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On the string-theoretic Euler number of a class of absolutely isolated singularities

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Abstract. An explicit computation of the so-called string-theoretic E-function $E_{\text{str}}(X; u, v)$ of a normal complex variety X with at most log-terminal singularities can be achieved by constructing one snc-desingularization of X, accompanied with the intersection graph of the exceptional prime divisors, and with the precise knowledge of their structure. In the present paper, it is shown that this is feasible for the case in which X is the underlying space of a class of absolutely isolated singularities (including both usual $\mathbf{A}_n$-singularities and Fermat singularities of arbitrary dimension). As byproduct of the exact evaluation of $e_{\text{str}}(X) = \lim_{u,v\to 1} E_{\text{str}}(X; u, v)$, for this class of singularities, one gets counterexamples to a conjecture of Batyrev concerning the boundedness of the string-theoretic index. Finally, the string-theoretic Euler number is also computed for global complete intersections in $\mathbb{P}^N_{\mathbb{C}}$ with prescribed singularities of the above type.

1. Introduction

The so-called E_{str}-*polynomials* $E_{\text{str}}(X; u, v)$ of normal complex varieties X with at most Gorenstein quotient or toroidal singularities were introduced in [5], and were used as main tools in [5] and [3] for the proof of several mirror-symmetry identities. More recently, Batyrev [1] generalized this notion also for X's having at most log-terminal singularities, by introducing E_{str}-*functions* instead which may be not even rational. These new invariants have already found lots of applications in the study of log-flips and of cohomological McKay correspondence. (See [2, 1.6, 4.11 and 8.4] and [9, Thm. 5.1].)

In the present paper we give explicit formulae for the evaluation of the function $E_{\text{str}}(X; u, v)$ for those X's which are the underlying spaces of two special series of $\mathbf{A}^{(r)}_{n,\ell}$-*singularities* (see below (d) for the precise definition) by constructing an appropriate snc-resolution $\varphi : \widetilde{X} \longrightarrow X$, by examining the nature of the arising exceptional prime divisors and, finally, by computing their E -polynomials. (In [7] this was carried out for all three-dimensional **A**-**D**-**E** singularities).

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(a) Log-terminal singularities. Let X be a normal complex variety. Suppose that X is $\mathbb{Q}$-Gorenstein, i.e., that a positive integer multiple of its canonical Weil divisor K_X is a Cartier divisor. X is said to have at most *log-terminal* (respectively, *canonical / terminal*) singularities if there exists an *snc*-desingularization $\varphi : \widetilde{X} \longrightarrow X$, i.e., a desingularization of X whose exceptional locus $\mathfrak{Ex}(\varphi) = \cup_{i=1}^{m} D_i$ consists of **s**mooth prime divisors $D_1, D_2, \ldots, D_m$ with only **n**ormal **c**rossings, such that the "discrepancy" w.r.t. φ is of the form $K_{\widetilde{X}} - \varphi^*(K_X) = \sum_{i=1}^{m} a_i \, D_i$, with all the a_i's > -1 ($\geq 0 / > 0$). These inequalities do not depend on the particular choice of φ.

(b) E-polynomials. Deligne proved in [8, §8] that the cohomology groups $H^i(X, \mathbb{Q})$ of any complex variety X are endowed with a natural *mixed Hodge structure* (MHS). The same remains true if one works with cohomologies $H_c^i(X, \mathbb{Q})$ with compact supports. There exist namely an increasing weight-filtration $\mathcal{W}_\bullet$ and a decreasing Hodge-filtration of $H^i(X, \mathbb{Q})$ (resp. $H_c^i(X, \mathbb{C})$) which induces a natural filtration $\mathcal{F}^\bullet$ on the complexification of the corresponding graded pieces $Gr_k^{\mathcal{W}_\bullet}(H^i(X, \mathbb{Q}))$ (resp. $Gr_k^{\mathcal{W}_\bullet}(H_c^i(X, \mathbb{Q}))$). Let

$$h^{p,q}(H^i(X, \mathbb{C})) := \dim_{\mathbb{C}} Gr^p_{\mathcal{F}^\bullet} Gr^{\mathcal{W}_\bullet}_{p+q}(H^i(X, \mathbb{C}))$$
$$(\text{ resp. } h^{p,q}(H_c^i(X, \mathbb{C})) := \dim_{\mathbb{C}} Gr^p_{\mathcal{F}^\bullet} Gr^{\mathcal{W}_\bullet}_{p+q}(H_c^i(X, \mathbb{C})) \text{ })$$

denote hereafter the corresponding *Hodge numbers*. The so-called E-**polynomial** of X is defined to be

$$E(X; u, v) := \sum_{p,q} e^{p,q}(X) \; u^p v^q \in \mathbb{Z}[u, v],$$

where $e^{p,q}(X) := \sum_{i \geq 0} (-1)^i \; h^{p,q}(H_c^i(X, \mathbb{C}))$. (If we set $u = v = 1$, then $E(X; 1, 1)$ equals the usual *topological* Euler characteristic $e(X)$ of X.)

(c) $E_{\mathbf{str}}$-functions. To pass to string-theoretic invariants, one takes essentially into account the "discrepancy coefficients".

Definition 1.1. *Let X be a normal complex variety with at most log-terminal singularities, $\varphi : \widetilde{X} \longrightarrow X$ an snc-desingularization of X as in* (a)*, $D_1, D_2, \ldots, D_m$ the prime divisors of the exceptional locus, and $I := \{1, 2, \ldots, m\}$. For any subset $J \subseteq I$ define*

$$D_J := \begin{cases} \widetilde{X}, & \text{if } J = \varnothing \\ \bigcap_{j \in J} D_j, & \text{if } J \neq \varnothing \end{cases} \quad \text{and} \quad D_J^\circ := D_J \smallsetminus \bigcup_{j \in I \smallsetminus J} D_j.$$

The algebraic function

$$E_{str}(X; u, v) := \sum_{J \subseteq I} E\left(D_J^\circ; u, v\right) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1} \tag{1.1}$$

(under the convention for $\prod_{j \in J}$ to be 1, if $J = \varnothing$, and $E(\varnothing; u, v) := 0$) is called the **string-theoretic E-function** *(or simply* E_{str}-**function**) *of* X.

The major result of [1] says that:

Theorem 1.2. $E_{\mathrm{str}}\,(X; u, v)$ *is independent of the choice of the snc- desingularization* $\varphi : \widetilde{X} \longrightarrow X$.

Remark 1.3. (i) Though the string-theoretic function $E_{\mathrm{str}}\,(X; u, v)$ enjoys this particularly important invariance property, to *evaluate* it by (1.1) one needs not only the *existence* of (at least one) snc-desingularization (which is guaranteed, e.g., by Hironaka's main theorems [18]), but also the precise knowledge of what kind of exceptional prime divisors are available on the corresponding smooth model, and which are their intersections. In general, there are several ways to resolve log-terminal singularities, involving different choices for the centers of the modifications of X and, sometimes, necessary extra normalizations, blow-ups of non-reduced subschemes etc. For this reason, a first realistic attempt to understand the behaviour of (1.1), from the computational point of view, cannot overlook the class of *absolutely isolated* singularities, i.e., isolated singularities resolvable by a finite sequence of (usual) blow-ups of closed points, for which one may keep the needed details (strict transforms after each step of the resolution procedure, snc-condition etc.) under control.

(ii) It is also worth mentioning that the "first summand" in (1.1), i.e., for $J = \varnothing$, equals

$$E(\widetilde{X} \smallsetminus \textstyle\bigcup_{j=1}^{m} D_j; u, v) = E(X \smallsetminus \mathrm{Sing}(X); u, v)$$

(where $\mathrm{Sing}(X)$ denotes the singular locus of X). This means that it can be described exclusively by the study of topological properties of X "around" the singularities without involving any resolution data.

Definition 1.4. *The rational number*

$$e_{\mathrm{str}}\,(X) := \lim_{u,v\to 1} E_{\mathrm{str}}\,(X; u, v) = \sum_{J\subseteq I} e\left(D_J^{\circ}\right) \prod_{j\in J} \frac{1}{a_j+1} \tag{1.2}$$

is called the **string-theoretic Euler number** *of* X. *Moreover, the* **string-theoretic index** $\mathrm{ind}_{\mathrm{str}}\,(X)$ *of* X *is defined to be the integer*

$$\mathrm{ind}_{\mathrm{str}}\,(X) := \min\left\{ l \in \mathbb{Z}_{\geq 1} \;\middle|\; e_{\mathrm{str}}\,(X) \in \frac{1}{l}\,\mathbb{Z} \right\}.$$

Examples 1.5. (i) For $\mathbb{Q}$-Gorenstein toric varieties X, $\mathrm{ind}_{\mathrm{str}}\,(X) = 1$, and $e_{\mathrm{str}}\,(X)$ is equal to the normalized volume of the defining fan. Moreover, for Gorenstein toric varieties X, $E_{\mathrm{str}}\,(X; u, v)$ is a polynomial (cf. [1, 4.4 and 4.10]).

(ii) Normal algebraic surfaces X with at most log-terminal singularities have string-theoretic index $\mathrm{ind}_{\mathrm{str}}\,(X) = 1$. There exist, however, normal complex varieties X of dimension $d \geq 3$ with at most Gorenstein canonical singularities having $\mathrm{ind}_{\mathrm{str}}\,(X) > 1$.

Batyrev's conjecture [1, 5.9], concerning the range of $\mathrm{ind}_{\mathrm{str}}\,(X)$, can be stated as follows:

Conjecture 1.6 (Boundedness of the string-theoretic index). Let X be an r-dimensional normal complex variety having at most Gorenstein canonical singularities. Then $\mathrm{ind}_{\mathrm{str}}(X)$ is bounded by a constant $C(r)$ depending only on r.

As it turns out (see below Remark 1.9), and in contrast to initial expectations due to some classes of examples (see, e.g., [1, 5.1, 5.10] for the case of cones over certain smooth projective Fano varieties), conjecture 1.6 is not true in general. Nevertheless, the characterization of those classes of X's, which admit bounded string-theoretic indices, remains an unsolved problem.

