7)$
(iii)	odd	2	$\frac{1}{4}(p+1)(p^2 + 5p + 8)$	(vi)	even	3	$(p+1)(p^2 + 4p + 7)$

Proof. To determine the area of $\ell\mathring{Q}_p^{[k]}$ one may work with its vertex set given in Lemma 6.3. Alternatively, using [38, Proposition 2.10, p. 79] and Formula (3.4) for $X_{Q_p^{[k]}}$ we deduce that

$$2\operatorname{area}(\mathring{Q}_p^{[k]}) = K^2_{X_{Q_p^{[k]}}} = 6 - k + p + \frac{4}{p+1},$$

and we read off $d_{Q_p^{[k]}}$ easier via Formula (6.1) which gives $d_{Q_p^{[k]}} = \ell^2 K^2_{X_{Q_p^{[k]}}}$. □

Lemma 6.7. *The dimension $\delta_{Q_p^{[k]}}$ of the projective space in which $\mathbb{V}(I_{\mathcal{A}_{Q_p^{[k]}}})$ is embedded equals*

$$\delta_{Q_p^{[k]}} = \frac{1}{2}(d_{Q_p^{[k]}} + \sharp(\partial(\ell\mathring{Q}_p^{[k]}) \cap \mathbb{Z}^2)). \tag{6.3}$$

Proof. Equation (6.3) is an immediate consequence of Pick's formula. □

Lemma 6.8. *The number $\beta_{Q_p^{[k]}}$ of the elements of any minimal generating system of $I_{\mathcal{A}_{Q_p^{[k]}}}$ is given by*

$$\beta_{Q_p^{[k]}} = \tfrac{1}{2}(\delta_{Q_p^{[k]}} + 1)(\delta_{Q_p^{[k]}} + 2) - (2d_{Q_p^{[k]}} + \sharp(\partial(\ell\mathring{Q}_p^{[k]}) \cap \mathbb{Z}^2) + 1). \tag{6.4}$$

Proof. By the main properties of the Ehrhart polynomial of the lattice polygon $\ell\mathring{Q}_p^{[k]}$, cf. [11, Example 9.4.4, p. 433], we obtain

$$\sharp(2(\ell\mathring{Q}_p^{[k]}) \cap \mathbb{Z}^2) = 4\mathrm{area}(\ell\mathring{Q}) + \sharp(\partial(\ell\mathring{Q}_p^{[k]}) \cap \mathbb{Z}^2) + 1.$$

Hence, (6.4) follows from (6.2) and (6.1). □

Hyperplanes $\mathcal{H} \subset \mathbb{P}_{\mathbb{C}}^{\delta_{Q_p^{[k]}}}$ give curves $\mathbb{V}(I_{\mathcal{A}_{Q_p^{[k]}}}) \cap \mathcal{H}$ which are linearly equivalent to $-\ell K_{X_{Q_p^{[k]}}}$. For *generic* hyperplanes $\mathcal{H}$ the intersection $\mathcal{C}_{Q_p^{[k]}} := \mathbb{V}(I_{\mathcal{A}_{Q_p^{[k]}}}) \cap \mathcal{H}$ is (by Bertini's Theorem) a smooth connected curve in the smooth locus of $\mathbb{V}(I_{\mathcal{A}_{Q_p^{[k]}}}) \cong X_{Q_p^{[k]}}$. The genus of $\mathcal{C}_{Q_p^{[k]}}$ is called the *sectional genus* $g_{Q_p^{[k]}}$ *of* $X_{Q_p^{[k]}}$.

Lemma 6.9. *The sectional genus of $X_{Q_p^{[k]}}$ is*

$$g_{Q_p^{[k]}} = \delta_{Q_p^{[k]}} - \sharp(\partial(\ell\mathring{Q}_p^{[k]}) \cap \mathbb{Z}^2) + 1. \tag{6.5}$$

Proof. Equation (6.5) follows from the fact that $g_{Q_p^{[k]}} = \sharp(\mathrm{int}(\ell\mathring{Q}_p^{[k]}) \cap \mathbb{Z}^2)$; see [11, Proposition 10.5.8, p. 509]. □

Proof of Theorem 1.5. The number $\sharp(\partial(\ell\mathring{Q}_p^{[k]}) \cap \mathbb{Z}^2)$ and the projective degree $d_{Q_p^{[k]}}$ are known from Lemmas 6.4 and 6.6, respectively, while $\delta_{Q_p^{[k]}}$ is computed via Formula (6.3), leading to Table (1.1), and consequently to Table (1.2) by making use of Formula (6.4). Finally, one obtains Table (1.3) by means of the Formula (6.5). □

Note 6.10. See [10] for a `Magma` code to compute a minimal generating system of the ideal defining the projective toric surface associated to an *arbitrary* lattice polygon. In our particular case (we deal only with *quadrics*) it is enough to collect all vectorial relations $(i_1, j_1) + (i_2, j_2) = (i_1', j_1') + (i_2', j_2')$ and to determine a $\mathbb{C}$-linearly independent subset of the set of the corresponding quadratic binomials $z_{(i_1,j_1)}z_{(i_2,j_2)} - z_{(i_1',j_1')}z_{(i_2',j_2')}$ by simply applying Gaussian elimination. For a short routine written in `Python` see [14].