(d) The $\mathbf{A}^{(r)}_{n,\ell}$'s. We define the r-dimensional $\mathbf{A}^{(r)}_{n,\ell}$**-singularities** as those isolated hypersurface singularities which have underlying spaces of the form

$$X^{(r)}_{n,\ell} := \mathrm{Spec}\left(\mathbb{C}\left[x_1,\ldots,x_{r+1}\right] / (f)\right),$$

where r, n, ℓ are integers, such that $r \geq \ell \geq 2,\ n+1 \geq \ell$, and

$$\boxed{f(x_1,\ldots,x_{r+1}) := x_1^{n+1} + x_2^{\ell} + x_3^{\ell} + \cdots + x_{r+1}^{\ell}}\,. \tag{1.3}$$

These are obviously singularities of *Brieskorn-Pham type*. In addition, by our assumptions about r, ℓ and n, they are *canonical* (see Reid [25, Prop. 4.3, p. 297]). The notation is chosen in this manner to remind that they include, in particular, both subseries of usual r-dimensional $\mathbf{A}_n$-singularities ($\mathbf{A}^{(r)}_{n,2}$'s) and of Fermat singularities ($\mathbf{A}^{(r)}_{\ell-1,\ell}$'s).

(e) Some auxiliary combinatorial functions. At first, for $p, q \in \mathbb{Z}_{\geq 0}$, let us denote Kronecker's symbol by

$$\delta_{p,q} = \begin{cases} 1, & \text{if } p = q \\ 0, & \text{if } p \neq q. \end{cases}$$

• Next, fixing r, ℓ and n, as in (d), we set $d := \mathrm{lcm}(n+1,\ell)$. and define three functions $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c} : \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{Z}_{\geq 0}$ by

$$\mathbb{Z}_{\geq 0} \ni i \longmapsto \mathbf{a}(i) = \sum_{p=0}^{n-1} \delta_{i,\frac{pd}{n+1}}, \tag{1.4}$$

by the multinomial coefficients

$$\mathbb{Z}_{\geq 0} \ni j \longmapsto \mathbf{b}(j) = \begin{cases} \displaystyle\sum_{(\nu_1,\nu_2,\ldots,\nu_{\ell-1}) \in \mathfrak{B}_j} \binom{r}{\nu_1,\nu_2,\ldots,\nu_{\ell-1}}, & \text{if } \mathfrak{B}_j \neq \varnothing \\ 0, & \text{otherwise} \end{cases} \tag{1.5}$$

(with $\binom{r}{\nu_1,\nu_2,\ldots,\nu_{\ell-1}} := \frac{r!}{\nu_1!\,\nu_2!\ldots\nu_{\ell-1}!}$), and by the convolutional formula

$$\mathbb{Z}_{\geq 0} \ni k \longmapsto \mathbf{c}(k) = \sum_{(i,j) \in \mathfrak{C}_k} \mathbf{a}(i)\ \mathbf{b}(j), \tag{1.6}$$

where for each $j \in \mathbb{Z}_{\geq 0}$,

$$\mathfrak{B}_j := \left\{ \left(\nu_1, \nu_2, \ldots, \nu_{\ell-1}\right) \in \left(\mathbb{Z}_{\geq 0}\right)^{\ell-1} \;\middle|\; \begin{array}{l} \nu_1 + \nu_2 + \cdots + \nu_{\ell-1} = r \\ \text{and (whenever } \ell \geq 3) \\ d\left(\nu_2 + 2\nu_3 + \cdots + (\ell-2)\,\nu_{\ell-1}\right) = j\,\ell \end{array} \right\},$$

and for each $k \in \mathbb{Z}_{\geq 0}$, $\mathfrak{C}_k := \left\{(i, j) \in \left(\mathbb{Z}_{\geq 0}\right)^2 \mid i + j = k\right\}$.

• Finally, for any four-tuple $(\kappa, \lambda, \nu, \xi) \in \left(\mathbb{Z}_{\geq 0}\right)^4$ with $\kappa \geq \lambda$, let us recall the definition of the *non-central Eulerian numbers* $\mathfrak{S}(\kappa, \lambda \mid \nu, \xi)$ *of generalized factorials* (with translation summand ξ). These are the coefficients which occur in the development of

$$\binom{\nu \cdot t + \xi}{\kappa} = \sum_{\lambda=0}^{\kappa} \mathfrak{S}(\kappa, \lambda \mid \nu, \xi) \binom{t + \kappa - \lambda}{\kappa}$$

and satisfy the recurrence relation

$$\begin{aligned}(\kappa+1)\; \mathfrak{S}(\kappa+1, \lambda \mid \nu, \xi) = {} & (\nu\lambda - \kappa + \xi)\; \mathfrak{S}(\kappa, \lambda \mid \nu, \xi) \\ & + (\nu(\kappa - \lambda + 1) + \kappa - \xi)\; \mathfrak{S}(\kappa, \lambda - 1 \mid \nu, \xi)\end{aligned}$$

with initial conditions $\mathfrak{S}(0, 0 \mid \nu, \xi) = 1$ and $\mathfrak{S}(\kappa, 0 \mid \nu, \xi) = \binom{\xi}{\kappa}$. In fact, it can be shown that

$$\mathfrak{S}(\kappa, \lambda \mid \nu, \xi) = \sum_{j=0}^{\lambda} (-1)^j \binom{\kappa+1}{j} \binom{\nu(\lambda - j) + \xi}{\kappa}.$$

(f) Main results. We can now state the main results.

Proposition 1.7. *Let $X = X^{(r)}_{n,\ell}$ be the underlying space of the $\mathbf{A}^{(r)}_{n,\ell}$- singularity. Then the E -polynomial $E\,(X \smallsetminus \{\mathbf{0}\}; u, v)$ equals*

$$(uv-1)\left[1 + (uv)^{r-1} + \sum_{p=1}^{r-2} \left((uv)^p + (-1)^r\, \mathbf{c}\left(d\left(p + \frac{n}{n+1} - \frac{r}{\ell}\right)\right) u^p\, v^{r-p-1}\right)\right] \tag{1.7}$$

(*with the* $\mathbf{c}$*-function as defined in* (1.6)).

Formula (1.7) provides the “first summand” of the E_{str}-function of X. On the other hand, if ℓ divides either n or $n+1$, $\mathbf{A}^{(r)}_{n,\ell}$’s are absolutely isolated (see below Proposition 3.1), and the E_{str}-function of $X^{(r)}_{n,\ell}$’s is computed as follows:

Theorem 1.8. *If the integer ℓ divides either n or $n+1$, then the $E_{\rm str}$-function of $X = X^{(r)}_{n,\ell}$ is given by the formula*

Cases	$E_{\rm str}(X; u, v)$
$\ell \mid n+1$	$E(X \smallsetminus \{\mathbf{0}\}; u, v)$ $+ (uv-1)\left(\frac{uv}{(uv)^{r-\ell+1}-1} + \sum_{i=2}^{m-1} \frac{uv-1}{(uv)^{i(r-\ell)+1}-1} - \frac{uv-1}{(uv)^{m(r-\ell)+1}-1}\right.$ $\left.\sum_{i=1}^{m-1} \frac{uv-1}{\left((uv)^{i(r-\ell)+1}-1\right)\left((uv)^{(i+1)(r-\ell)+1}-1\right)}\right)$ $\times \left[\sum_{p=0}^{r-2} u^p \left(v^p + (-1)^{r-2}\,\mathfrak{S}(r-1, p+1 \mid \ell-1, p)\; v^{r-2-p}\right)\right]$ $+ \frac{(uv-1)}{(uv)^{m(r-\ell)+1}-1}\left[\sum_{p=0}^{r-1} u^p \left(v^p + (-1)^{r-1}\,\mathfrak{S}(r, p+1 \mid \ell-1, p)\; v^{r-1-p}\right)\right]$
$\ell \mid n$	$E(X \smallsetminus \{\mathbf{0}\}; u, v) + \frac{(uv)^r-1}{(uv)^{(m-1)\,\ell\,(r-\ell)+r}-1}$ $+ (uv-1)\left(\frac{uv}{(uv)^{r-\ell+1}-1} + \sum_{i=2}^{m-1} \frac{uv-1}{(uv)^{i(r-\ell)+1}-1}\right.$ $- \frac{uv-1}{(uv)^{(m-1)\,\ell\,(r-\ell)+r}-1} + \sum_{i=1}^{m-2} \frac{uv-1}{\left((uv)^{i(r-\ell)+1}-1\right)\left((uv)^{(i+1)(r-\ell)+1}-1\right)}$ $\left.+ \frac{uv-1}{\left((uv)^{(m-1)(r-\ell)+1}-1\right)\left((uv)^{(m-1)\,\ell\,(r-\ell)+r}-1\right)}\right)$ $\times \left[\sum_{p=0}^{r-2} u^p \left(v^p + (-1)^{r-2}\,\mathfrak{S}(r-1, p+1 \mid \ell-1, p)\; v^{r-2-p}\right)\right]$

In particular, for the string-theoretic Euler number we obtain:

Cases	$e_{\rm str}(X)$
$\ell \mid n+1$	$\frac{m-1}{m(r-\ell)+1}\left[\frac{1}{\ell}\left((1-\ell)^r - 1\right) + r\right]$ $+ \frac{1}{m(r-\ell)+1}\left[\frac{1}{\ell}((1-\ell)^{r+1} - 1) + r + 1\right]$
$\ell \mid n$	$\frac{r}{(m-1)\,\ell\,(r-\ell)+r} + \frac{(m-1)\ell}{(r-\ell)(m-1)\ell+r}\left[\frac{1}{\ell}\left((1-\ell)^r - 1\right) + r\right]$

The above number m is defined to be

$$m := \begin{cases} \frac{n+1}{\ell}, & \text{if } n+1 \equiv 0 \,(\mathrm{mod}\ \ell) \\ \frac{n}{\ell}+1, & \text{if } n \equiv 0 \,(\mathrm{mod}\ \ell) \end{cases} \tag{1.8}$$

Remark 1.9. Counterexamples to conjecture 1.6 occur already for $\ell = 2$, as we have:

$$e_{\mathrm{str}}\left(X^{(r)}_{n,2}\right) = \begin{cases} \frac{m(r-1)+2}{m(r-2)+1} = \frac{n(r-1)+r+3}{n(r-2)+r}, & \text{if both } n \text{ and } r \text{ are odd} \\ \frac{m\,r}{m(r-2)+1} = \frac{r(n+1)}{(r-2)(n+1)+2}, & \text{if } n \text{ is odd and } r \text{ even} \\ \frac{2(m-1)(r-1)+r}{2(m-1)(r-2)+r} = \frac{(r-1)n+r}{(r-2)n+r}, & \text{if } n \text{ is even and } r \text{ odd} \\ \frac{(2m-1)r}{2(m-1)(r-2)+r} = \frac{r(n+1)}{(r-2)n+r}, & \text{if both } n \text{ and } r \text{ are even} \end{cases}$$

For instance, in dimension $r = 3$, we obtain:

$$\lim_{n\to\infty,\ n \text{ even}} \mathrm{ind}_{\mathrm{str}}\left(X^{(3)}_{n,2}\right) = \infty.$$

On the other hand, for *all* odd n's, $e_{\mathrm{str}}(X^{(3)}_{n,2}) = 2$ and $\mathrm{ind}_{\mathrm{str}}(X^{(3)}_{n,2}) = 1$.

2. On the MHS of the cohomology groups of links

At first, we shall exploit the fact that $\mathbf{A}^{(r)}_{n,\ell}$'s are quasihomogeneous singularities, and show that Proposition 1.7 is a byproduct of a more general result concerning isolated singularities of this sort. (See below Proposition 2.8).

(a) Links and Milnor fibers. Let $(W,\mathbf{0}) \subseteq \left(\mathbb{C}^N,\mathbf{0}\right)$ be the germ of a complex analytic set W having pure dimension $r+1$ and the origin as isolated singularity. Assume that $f : W \to \mathbb{C}$ is a holomorphic function, such that $f\big|_{W\smallsetminus\{\mathbf{0}\}}$ is non-singular. Obviously, $X := f^{-1}(\mathbf{0})$ is a complex analytic subset of $\mathbb{C}^N$ of pure dimension r with the origin as isolated singularity. Let $L := L(X,\mathbf{0}) := \mathbb{S}_\varepsilon \cap X$ denote its *link*, where $\mathbb{S}_\varepsilon := \left\{\mathbf{z} \in \mathbb{C}^N \mid \|\mathbf{z}\| = \varepsilon\right\}, 0 < \varepsilon \ll 1$. L is a differentiable, compact, oriented manifold of dimension $2r-1$, and there are isomorphisms:

$$H^{i+1}\left(X, X\smallsetminus\{\mathbf{0}\},\mathbb{Q}\right) \cong H^{i}\left(X\smallsetminus\{\mathbf{0}\},\mathbb{Q}\right) \cong H^{i}\left(L,\mathbb{Q}\right). \tag{2.1}$$

If $\mathbb{B}_{\varepsilon'}$ is the open ball with $\mathbf{0}$ as its center and ε' as its radius, where $\varepsilon < \varepsilon' \ll 1$, it is known that the map

$$f\big|_{\mathbb{B}_{\varepsilon'}\cap f^{-1}(\mathbb{D}^*_\alpha)} : \mathbb{B}_{\varepsilon'} \cap f^{-1}(\mathbb{D}^*_\alpha) \longrightarrow \mathbb{D}^*_\alpha$$

determines a differentiable fibre bundle, where $\mathbb{D}^*_\alpha := \{t \in \mathbb{C} \mid 0 < |t| < \alpha\}$ is a small punctured disc in $\mathbb{C}$ with $0 < \alpha < \varepsilon$. Let $F = F_t$ be the corresponding fiber, the so-called (*open*) *Milnor fiber*. The study of the relation between the MH-structures of the cohomology groups of L and F relies on certain corollaries of a theorem of Steenbrink [27, (2.3)] and Hamm [17, Thm. 1.6.1]. (The coefficients of the cohomology groups are always taken from $\mathbb{C}$.)

Theorem 2.1 (Steenbrink–Hamm). *For all i, there exists an exact MHS-sequence*:

$$\cdots \longrightarrow H^{i-1}(L) \longrightarrow H_c^i(F) \longrightarrow H^i(F) \longrightarrow H^i(L) \longrightarrow \cdots$$

Corollary 2.2. *We have the following exact sequence and isomorphisms of MHS:*

(i) $0 \to H^{r-1}(F) \to H^{r-1}(L) \to H_c^r(F) \to H^r(F)$
$\to H^r(L) \to H_c^{r+1}(F) \to 0,$
(ii) $H^i(L) \cong H^i(F)$, *for all* $i < r-1$,
(iii) $H^{i-1}(L) \cong H_c^i(F)$, *for all* $i > r-1$.

Proof. Since $\mathbb{B}_{\varepsilon'}$ is a complex Stein manifold, F is a complex Stein manifold too. Hence, F has the homotopy type of a CW-complex of real dimension r (see [16]), which means that $H^i(F) \cong H_c^{2r-i}(F) = 0$ for all $i \geq r+1$. The exactness in (i) and the existence of MHS-isomorphisms (ii) and (iii) follow from the long exact sequence of Theorem 2.1, combined with the vanishing of these cohomology groups. □

Corollary 2.3. *For all p, q, the Hodge numbers of the two "middle" cohomology groups of F satisfy the equalities:*

$$\begin{aligned} h^{p,q}(H^r(F)) &= h^{p,q}(H^{r-1}(F)) + h^{p,q}(H_c^r(F)) - h^{p,q}(H_c^{r+1}(F)) \\ &\quad + h^{p,q}(H^r(L)) - h^{p,q}(H^{r-1}(L)) \\ &= h^{p,q}(H^{r-1}(F)) + h^{r-p,r-q}(H^r(F)) - h^{r-p,r-q}(H^{r-1}(F)) \\ &\quad + h^{p,q}(H^r(L)) - h^{p,q}(H^{r-1}(L)) \end{aligned}$$

Proof. The first equality is obvious by 2.2 (i), and the second one follows from Poincaré duality. □

Proposition 2.4. *If $N = r+1$, $W = \mathbb{C}^{r+1}$ and $(X, \mathbf{0})$ is a purely r-dimensional isolated hypersurface singularity, with $r \geq 2$, then the only "non-trivial" Hodge numbers of the cohomology groups of its link $L = L(X, \mathbf{0})$ are*

$$h^{p,q}(H^{r-1}(L)) = h^{r-p,r-q}(H^r(L)), \text{ with } p+q \leq r-1,$$

as we have:

(i) $h^{p,q}(H^i(L)) = 0$, *for all* p, q *whenever* $i \notin \{0, r-1, r, 2r-1\}$.
(ii) $h^{p,q}(H^0(L)) = 1$, *for* $p = q = 0$, *and* $= 0$, *otherwise.*
(iii) $h^{p,q}(H^{2r-1}(L)) = 1$, *for* $p = q = r$, *and* $= 0$, *otherwise.*
(iv) $h^{p,q}(H^{r-1}(L)) = h^{r-p,r-q}(H^r(L))$, *for all* p, q,
and equals 0 *whenever* $p+q > r-1$.

Proof. L is $(r-2)$-connected (cf. [24, Thm. 5.2]), and the local Lefschetz Theorem gives (i), (ii) and (iii) because $H^i(L) = 0$ for all indices $i \notin \{0, r-1, r, 2r-1\}$ and $H^0(L) \cong H^{2r-1}(L) \cong \mathbb{C}$. For (iv) use Poincaré duality and the fact, that the natural MHS on $H^i(L)$ has weights $Gr_j^{\mathcal{W}_\bullet}(H^i(L)) = 0$ for $j > i$ (by the Semipurity Theorem, cf. [27, Cor. (1.12), p. 518]). □

(b) Quasihomogeneous isolated singularities. A polynomial

$$f \in \mathbb{C}\left[x_1, x_2, \ldots, x_{r+1}\right]$$

is *quasihomogeneous* of degree d with respect to the *weights*

$$\mathbf{w} = (w_1, \ldots, w_{r+1}) \in \left(\mathbb{Z}_{\geq 1}\right)^{r+1}$$

if

$$f\left(\lambda^{w_1} x_1, \ldots, \lambda^{w_{r+1}} x_{r+1}\right) = \lambda^d\, f\left(x_1, \ldots, x_{r+1}\right), \quad \forall\, \lambda, \quad \lambda \in \mathbb{C}^*.$$

Hereafter we consider such an f, assume that $r \geq 2$ and that

$$X_f := \left\{ (x_1, \ldots, x_{r+1}) \in \mathbb{C}^{r+1} \mid f\left(x_1, \ldots, x_{r+1}\right) = 0 \right\}$$

has no other singularities than $\mathbf{0} \in \mathbb{C}^{r+1}$. Note that the Milnor algebra

$$M(f) := \mathbb{C}\left[x_1, x_2, \ldots, x_{r+1}\right] \Big/ \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{r+1}} \right)$$

associated to f is a graded $\mathbb{C}$-algebra of finite type (with $\deg(x_i) = w_i$, $i = 1, \ldots, r+1$) whose Poincaré series $P_{M(f)}(t)$ equals

$$\begin{aligned} P_{M(f)}(t) &= \sum_{k \geq 0} \dim_{\mathbb{C}} \left(M(f)_k \right) t^k \\ &= \frac{\left(1 - t^{d - w_1}\right)\left(1 - t^{d - w_2}\right) \cdots \left(1 - t^{d - w_{r+1}}\right)}{\left(1 - t^{w_1}\right)\left(1 - t^{w_2}\right) \cdots \left(1 - t^{w_{r+1}}\right)} \end{aligned} \tag{2.2}$$

(cf. [10, (7.27), p. 112]). Next, we define the *quasismooth* weighted projective hypersurfaces

$$Z = \left\{ \left[x_0 : x_1 : \ldots : x_{r+1}\right] \in \mathbb{P}_{\mathbb{C}}^{r+1}(1, \mathbf{w}) \mid \overline{f}\left(x_0, \ldots, x_{r+1}\right) = 0 \right\}$$

where $\overline{f}(x_0, \ldots, x_{r+1}) := x_0^d - f(x_1, \ldots, x_{r+1})$, and

$$\begin{aligned} Z_\infty &= \left\{ \left[x_0 : x_1 : \ldots : x_{r+1}\right] \in Z \mid x_0 = 0 \right\} \\ &\cong \left\{ \left[x_1 : \ldots : x_{r+1}\right] \in \mathbb{P}_{\mathbb{C}}^{r}(\mathbf{w}) \mid f\left(x_1, \ldots, x_{r+1}\right) = 0 \right\}. \end{aligned}$$

We have

$$M\left(\overline{f}\right) = M(f) \otimes \mathbb{C}[x_0] \Big/ \left(x_0^{d-1}\right) \tag{2.3}$$

and the map $(x_1, \ldots, x_{r+1}) \longmapsto \left[1 : x_1 : \ldots : x_{r+1}\right]$ induces a diffeomorphism between

$$F = \left\{ (x_1, \ldots, x_{r+1}) \in \mathbb{C}^{r+1} \mid f\left(x_1, \ldots, x_{r+1}\right) = 1 \right\}$$

and the complement $Z \smallsetminus Z_\infty$, where this F is diffeomorphic to the (usual) Milnor fiber of the singularity $(X_f, \mathbf{0})$ (see [11, (1.13), p. 72]). Moreover, F has the homotopy type of a bouquet of $\mu(f)$ r-spheres, with

$$\mu(f) = \lim_{t \to 1} P_{M(f)}(t) = \prod_{i=1}^{r+1} \left(\frac{d}{w_i} - 1 \right) \tag{2.4}$$

denoting the corresponding *Milnor number*. The *primitive* cohomology groups of Z_∞ are defined by the exact sequence

$$0 \longrightarrow H^{r-1}(\mathbb{P}^r_{\mathbb{C}}(\mathbf{w}), \mathbb{C}) \longrightarrow H^{r-1}(Z_\infty, \mathbb{C}) \longrightarrow H^{r-1}_{\text{prim}}(Z_\infty, \mathbb{C}) \longrightarrow 0.$$

Since both $\mathbb{P}^r_{\mathbb{C}}(\mathbf{w})$ and Z_∞ are orbifolds, they are equipped with pure Hodge structure, and therefore both $H^{r-1}(Z_\infty, \mathbb{C})$ and $H^{r-1}_{\text{prim}}(Z_\infty, \mathbb{C})$ decompose, say as

$$H^{r-1}(Z_\infty, \mathbb{C}) = \bigoplus_{p+q=r-1} H^{p,q}(Z_\infty), \; H^{r-1}_{\text{prim}}(Z_\infty, \mathbb{C}) = \bigoplus_{p+q=r-1} H^{p,q}_{\text{prim}}(Z_\infty),$$

(The same is also valid for $H^r_{[\text{prim}]}(Z, \mathbb{C})$).