Examples 6.11. (i) The ideal $I_{\mathcal{A}_{Q_1^{[2]}}}$ (with $\mathbb{V}(I_{\mathcal{A}_{Q_1^{[2]}}}) \subset \mathbb{P}_{\mathbb{C}}^7$) is minimally generated by the following 14 quadrics:

$$\begin{array}{lll}
z_{(-1,0)}z_{(1,-1)} - z_{(0,-1)}z_{(0,0)}, & z_{(-1,0)}z_{(1,0)} - z_{(0,-1)}z_{(0,1)}, & z_{(1,0)}^2 - z_{(1,1)}z_{(1,-1)}, \\
z_{(-1,0)}z_{(1,1)} - z_{(0,0)}z_{(0,1)}, & z_{(1,1)}z_{(1,0)} - z_{(1,2)}z_{(1,-1)}, & z_{(1,1)}^2 - z_{(1,2)}z_{(1,0)}, \\
z_{(0,1)}^2 - z_{(-1,0)}z_{(1,2)}, & z_{(0,1)}z_{(1,-1)} - z_{(0,-1)}z_{(1,1)}, & z_{(0,1)}z_{(1,0)} - z_{(0,-1)}z_{(1,2)}, \\
z_{(0,1)}z_{(1,1)} - z_{(0,0)}z_{(1,2)}, & z_{(0,0)}z_{(1,-1)} - z_{(0,-1)}z_{(1,0)}, & z_{(0,0)}z_{(1,0)} - z_{(0,-1)}z_{(1,1)}, \\
z_{(0,0)}z_{(1,1)} - z_{(0,-1)}z_{(1,2)}, & z_{(0,0)}^2 - z_{(0,-1)}z_{(0,1)}. &
\end{array}$$

(ii) Correspondingly, the 9 quadrics

$$\begin{array}{lll}
z_{(-1,0)}z_{(1,0)} - z_{(0,1)}z_{(0,-1)}, & z_{(1,0)}^2 - z_{(1,1)}z_{(1,-1)}, & z_{(-1,0)}z_{(1,-1)} - z_{(0,0)}z_{(0,-1)}, \\
z_{(-1,0)}z_{(1,1)} - z_{(0,1)}z_{(0,0)}, & z_{(0,-1)}z_{(1,0)} - z_{(0,0)}z_{(1,-1)}, & z_{(0,-1)}z_{(1,1)} - z_{(0,1)}z_{(1,-1)}, \\
z_{(0,0)}z_{(1,0)} - z_{(0,1)}z_{(1,-1)}, & z_{(0,0)}z_{(1,1)} - z_{(0,1)}z_{(1,0)}, & z_{(0,0)}^2 - z_{(0,1)}z_{(0,-1)}
\end{array}$$

form a minimal set of generators of the ideal $I_{\mathcal{A}_{Q_1^{[3]}}}$, and $\mathbb{V}(I_{\mathcal{A}_{Q_1^{[3]}}}) \subset \mathbb{P}_{\mathbb{C}}^6$. The surface $X_{Q_1^{[3]}}$ is obtained by blowing up $X_{Q_1^{[2]}}$ at one non-singular point, cf. Figure 5.

(iii) The next example is much more complicated; it is created by the LDP-polygon $Q_3^{[3]}$, cf. Figure 6, in which $2\mathring{Q}_3^{[3]} \cap \mathbb{Z}^2$ consists of 22 lattice points, and $\mathbb{V}(I_{\mathcal{A}_{Q_3^{[3]}}}) \subset \mathbb{P}_{\mathbb{C}}^{21}$.

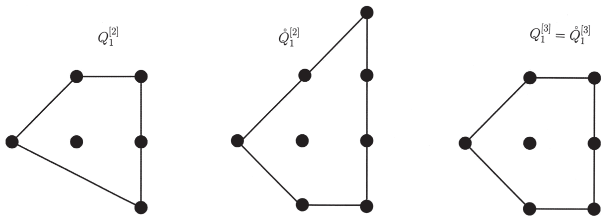

Figure 5

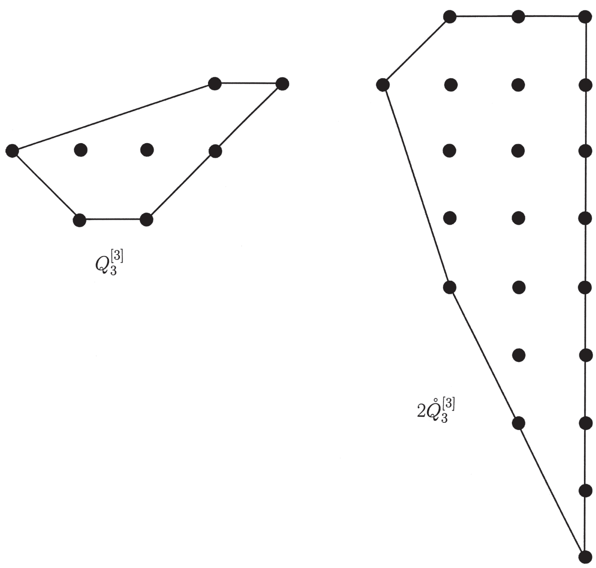

Figure 6

Using [14] we see that $I_{\mathcal{A}_{Q_3^{[3]}}}$ is minimally generated by the following 182 quadrics:

$$
\begin{array}{llll}
z_{(0,-2)}z_{(2,-2)} - z_{(1,-4)}z_{(1,0)}, & z_{(1,-4)}z_{(2,-4)} - z_{(1,-2)}z_{(2,-6)}, & z_{(-1,1)}z_{(1,-1)} - z_{(0,-2)}z_{(0,2)}, & z_{(0,-2)}z_{(2,0)} - z_{(1,-1)}z_{(1,-1)}, \\
z_{(-1,1)}z_{(1,-4)} - z_{(0,-2)}z_{(0,-1)}, & z_{(1,-4)}z_{(2,-1)} - z_{(1,-1)}z_{(2,-4)}, & z_{(0,-2)}z_{(2,-4)} - z_{(1,-4)}z_{(1,-2)}, & z_{(2,-6)}z_{(2,-1)} - z_{(2,-5)}z_{(2,-2)}, \\
z_{(0,-2)}z_{(2,-3)} - z_{(0,0)}z_{(2,-5)}, & z_{(-1,1)}z_{(2,0)} - z_{(0,2)}z_{(1,-1)}, & z_{(0,-1)}z_{(2,2)} - z_{(0,0)}z_{(2,1)}, & z_{(-1,1)}z_{(2,-6)} - z_{(0,-1)}z_{(1,-4)}, \\
z_{(0,-2)}z_{(2,-5)} - z_{(0,-1)}z_{(2,-6)}, & z_{(1,-4)}z_{(2,2)} - z_{(1,-1)}z_{(2,-1)}, & z_{(1,-4)}z_{(2,1)} - z_{(1,0)}z_{(2,-3)}, & z_{(-1,1)}z_{(2,-5)} - z_{(0,-2)}z_{(1,-2)}, \\
z_{(-1,1)}z_{(2,1)} - z_{(0,2)}z_{(1,0)}, & z_{(-1,1)}z_{(2,-3)} - z_{(0,1)}z_{(1,-3)}, & z_{(-1,1)}z_{(2,-5)} - z_{(0,-1)}z_{(1,-3)}, & z_{(1,-4)}z_{(2,0)} - z_{(1,-2)}z_{(2,-2)}, \\
z_{(1,-4)}z_{(2,1)} - z_{(1,-1)}z_{(2,-2)}, & z_{(1,-4)}z_{(2,-3)} - z_{(1,-1)}z_{(2,-6)}, & z_{(1,-4)}z_{(2,2)} - z_{(1,-3)}z_{(2,1)}, & z_{(-1,1)}z_{(2,-1)} - z_{(0,1)}z_{(1,-1)}, \\
z_{(0,0)}z_{(2,2)} - z_{(0,2)}z_{(2,0)}, & z_{(-1,1)}z_{(2,-4)} - z_{(0,1)}z_{(1,-4)}, & z_{(1,-3)}z_{(2,2)} - z_{(1,0)}z_{(2,-1)}, & z_{(0,-2)}z_{(2,-2)} - z_{(1,-2)}^2, \\
z_{(1,-4)}z_{(2,0)} - z_{(1,1)}z_{(2,-5)}, & z_{(1,-4)}z_{(2,-2)} - z_{(1,0)}z_{(2,-6)}, & z_{(2,-6)}z_{(2,-1)} - z_{(2,-4)}z_{(2,-3)}, & z_{(2,-6)}z_{(2,2)} - z_{(2,-4)}z_{(2,0)}, \\
z_{(2,-2)}z_{(2,2)} - z_{(2,0)}^2, & z_{(1,-3)}z_{(2,2)} - z_{(1,-2)}z_{(2,1)}, & z_{(0,0)}z_{(2,2)} - z_{(0,1)}z_{(2,1)}, & z_{(1,-1)}z_{(2,2)} - z_{(1,2)}z_{(2,-1)}, \\
z_{(-1,1)}z_{(2,-2)} - z_{(0,-2)}z_{(1,1)}, & z_{(2,-6)}z_{(2,-2)} - z_{(2,-4)}z_{(2,-4)}, & z_{(1,-1)}z_{(2,2)} - z_{(1,1)}z_{(2,0)}, & z_{(-1,1)}z_{(2,-4)} - z_{(0,-1)}z_{(1,-2)}, \\
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z_{(0,-2)}z_{(2,1)} - z_{(1,-1)}z_{(1,0)}, & z_{(-1,1)}z_{(2,-3)} - z_{(0,0)}z_{(1,-2)}, & z_{(-1,1)}z_{(2,-2)} - z_{(0,0)}z_{(1,-1)}, & z_{(0,-2)}z_{(2,-1)} - z_{(1,-2)}z_{(1,-1)},
\end{array}
$$

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