Lemma 2.5. *For the Milnor fiber F of $(X_f, \mathbf{0})$ we have*

(i) $h^{p,q}(H^0(F, \mathbb{C})) = 1$, *for* $p = q = 0$, *and* $= 0$, *otherwise*

(ii) $H^i(F, \mathbb{C}) = 0$, *for all* $i \notin \{0, r\}$.

(iii) $h^{p,q}(H^r(F, \mathbb{C})) = 0$, *for* $p + q \notin \{r, r+1\}$.

(iv) $h^{p,r-p}(H^r(F, \mathbb{C})) = h^{p,r-p}_{\text{prim}}(Z) = h^{p,r-p}(Z) - \delta_{p,r-p}$, *for* $0 \le p \le r$.

(v) $h^{p,r+1-p}(H^r(F, \mathbb{C})) = h^{p-1,r-p}_{\text{prim}}(Z_\infty) = h^{p-1,r-p}(Z_\infty) - \delta_{p-1,r-p}$, *for* $1 \le p \le r$.

Proof. (i) This follows from 2.2 (ii) and 2.4 (ii).

(ii)–(v). At first note that $H^{p,q}(\mathbb{P}^r_{\mathbb{C}}(\mathbf{w}))$ (resp., $H^{p,q}(\mathbb{P}^{r+1}_{\mathbb{C}}(1, \mathbf{w}))$) is $\cong \mathbb{C}$, whenever $p = q$, and $= 0$, otherwise. As Steenbrink points out in [26, p. 216], there is an exact MHS-sequence of Gysin-type:

$$\cdots \to H^i(Z, \mathbb{C}) \to H^i(Z \smallsetminus Z_\infty, \mathbb{C}) \to H^{i-1}(Z_\infty, \mathbb{C})(-1) \xrightarrow{\theta} H^{i+1}(Z, \mathbb{C}) \to \cdots$$

By the Weak Lefschetz Theorem [12, 4.2.2], the homomorphism

$$H^{p,q}(Z_\infty) \longrightarrow H^{p,q+1}\left(\mathbb{P}^r_{\mathbb{C}}(\mathbf{w})\right) \left[\text{resp., } H^{p,q}(Z) \longrightarrow H^{p,q+1}\left(\mathbb{P}^{r+1}_{\mathbb{C}}(1, \mathbf{w})\right)\right]$$

is an isomorphism for $p + q > r - 1$ (resp., $p + q > r$) and an epimorphism for $p + q = r - 1$ (resp., $p + q = r$). Thus, θ is an isomorphism for all $i \notin \{0, r\}$, proving (ii). Moreover, since

$$\mathcal{W}_j\left(H^r(F, \mathbb{C})\right) = \begin{cases} 0, & \text{if } j < r \\ H^r(F, \mathbb{C}), & \text{if } j > r \end{cases}$$

i.e., $Gr_j^{\mathcal{W}_\bullet}(H^r(F,\mathbb{C})) = 0$, for $j \notin \{r, r+1\}$, (cf. [8, §8.2]), (iii) is obvious, and the above exact MHS-sequence gives the isomorphisms

$$\begin{aligned}\mathcal{W}_r\left(H^r(F,\mathbb{C})\right) &= \operatorname{Im}\left(H^r(Z,\mathbb{C}) \longrightarrow H^r(Z\smallsetminus Z_\infty,\mathbb{C})\right)\\ &\cong \operatorname{CoKer}\left(H^{r-2}(Z_\infty,\mathbb{C})(-1) \longrightarrow H^r(Z,\mathbb{C})\right)\\ &\cong \operatorname{CoKer}\left(H^r(\mathbb{P}_{\mathbb{C}}^{r+1}(1,\mathbf{w})) \longrightarrow H^r(Z,\mathbb{C})\right) = H^r_{\text{prim}}(Z,\mathbb{C})\end{aligned}$$

and

$$\begin{aligned}&Gr_{r+1}^{\mathcal{W}_\bullet}\left(H^r(F,\mathbb{C})\right)\\ &\quad = H^r(Z\smallsetminus Z_\infty,\mathbb{C}) \;/\; \operatorname{Ker}\left(H^r(Z\smallsetminus Z_\infty,\mathbb{C}) \longrightarrow H^{r-1}(Z_\infty,\mathbb{C})(-1)\right)\\ &\quad \cong \operatorname{Ker}\left(H^{r-1}(Z_\infty,\mathbb{C})(-1) \longrightarrow H^{r+1}(Z,\mathbb{C})\right)\\ &\quad \cong \operatorname{CoKer}\left(H^{r-1}(\mathbb{P}^r_{\mathbb{C}}(\mathbf{w}),\mathbb{C})(-1) \longrightarrow H^{r-1}(Z_\infty,\mathbb{C})(-1)\right)\\ &\quad = H^{r-1}_{\text{prim}}(Z_\infty,\mathbb{C})(-1),\end{aligned}$$

respectively, proving (iv) and (v). $\square$

Theorem 2.6 (Griffiths–Steenbrink). *If $(X_f, \mathbf{0})$ is an r-dimensional isolated quasihomogeneous hypersurface singularity of degree d w.r.t. the weights $w_1, \ldots, w_{r+1}$, then*

$$H^{p-1,r-p}_{\text{prim}}(Z_\infty) \cong M(f)_{pd-(w_1+\ldots+w_{r+1})}.$$

Hence,

$$\begin{cases} h^{p,r-p}\left(H^r(F,\mathbb{C})\right) = \sum\limits_{i=1}^{d-1} \dim_{\mathbb{C}}\left(M(f)_{pd-(w_1+\ldots+w_{r+1})+i}\right) \\ h^{p+1,r-p}\left(H^r(F,\mathbb{C})\right) = \dim_{\mathbb{C}}\left(M(f)_{(p+1)d-(w_1+\ldots+w_{r+1})}\right). \end{cases} \tag{2.5}$$

Proof. Extending Griffiths' results [14] to the case of weighted homogeneous hypersurfaces, the global sections of the sheaves

$$\Omega^p_{\mathbb{P}^r_{\mathbb{C}}(\mathbf{w})}(Z_\infty) = \Omega^p_{\mathbb{P}^r_{\mathbb{C}}(\mathbf{w})} \otimes \mathcal{O}_{\mathbb{P}^r_{\mathbb{C}}(\mathbf{w})}(Z_\infty),$$

as well as the graded pieces of middle cohomology of $\mathbb{P}^r_{\mathbb{C}}(\mathbf{w}) \smallsetminus Z_\infty$, are described by means of special auxiliary differential forms with poles along Z_∞. In particular,

$$H^0(\mathbb{P}^r_{\mathbb{C}}(\mathbf{w}), \Omega^r_{\mathbb{P}^r_{\mathbb{C}}(\mathbf{w})}(Z_\infty)) = \left\{ \frac{g\cdot\Omega_0}{f} \;\middle|\; g \in \mathbb{C}[x_1,\ldots,x_{r+1}]_{d-(w_1+\ldots+w_{r+1})} \right\},$$

where

$$\Omega_0 := \sum_{i=1}^{r+1} (-1)^i \; w_i\, x_i \; dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{r+1},$$

and

$$Gr^p_{\mathcal{F}\bullet}(H^r(\mathbb{P}^r_{\mathbb{C}}(\mathbf{w}) \setminus Z_\infty, \mathbb{C})) \cong H^{r-p}(\mathbb{P}^r_{\mathbb{C}}(\mathbf{w}), \Omega^p_{\mathbb{P}^r_{\mathbb{C}}(\mathbf{w})}(\log Z_\infty))$$

$$\cong \frac{H^0(\mathbb{P}^r_{\mathbb{C}}(\mathbf{w}), \Omega^r_{\mathbb{P}^r_{\mathbb{C}}(\mathbf{w})}((r-p-1)Z_\infty))}{H^0(\mathbb{P}^r_{\mathbb{C}}(\mathbf{w}), \Omega^r_{\mathbb{P}^r_{\mathbb{C}}(\mathbf{w})}((r-p)Z_\infty)) + \partial(H^0(\mathbb{P}^r_{\mathbb{C}}(\mathbf{w}), \Omega^{r-1}_{\mathbb{P}^r_{\mathbb{C}}(\mathbf{w})}((r-p)Z_\infty)))}$$

(∂ denotes the corresponding differential operator). Since the map

$$\begin{array}{ccc} H^0(\mathbb{P}^r_{\mathbb{C}}(\mathbf{w}), \Omega^r_{\mathbb{P}^r_{\mathbb{C}}(\mathbf{w})}((r-p-1)Z_\infty)) & \xrightarrow{\Phi} & \mathbb{C}[x_1, .., x_{r+1}]_{(r-p+1)d-(w_1+\cdots+w_{r+1})} \\ \dfrac{g \cdot \Omega_0}{f^{r-p+1}} & \longmapsto & g \end{array}$$

defines an isomorphism, and

$$H^0(\mathbb{P}^r_{\mathbb{C}}(\mathbf{w}), \Omega^r_{\mathbb{P}^r_{\mathbb{C}}(\mathbf{w})}((r-p)Z_\infty)) + \partial(H^0(\mathbb{P}^r_{\mathbb{C}}(\mathbf{w}), \Omega^{r-1}_{\mathbb{P}^r_{\mathbb{C}}(\mathbf{w})}((r-p)Z_\infty)))$$

has

$$\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{r+1}}\right)_{(r-p+1)d-(w_1+\ldots+w_{r+1})}$$

as its image under Φ (see [4, §11]), we get

$$Gr^p_{\mathcal{F}\bullet}(H^r(\mathbb{P}^r_{\mathbb{C}}(\mathbf{w}) \setminus Z_\infty, \mathbb{C})) \cong M(f)_{(r-p+1)d-(w_1+\ldots+w_{r+1})}.$$

Using Hard Lefschetz Theorem one deduces the exact MHS-sequence:

$$\begin{aligned} 0 \to H^{r-2}(\mathbb{P}^r_{\mathbb{C}}(\mathbf{w}), \mathbb{C}) &\xrightarrow{\cong} H^r(\mathbb{P}^r_{\mathbb{C}}(\mathbf{w}), \mathbb{C}) \\ &\to H^r(\mathbb{P}^r_{\mathbb{C}}(\mathbf{w}) \setminus Z_\infty, \mathbb{C}) \to H^{r-1}_{\text{prim}}(Z_\infty, \mathbb{C}) \to 0, \end{aligned}$$

giving

$$H^{p,r-1-p}_{\text{prim}}(Z_\infty) \cong M(f)_{(r-p)d-(w_1+\ldots+w_{r+1})} \cong M(f)_{(p+1)d-(w_1+\ldots+w_{r+1})}.$$

Formulae (2.5) follow from Lemma 2.5 (iv), (v), and (2.3). □

Lemma 2.7. *If* $(X_f, \mathbf{0})$ *is an* r*-dimensional isolated quasihomogeneous hypersurface singularity with* L *as its link and* F *as its Milnor fiber, then*

$$h^{p,q}\left(H^{r-1}(L, \mathbb{C})\right) = 0, \text{ whenever } p + q \neq r - 1,$$

and the "non-trivial" Hodge numbers of the cohomology groups of its link L *are*

$$h^{p,r-1-p}\left(H^{r-1}(L, \mathbb{C})\right) = h^{r-p,p+1}\left(H^r(L, \mathbb{C})\right) = h^{p+1,r-p}\left(H^r(F, \mathbb{C})\right) \quad (2.6)$$

for $p = 0, 1, \ldots, r - 1$, *and can be therefore read off from* (2.5).

Proof. If $p+q \notin \{r-1, r+1\}$, then by 2.3, 2.4 (iv) and 2.5 (i), (iii), we obtain

$$h^{r-p,r-q}\left(H^{r-1}(L,\mathbb{C})\right) = h^{p,q}\left(H^{r}(L,\mathbb{C})\right) = h^{p,q}\left(H^{r-1}(L,\mathbb{C})\right) = 0,$$

because the corresponding Hodge numbers of $H^{r-1}(F,\mathbb{C})$ and $H^{r}(F,\mathbb{C})$ vanish, and $p+q < r-1$ (resp., $= r \mid > r+1$) iff $(r-p)+(r-q) > r+1$ (resp., $= r \mid < r-1$). On the other hand, if $p+q \in \{r-1, r+1\}$, Cor. 2.3 gives:

$$h^{p,q}(H^{r-1}(L,\mathbb{C})) - h^{p,q}(H^{r}(L,\mathbb{C})) = h^{r-p,r-q}(H^{r}(F,\mathbb{C})) - h^{p,q}(H^{r}(F,\mathbb{C})). \tag{2.7}$$

If $p+q = r-1$, the Hodge numbers $h^{p,q}(H^{r}(L,\mathbb{C})) = h^{r-p,r-q}(H^{r-1}(L,\mathbb{C}))$ vanish by 2.4 (iv). Analogously, $h^{p,q}(H^{r-1}(L,\mathbb{C}))$ vanishes whenever $p+q = r+1$. Finally, (2.6) follows from Lemma 2.5 (iii) and (2.7). □

Proposition 2.8. *If $(X_f, \mathbf{0})$ is an r-dimensional isolated quasihomogeneous hypersurface singularity of degree d w.r.t. the weights $w_1, \ldots, w_{r+1}$, and L its link, then the E-polynomial $E\left(X_f \smallsetminus \{\mathbf{0}\}; u, v\right)$ equals*

$$(uv-1)\left[\sum_{p=0}^{r-1}\left((uv)^p + (-1)^{r-1}\ h^{p,r-1-p}(H^{r-1}(L,\mathbb{C}))\, u^p\, v^{r-p-1}\right)\right] \tag{2.8}$$

and its coefficients are therefore computable in terms of d and $w_1, \ldots, w_{r+1}$ via (2.6) *and* (2.5).

Proof. Using (2.1) and Poincaré duality, we obtain:

$$h^{p,q}(H^{i}(L,\mathbb{C})) = h^{p,q}(H^{i}\left(X_f \smallsetminus \{\mathbf{0}\},\mathbb{C}\right)) = h^{d-p,d-q}(H_c^{2d-i}\left(X_f \smallsetminus \{\mathbf{0}\},\mathbb{C}\right)).$$

Hence,

$$E\left(X_f \smallsetminus \{\mathbf{0}\}; u, v\right) = (uv)^r\, E\left(L; u^{-1}, v^{-1}\right). \tag{2.9}$$

On the other hand, Proposition 2.4 gives

$$\begin{aligned} E(L;u,v) &= \sum_{0\le p,q\le r} e^{p,q}(L)\ u^p\, v^q \\ &= \sum_{0\le p,q\le r}\left[h^{p,q}(H^0(L)) - h^{p,q}(H^{2r-1}(L))\right]\ u^p\, v^q \\ &\quad + (-1)^{r-1}\sum_{0\le p,q\le r}\left[h^{p,q}(H^{r-1}(L)) - h^{p,q}(H^{r}(L))\right]\ u^p\, v^q \\ &= \sum_{0\le p,q\le r}\left[h^{p,q}(H^0(L)) - h^{p,q}(H^{2r-1}(L))\right]\ u^p\, v^q \\ &\quad + (-1)^{r-1}\sum_{0\le p,q\le r}\left[h^{p,q}(H^{r-1}(L)) - h^{r-p,r-q}(H^{r-1}(L))\right]\ u^p\, v^q \end{aligned}$$

$$= 1-(uv)^r+(-1)^{r-1}\,[\sum_{0\le p,q\le r} h^{p,q}(H^{r-1}(L))u^p v^q]$$

$$+(-1)^r\;[\sum_{0\le p,q\le r} h^{r-p,r-q}(H^{r-1}(L))u^p v^q]$$

$$= 1-(uv)^r+(-1)^{r-1}\left[\sum_{\substack{0\le p,q\le r-1\\ 0\le p+q\le r-1}} h^{p,q}(H^{r-1}(L))u^p v^q\right]$$

$$+(-1)^r\left[\sum_{\substack{1\le p,q\le r\\ r+1\le p+q\le 2r-1}} h^{r-p,r-q}(H^{r-1}(L))\;u^p\;v^q\right].$$

(The terms containing coefficients $h^{p,q}(H^{r-1}(L))$, with $p+q=r$, cancel out, as they occur in both summands). Since $(X_f,\mathbf{0})$ is an isolated quasihomogeneous hypersurface singularity, we may use Lemma 2.7 to write

$$E(L;u,v) = 1-(uv)^r+(-1)^{r-1}\left[\sum_{\substack{0\le p,q\le r-1\\ p+q=r-1}} h^{p,q}(H^{r-1}(L))\;u^p\;v^q\right]$$

$$+(-1)^r\left[\sum_{\substack{1\le p,q\le r\\ p+q=r+1}} h^{r-p,r-q}(H^{r-1}(L))\;u^p\;v^q\right]$$

$$= 1-(uv)^r+(-1)^{r-1}\left[\sum_{p=0}^{r-1} h^{p,r-1-p}(H^{r-1}(L))\;u^p\;v^{r-1-p}\right]$$

$$+(-1)^r\left[\sum_{p=0}^{r-1} h^{p,r-1-p}(H^{r-1}(L))\;u^{p+1}\;v^{r-p}\right]$$

$$= 1-(uv)^r+(-1)^{r-1}\left[\sum_{p=0}^{r-1} h^{p,r-1-p}(H^{r-1}(L))\;u^p\;v^{r-1-p}\right](1-uv)$$

$$= (1-uv)\sum_{p=0}^{r-1}(uv)^p+(-1)^{r-1}\,(1-uv)$$

$$\times\left[\sum_{p=0}^{r-1} h^{p,r-1-p}(H^{r-1}(L))\;u^p\;v^{r-1-p}\right]$$

$$= (1-uv)\left[\sum_{p=0}^{r-1}\Big((uv)^p + (-1)^{r-1}\, h^{p,r-1-p}(H^{r-1}(L))\, u^p\, v^{r-1-p}\Big)\right]$$

Combining the last equality with (2.9), using

$$\begin{aligned} h^{p,r-1-p}(H^{r-1}(L,\mathbb{C})) &= (-1)^{r-1}\, e^{p,r-1-p}\,(L) \\ &= (-1)^{r-1}\, e^{r-1-p,p}\,(L) = h^{r-1-p,p}(H^{r-1}(L,\mathbb{C})) \end{aligned}$$

and substituting $r-p-1$ for p, we deduce formula (2.8). □

Remark 2.9. (i) By (2.8), $e(X_f \smallsetminus \{\mathbf{0}\}) = E\left(X_f \smallsetminus \{\mathbf{0}\}\,;1,1\right) = e(L) = 0$, which is also obvious from the fact, that L is an *odd*-dimensional differentiable manifold.

(ii) If the singularity $(X_f,\mathbf{0})$ in 2.8 is, in addition, a *rational* singularity, then

$$h^{0,r-1}(H^{r-1}(L)) = h^{r-1,0}(H^{r-1}(L)) = 0.$$

(See the proof of Proposition 4.1 of [7].)

(iii) The defining polynomial (1.3) of an $\mathbf{A}^{(r)}_{n,\ell}$-singularity is quasihomogeneous of degree $d = \mathrm{lcm}\,(n+1,\ell)$ w.r.t. the weights $\left(\frac{d}{n+1},\frac{d}{\ell},\frac{d}{\ell},\ldots,\frac{d}{\ell}\right)$, with Poincaré polynomial

$$P_{M(f)}\,(t) = \left(1+\sum_{j=1}^{n-1} t^{\frac{jd}{n+1}}\right)\left(1+\sum_{\kappa=1}^{\ell-2} t^{\frac{\kappa d}{\ell}}\right)^r \tag{2.10}$$

and Milnor number $\mu\,(f) = n\,(\ell-1)^r$ (see (2.2) and (2.4)). Moreover, since $\mathbf{A}^{(r)}_{n,\ell}$'s are canonical, they are also rational singularities.

Proof of Proposition 1.7. To produce formula (1.7) for the E-polynomial of $X^{(r)}_{n,\ell} \smallsetminus \{\mathbf{0}\}$, it suffices to evaluate (2.8) via (2.6), (2.5) and 2.9 (ii)–(iii) in terms of n, ℓ and d. Since the function $\mathbf{a}$, defined in (1.4), can be expressed as

$$\mathbf{a}\,(i) = \begin{cases} 1, & \text{if } i \in \left\{0, \frac{d}{n+1}, \frac{2d}{n+1}, \ldots, \frac{(n-1)d}{n+1}\right\} \\ 0, & \text{otherwise,} \end{cases}$$

and since $\mathbf{b}\,(j)$, as defined in (1.5), gives the coefficient of t^j in the multinomial expansion of the second factor of (2.10), we need the convolutional function (1.6) in order to write the required dimensions as

$$\begin{aligned} h^{p,r-1-p}\big(H^{r-1}\,(L,\mathbb{C})\big) &= h^{p+1,r-p}\left(H^r\,(F,\mathbb{C})\right) = \dim_{\mathbb{C}}\left(M\,(f)_{d\left(p+1-\frac{1}{n+1}-\frac{r}{\ell}\right)}\right) \\ &= \mathbf{c}\left(d\left(p+1-\tfrac{1}{n+1}-\tfrac{r}{\ell}\right)\right), \end{aligned}$$

and to end up to (1.7). □

3. Desingularization and theorem's proof

Next, using blow-ups of closed points, we shall construct snc-resolutions for all $\mathbf{A}_{n,\ell}^{(r)}$-singularities for which either $\ell \mid n$ or $\ell \mid n+1$. Let $X = X_{n,\ell}^{(r)}$ be their underlying spaces and denote by

$$Y_{\ell}^{(r-1)} := \left\{ \left[z_1 : z_2 : \ldots : z_{r+1} \right] \in \mathbb{P}_{\mathbb{C}}^{r} \;\middle|\; \sum_{j=1}^{r+1} z_j^{\ell} = 0 \right\}$$

the $(r-1)$-dimensional Fermat hypersurface of degree $\ell \geq 2$ in the projective space $\mathbb{P}_{\mathbb{C}}^{r}$.

Proposition 3.1.

(i) *If* $n+1 \equiv 0 \pmod{\ell}$ *, then there exists an snc-desingularization* $\varphi : \widetilde{X} \longrightarrow X$ *with discrepancy*

$$K_{\widetilde{X}} - \varphi^{*}(K_X) = \sum_{i=1}^{m} i \; (r-\ell) \; D_i \tag{3.1}$$

where

$$D_i \cong \mathbb{P}(\mathcal{O}_{Y_{\ell}^{(r-2)}} \oplus \mathcal{O}_{Y_{\ell}^{(r-2)}}(1)), \; \forall\, i, \; 1 \leq i \leq m-1, \quad \text{and} \quad D_m \cong Y_{\ell}^{(r-1)}.$$

(ii) *If* $n \equiv 0 \pmod{\ell}$ *, then there is an snc-desingularization* $\varphi : \widetilde{X} \longrightarrow X$ *with discrepancy*

$$K_{\widetilde{X}} - \varphi^{*}(K_X) = \sum_{i=1}^{m-1} i \; (r-\ell) \; D_i + [(m-1)\; \ell \; (r-\ell) + (r-1)] \; D_m \tag{3.2}$$

where

$$D_i \cong \mathbb{P}(\mathcal{O}_{Y_{\ell}^{(r-2)}} \oplus \mathcal{O}_{Y_{\ell}^{(r-2)}}(1)), \; \forall\, i, \; 1 \leq i \leq m-1, \quad \text{and} \quad D_m \cong \mathbb{P}_{\mathbb{C}}^{r-1}.$$

In both cases $D_i \cap D_{i+1} \cong Y_{\ell}^{(r-2)}$ *for all* i, $1 \leq i \leq m-1$, *and* $D_i \cap D_j = \varnothing$ *for all* i, j, $1 \leq i, j \leq m$, *with* $|i-j| \neq 1$. (*The number* m *is defined as in* (1.8)).

Proof. Let f be the polynomial (1.3) and X $(= X_f = X_{n,\ell}^{(r)})$ the underlying space of the $\mathbf{A}_{n,\ell}^{(r)}$-singularity.

Construction of the desingularization. Let $\pi : \mathbf{Bl}_{\mathbf{0}}(\mathbb{C}^{r+1}) \longrightarrow \mathbb{C}^{r+1}$ be the blow up of $\mathbb{C}^{r+1}$ at the origin, with

$$\mathbf{Bl}_{\mathbf{0}}(\mathbb{C}^{r+1}) = \left\{ \left((x_1, .., x_{r+1}), [t_1 : \cdots : t_{r+1}] \right) \in \mathbb{C}^{r+1} \times \mathbb{P}_{\mathbb{C}}^{r} \;\middle|\; \begin{array}{l} x_i\, t_j = x_j\, t_i, \\ \forall\, i, j, \\ 1 \leq i, j \leq r+1 \end{array} \right\}$$

$\mathcal{E} := \pi^{-1}(\mathbf{0}) = \{\mathbf{0}\} \times \mathbb{P}^r_{\mathbb{C}}$, and let U_i denote the open set given by $(t_i \neq 0)$. In terms of analytic coordinates,

$$U_i = \left\{ \left((x_1, .., x_{r+1}), (\xi_1, .., \widehat{\xi_i}, .., \xi_{r+1}) \right) \in \mathbb{C}^{r+1} \times \mathbb{C}^r \;\middle|\; \begin{array}{l} x_j = x_i\, \xi_j,\ \forall\, j, \\ j \in \{1, .., r+1\} \smallsetminus \{i\} \end{array} \right\},$$

where $\xi_j = \frac{t_j}{t_i}$. Identifying U_i with a copy of $\mathbb{C}^{r+1}$ w.r.t. the coordinates x_i, $\xi_1, \ldots, \widehat{\xi_i}, \ldots, \xi_{r+1}$, the restriction $\pi\,|_{U_i}$ is given by mapping

$$\begin{array}{c} \mathbb{C}^{r+1} \ni (x_i, \xi_1, .., \widehat{\xi_i}, .., \xi_{r+1}) \\ \downarrow \cong \\ \left((x_i\, \xi_1, .., x_i\, \xi_{i-1}, x_i, x_i\, \xi_{i+1}, .., x_i\, \xi_{r+1}), [\xi_1 : .. : \underbrace{1}_{i\text{-th pos.}} : .. : \xi_{r+1}] \right) \in U_i \\ \downarrow\ \pi\,|_{U_i} \\ (x_i\, \xi_1, \ldots, x_i\, \xi_{i-1}, x_i, x_i\, \xi_{i+1}, \ldots, x_i\, \xi_{r+1}) \end{array}$$

Further, $\mathcal{E}_i := \mathcal{E} \cap U_i$ is described as the coordinate hyperplane $(x_i = 0)$; i.e., the open cover $\{U_i\}_{1 \leq i \leq r+1}$ of $\mathbf{Bl_0}(\mathbb{C}^{r+1})$ restricts to $\mathcal{E}$ to provide the standard open cover of $\mathbb{P}^r_{\mathbb{C}}$ by affine spaces $\mathbb{C}^{r+1}$, with $\{\xi_j\}_{j \in \{1, \ldots, r+1\} \smallsetminus \{i\}}$ being the analytic coordinates of $\mathcal{E}_i$.

Notation. To work with a more convenient notation we define

$$\mathbf{Bl_0}(\mathbb{C}^{r+1}) = \bigcup_{i=1}^{r+1} U_i, \qquad U_i = \operatorname{Spec}\left(\mathbb{C}\left[y_{i,1}, \ldots, y_{i,r+1} \right] \right),$$

by setting as coordinates for U_i's:

$$y_{i,k} := \begin{cases} x_k, & \text{for } i = k \\ \xi_k, & \text{for } i \neq k \end{cases}$$

• *The first blow-up.* Blowing up X at the origin, we take the diagram

$$\begin{array}{ccccc} \mathcal{E} & \subset & \mathbf{Bl_0}(\mathbb{C}^{r+1}) & \xrightarrow{\pi} & \mathbb{C}^{r+1} \\ \cup & & \cup & & \cup \\ \mathcal{E}_X := \mathcal{E} \cap \mathbf{Bl_0}(X) & \subset & \mathbf{Bl_0}(X) & \xrightarrow{\pi|_{\text{restr.}}} & X \end{array}$$

and consider the strict transform

$$\mathbf{Bl_0}(X) = \overline{\pi^{-1}(X \cap (\mathbb{C}^{r+1} \smallsetminus \{\mathbf{0}\}))} = \overline{\pi^{-1}(X) \cap (\mathbf{Bl_0}(\mathbb{C}^{r+1}) \smallsetminus \mathcal{E})}$$

of X in $\mathbb{C}^{r+1}$ under π, and the corresponding exceptional divisor $\mathcal{E}_X$.

• *Local description of* $\mathbf{Bl_0}(X)$ *and* $\mathcal{E}_X$. Pulling back f, we get

$$\pi^*(f)\,|_{U_i} = x_i^{\ell}\, \widetilde{f}_i = y_{i,i}^{\ell}\, \widetilde{f}_i$$

with

$$
\begin{aligned}
&\widetilde{f}_i\left(y_{i,1},\ldots,y_{i,r+1}\right)\\
&=\begin{cases} y_{1,1}^{(n+1)-\ell}+y_{1,2}^{\ell}+\cdots+y_{1,r+1}^{\ell}, & \text{if } i=1\\ y_{i,1}^{n+1}\,y_{i,i}^{(n+1)-\ell}+y_{i,2}^{\ell}+\cdots+y_{i,i-1}^{\ell}+1+y_{i,i+1}^{\ell}+\cdots+y_{i,r+1}^{\ell}, & \text{otherwise}\end{cases}
\end{aligned}
$$

Locally,

$$
\mathbf{Bl}_{\mathbf{0}}(X)\left|_{U_i}\right.\cong\left\{\left(y_{i,1},\ldots,y_{i,r+1}\right)\in\mathbb{C}^{r+1}\;\middle|\;\widetilde{f}_i\left(y_{i,1},\ldots,y_{i,r+1}\right)=0\right\},
$$

and the equations for $\mathcal{E}_X\left|_{U_i}\right.$ read as follows:

$$
\begin{aligned}
\mathbf{Bl}_{\mathbf{0}}(X)\cap\mathcal{E}_i&=\mathcal{E}_X\left|_{U_i}\right.\\
&\cong\left\{\left(y_{i,1},\ldots,y_{i,r+1}\right)\in\mathbb{C}^{r+1}\;\middle|\;y_{i,i}=\widetilde{f}_i\left(y_{i,1},\ldots,y_{i,r+1}\right)=0\right\}.
\end{aligned}
$$

Thus, the only singular affine patch is $U_1=\mathrm{Spec}\big(\mathbb{C}\left[y_{1,1},\ldots,y_{1,r+1}\right]\big)$ whenever $n>\ell$.

• *Global description of* $\mathbf{Bl}_{\mathbf{0}}(X)$ *and* $\mathcal{E}_X$. Passing to global coordinates, we can write

$$
\begin{aligned}
&\mathbf{Bl}_{\mathbf{0}}(X)\\
&=\left\{\left((x_1,..,x_{r+1}),\left[t_1:\cdots:t_{r+1}\right]\right)\in\mathbf{Bl}_{\mathbf{0}}(\mathbb{C}^{r+1})\;\middle|\;x_1^{(n+1)-\ell}\,t_1^{\ell}+\sum_{j=2}^{r+1}t_j^{\ell}=0\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{E}_X\\
&=\begin{cases}\left\{\left(\mathbf{0},\left[t_1:\cdots:t_{r+1}\right]\right)\in\{\mathbf{0}\}\times\mathbb{P}_{\mathbb{C}}^{r}\;\middle|\;t_1^{\ell}+t_2^{\ell}+\cdots+t_{r+1}^{\ell}=0\right\}, & \text{if } n=\ell-1\\ \left\{\left(\mathbf{0},\left[t_1:\cdots:t_{r+1}\right]\right)\in\{\mathbf{0}\}\times\mathbb{P}_{\mathbb{C}}^{r}\;\middle|\;t_2^{\ell}+t_3^{\ell}+\cdots+t_{r+1}^{\ell}=0\right\}, & \text{otherwise.}\end{cases}
\end{aligned}
$$

• *The (Fermat) singularity* $\mathbf{A}_{\ell-1,\ell}^{(r)}$ $(m=1)$. Blowing up the origin once, we achieve immediately the required desingularization, having exceptional divisor $\mathcal{E}_X\cong Y_{\ell}^{(r-1)}$.

• *The singularity* $\mathbf{A}_{\ell,\ell}^{(r)}$ $(m=2)$. In this case, $\mathbf{Bl}_{\mathbf{0}}(X)$ is smooth, whereas $\mathcal{E}_X$ on $\mathbf{Bl}_{\mathbf{0}}(X)$ has a singular, ordinary ℓ-fold point at $Q=(\mathbf{0},[1:0:\cdots:0])\in\mathcal{E}_X|_{U_1}$. To obtain an snc-resolution of the original singularity, we blow up once more at Q, and consider $\varphi=\pi_1\circ\pi_2$,

$$
\widetilde{X}=\mathbf{Bl}_{Q}(\mathbf{Bl}_{\mathbf{0}}(X))\xrightarrow{\pi_2}\mathbf{Bl}_{\mathbf{0}}(X)\xrightarrow{\pi_1=\pi}X.
$$

The new exceptional divisor D_2 is a $\mathbb{P}_{\mathbb{C}}^{r-1}$, and the strict transform D_1 of $\mathcal{E}_X$ is nothing but the ($(r-1)$-dimensional) blow-up of $\mathcal{E}_X$ at Q. Since $\mathcal{E}_X$ can be viewed as the projective cone $C^{\mathrm{pr}}\big(Y_\ell^{(r-2)}\big) \subset \mathbb{P}^r_{\mathbb{C}}$ over the Fermat hypersurface $Y_\ell^{(r-2)} \subset \mathbb{P}^{r-1}_{\mathbb{C}}$ with $[1:0:\cdots:0]$ as its vertex, blowing up $[1:0:\cdots:0]$, the diagram

$$\begin{array}{ccccc} \mathbb{P}(\mathcal{O}_{\mathbb{P}^{r-1}_{\mathbb{C}}} \oplus \mathcal{O}_{\mathbb{P}^{r-1}_{\mathbb{C}}}(1)) \cong & \mathbf{Bl}_{[1:0:\cdots:0]}(\mathbb{P}^r_{\mathbb{C}}) & \longrightarrow & \mathbb{P}^r_{\mathbb{C}} & \\ & \cup & & \cup & \\ & \mathbf{Bl}_{[1:0:\cdots:0]}\big(C^{\mathrm{pr}}\big(Y_\ell^{(r-2)}\big)\big) & \longrightarrow & C^{\mathrm{pr}}\big(Y_\ell^{(r-2)}\big) \cong \mathcal{E}_X & \end{array}$$

yields the isomorphism

$$\mathbf{Bl}_{[1:0:\cdots:0]}\big(C^{\mathrm{pr}}\big(Y_\ell^{(r-2)}\big)\big) \cong \mathbb{P}\big(\mathcal{O}_{Y_\ell^{(r-2)}} \oplus \mathcal{O}_{Y_\ell^{(r-2)}}(1)\big).$$

Hence, D_1 is a $\mathbb{P}^1_{\mathbb{C}}$-bundle of rank 2 over $Y_\ell^{(r-2)}$ meeting D_2 along

$$(D_1 \cdot D_2)\big|_{D_1} = \mathbb{P}\big(\mathcal{O}_{Y_\ell^{(r-2)}}(1)\big) \cong Y_\ell^{(r-2)}$$

(see Fig. 1).

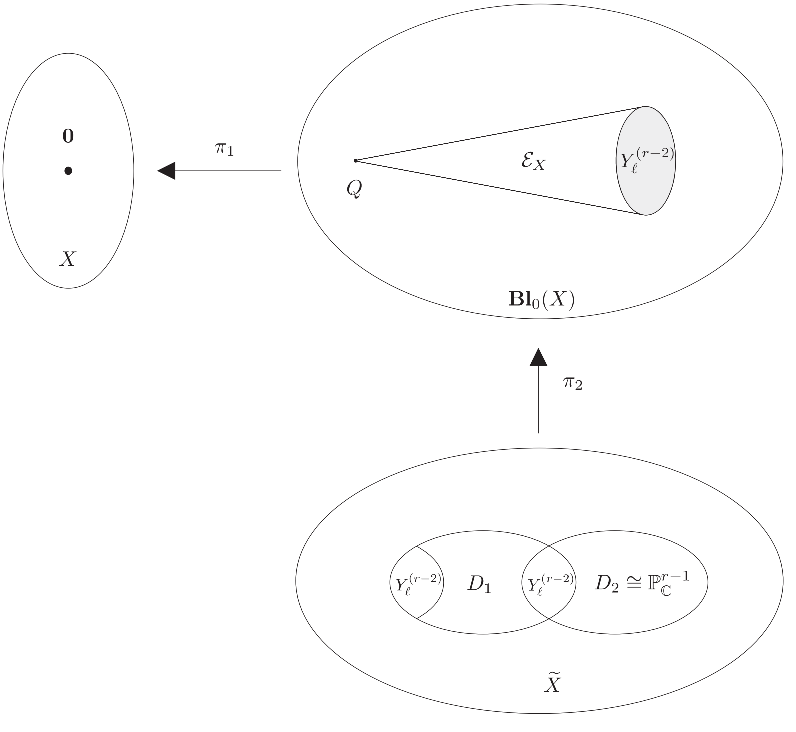

Fig. 1.

• *Singularities* $\mathbf{A}_{n,\ell}^{(r)}$ *with* $n > \ell$, $m \geq 2$, *either* $\ell \mid n$ *or* $\ell \mid n+1$. Locally, these singularities can be reduced successively to one of the above types as follows:

$$\boxed{\begin{array}{l} \mathbf{A}_{n,\ell}^{(r)} \rightsquigarrow \mathbf{A}_{n-\ell,\ell}^{(r)} \rightsquigarrow \mathbf{A}_{n-2\ell,\ell}^{(r)} \rightsquigarrow \cdots \rightsquigarrow \mathbf{A}_{2\ell-1,\ell}^{(r)} \rightsquigarrow \mathbf{A}_{\ell-1,\ell}^{(r)} \rightsquigarrow \mathbf{A}_{-1,\ell}^{(r)} \\ \qquad\qquad\qquad\qquad\qquad\qquad (\text{if } \ n+1 \equiv 0 \ (\mathrm{mod}\ \ell)) \\ \mathbf{A}_{n,\ell}^{(r)} \rightsquigarrow \mathbf{A}_{n-\ell,\ell}^{(r)} \rightsquigarrow \mathbf{A}_{n-2\ell,\ell}^{(r)} \rightsquigarrow \cdots \rightsquigarrow \mathbf{A}_{2\ell,\ell}^{(r)} \rightsquigarrow \mathbf{A}_{\ell,\ell}^{(r)} \rightsquigarrow \mathbf{A}_{0,\ell}^{(r)} \rightsquigarrow \mathbf{A}_{0,\ell}^{(r)} \\ \qquad\qquad\qquad\qquad\qquad\qquad (\text{if } \ n \equiv 0 \ (\mathrm{mod}\ \ell)) \end{array}}$$

(Each "$\rightsquigarrow$" denotes the result of a local blow-up, and $\mathbf{A}_{-1,\ell}^{(r)}, \mathbf{A}_{0,\ell}^{(r)}$ stand for "smooth charts"). But also globally, $\varphi : \widetilde{X} \longrightarrow X$ is decomposed into just m blow-ups

$$\begin{aligned} &\widetilde{X} = X_m \xrightarrow{\pi_m} X_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_3} X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = X \\ &X_i := \mathbf{Bl}_{Q_i}(\mathbf{Bl}_{Q_{i-1}}(\cdots(\mathbf{Bl}_{Q_1}(X)))), \ \ \forall i, \ 1 \leq i \leq m, \end{aligned} \tag{3.3}$$

of m "separated" points $Q_1 = \mathbf{0}$, $Q_2 = (\mathbf{0}, [1:0:0:\cdots:0]), \ldots, Q_m$, in the sense, that all the appearing exceptional divisors are prime (by construction) and, in addition, if $E_1 = \mathcal{E}_X, E_2, \ldots, E_m$ are the exceptional loci of $\pi_1, \pi_2, \ldots, \pi_m$, respectively, the singular point Q_i is resolved by π_i and the (possibly existing) new singular point Q_{i+1} is *not* contained in the strict transforms of $E_1, E_2, \ldots, E_{i-1}$ under π_i. Thus, defining the divisor D_i on $\widetilde{X}$ to be the strict transform of E_i under $\pi_{i+1} \circ \pi_{i+2} \circ \cdots \circ \pi_{m-1} \circ \pi_m$, we obtain the intersection graph of Fig. 2 with $D_i \cap D_{i+1} \cong Y_\ell^{(r-2)}$.

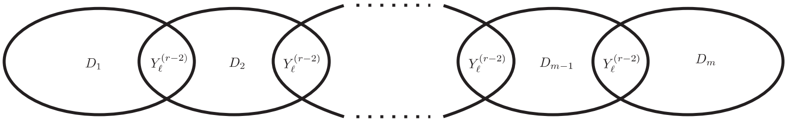

Fig. 2.

Computation of the discrepancy coefficients. Consider the Poincaré residue map

$$\mathrm{Res}_X : H^0(\mathbb{C}^{r+1}, \omega_{\mathbb{C}^{r+1}}(X)) \longrightarrow H^0(X, \omega_X),$$

where $\omega_X = \mathcal{O}_X(K_X) = (\Omega_X^r)^{\vee\vee} \subset \Omega^r_{\mathbb{C}(X)/\mathbb{C}}$. The rational canonical differential

$$\mathfrak{s} := \mathrm{Res}_X \left(\frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge \cdots \wedge dx_{r+1}}{f} \right) = \frac{dx_2 \wedge dx_3 \wedge \cdots \wedge dx_{r+1}}{(\partial f \,/\, \partial x_1)}$$

can be viewed as a (local) generator of $H^0(X, \omega_X)$. Assume that $n \neq \ell$ and that you have performed the first blow-up of X at $\mathbf{0}$. Then the new singularity (if any) on $\mathbf{Bl_0}(X)$ will belong to $\mathcal{E}_X \left|_{U_1}\right.$. For this reason, to find the discrepancy coefficient w.r.t. $\pi : \mathbf{Bl_0}(X) \longrightarrow X$, it suffices to compare $\mathfrak{s}$ with the rational canonical differential

$$\bar{\mathfrak{s}} := \frac{dy_{1,2} \wedge dy_{1,3} \wedge \cdots \wedge dy_{1,r+1}}{(\partial \widetilde{f}_1 \,/\, \partial y_{1,1})} \in \Omega^r_{\mathbb{C}(U_1)/\mathbb{C}}.$$

(U_1 is non-singular with local coordinates $y_{1,2}, \ldots, y_{1,r+1}$ at any point P for which $\partial \widetilde{f}_1(P) \,/\, \partial y_{1,1} \neq 0$). In U_1 we have $x_1 = y_{1,1}$ and

$$x_j = x_1 \, \xi_j = y_{1,1} \, y_{1,j}, \quad \text{for all} \quad j \in \{2, 3, \ldots, r+1\}.$$

Hence,

$$\begin{aligned} dx_2 \wedge dx_3 \wedge \cdots \wedge dx_{r+1} \\ = y_{1,1}^{r-1} \, (y_{1,1} \left(dy_{1,2} \wedge dy_{1,3} \wedge \cdots \wedge dy_{1,r+1}\right) \\ + \sum_{i=2}^{r+1} (-1)^i \;\; y_{1,i} \, dy_{1,2} \wedge \cdots \wedge \widehat{dy_{1,i}} \wedge \cdots \wedge dy_{1,r+1}) \end{aligned} \tag{3.4}$$

and

$$\partial f \,/\, \partial x_1 = (n+1)\; x_1^n = (n+1)\; y_{1,1}^n = \left(\frac{n+1}{n+1-\ell}\right)\; y_{1,1}^{\ell} \; (\partial \widetilde{f}_1 \,/\, \partial y_{1,1}) \tag{3.5}$$

On the other hand,

$$d\widetilde{f}_1 = (n+1-\ell)\;\; y_{1,1}^{n-\ell} \, dy_{1,1} + \ell \, \left(y_{1,2}^{\ell-1} \, dy_{1,2} + \cdots + y_{1,r+1}^{\ell-1} \, dy_{1,r+1}\right) = 0$$

if and only if

$$dy_{1,1} = -\frac{\ell}{n+1-\ell} \; y_{1,1}^{\ell-n} \; \left(y_{1,2}^{\ell-1} \, dy_{1,2} + \cdots + y_{1,r+1}^{\ell-1} \, dy_{1,r+1}\right). \tag{3.6}$$

Substituting the expression (3.6) for $dy_{1,1}$ into the right-hand side of (3.4), we obtain

$$\begin{aligned} dx_2 \wedge dx_3 \wedge \cdots \wedge dx_{r+1} \\ = \left(-\frac{\ell}{n+1-\ell} \; y_{1,1}^{r-1+\ell-n} \; (y_{1,2}^{\ell} + \cdots + y_{1,r+1}^{\ell}) + y_{1,1}^{r}\right) dy_{1,2} \wedge \cdots \wedge dy_{1,r+1}. \end{aligned} \tag{3.7}$$

Combining now (3.7) with $y_{1,2}^{\ell} + \cdots + y_{1,r+1}^{\ell} = -y_{1,1}^{(n+1)-\ell}$ and (3.5), we get

$$\mathfrak{s} = \frac{y_{1,1}^{r} \; dy_{1,2} \wedge dy_{1,3} \wedge \cdots \wedge dy_{1,r+1}}{y_{1,1}^{\ell} \; (\partial \widetilde{f}_1 \,/\, \partial y_{1,1})} = y_{1,1}^{r-\ell} \; \bar{\mathfrak{s}}. \tag{3.8}$$

The equality (3.8) shows that the discrepancy coefficient of $\mathcal{E}_X$ with respect to $\pi : \mathbf{Bl_0}(X) \longrightarrow X$ equals $r-\ell$. Using the notation introduced in (3.3), one proves analogously that

$$K_{X_i} - \pi_i^* \left(K_{X_{i-1}}\right) = (r-\ell)\ E_i,\ \ \forall i,\ 1 \le\ i \le m-1. \tag{3.9}$$

Moreover,

$$K_{X_m} - \pi_m^* \left(K_{X_{m-1}}\right) = \begin{cases} (r-\ell)\ D_m,\ \text{if}\ \ \ell\,|n+1 \\ \\ (r-1)\ D_m,\ \text{if}\ \ \ell\,|n \end{cases} \tag{3.10}$$

Note that if $\ell\,|n$, then we have to pass through $\mathbf{A}^{(r)}_{\ell,\ell}$. The additional blow-up which resolves the singularity of the exceptional locus (so that $\varphi : \widetilde{X} \longrightarrow X$ fulfills the snc-condition) has a *smooth* point on the *r-fold* as its centre. Consequently, the discrepancy coefficient of $D_m = D_{\frac{n}{\ell}+1}$ equals $r-1$ (see [15, p. 187]). Now (3.9) gives:

$$\begin{aligned} &K_{\widetilde{X}} - \varphi^* \left(K_X\right) \\ &= \sum_{i=1}^{m-1} \left(\pi_{i+1} \circ \pi_{i+2} \circ \cdots \circ \pi_m\right)^* \left((r-\ell)\ E_i\right) + \left[K_{X_m} - \pi_m^* \left(K_{X_{m-1}}\right)\right] \end{aligned} \tag{3.11}$$

Since

$$\left(\pi_{i+1} \circ \pi_{i+2} \circ \cdots \circ \pi_m\right)^* \left(\,E_i\right) = \begin{cases} \sum\limits_{j=i}^{m} D_j, & \text{if}\ \ \ell\,|n+1 \\ \sum\limits_{j=i}^{m-1} D_j + \ell\, D_m, & \text{if}\ \ \ell\,|n \end{cases} \tag{3.12}$$

for all i, $1 \le i \le m-1$, the formulae (3.1) and (3.2) follow from (3.10), (3.11) and (3.12). $\square$

Remark 3.2. (i) If $n+1 \equiv 0 (\text{mod}\ \ell)$, then $\varphi : \widetilde{X} \longrightarrow X$ is crepant.

(ii) Obviously,

$$E(\mathbb{P}^{r-1}_{\mathbb{C}}; u, v) = \sum_{p=0}^{r-1} (uv)^p\,. \tag{3.13}$$

(iii) To complete the catalogue of the E-polynomials of our exceptional divisors, it suffices to find out those of $Y_\ell^{(r-2)}$ (or, equivalently, of $Y_\ell^{(r-1)}$), as we have

$$E(\mathbb{P}(\mathcal{O}_{Y_\ell^{(r-2)}} \oplus \mathcal{O}_{Y_\ell^{(r-2)}}(1)); u, v) = E(Y_\ell^{(r-2)}; u, v) \cdot (1+uv)\,. \tag{3.14}$$

(iv) According to the classical Lefschetz Hyperplane Theorem, the Fermat hypersurface $Y_\ell^{(r-1)}$ has "non-trivial" Hodge (p,q)-numbers only if $p+q = r-1$. Next lemma expresses them by means of the non-central Eulerian numbers of generalized factorials (as defined in §1 (e)), and can be easily proven, e.g., by determining the χ_y-characteristic of $Y_\ell^{(r-1)}$ via Riemann–Roch Theorem (see [19, §2]), or, alternatively, by writing down the exact sequences involving the cohomology groups of $\mathbb{P}^r_{\mathbb{C}}$ and $Y_\ell^{(r-1)}$ with coefficients taken from the twisted sheaves $\Omega^p_{\mathbb{P}^r_{\mathbb{C}}}(-\ell)$ and $\Omega^p_{Y_\ell^{(r-1)}}(-\ell)$, respectively. (Note that both proofs are valid for *any* smooth hypersurface of degree ℓ. On the other hand, the formula for the Euler number is simpler and can be derived directly by evaluating the highest Chern class of $Y_\ell^{(r-1)}$ and applying Gauss–Bonnet Theorem; see, e.g., [11, p. 152].)

Lemma 3.3. *The Hodge numbers of the $(r-1)$-dimensional Fermat hypersurface $Y_\ell^{(r-1)}$ of degree $\ell \geq 2$ are given by the formula*

$$h^{p,q}(Y_\ell^{(r-1)}) = \begin{cases} \mathfrak{S}\,(r,\, p+1 \mid \ell-1,\, p) + \delta_{2p,r-1}, & \text{if} \quad p+q = r-1 \\ \delta_{p,q}, & \text{if} \quad p+q \neq r-1 \end{cases}$$

Hence, $Y_\ell^{(r-1)}$ has E-polynomial

$$\begin{aligned} E(Y_\ell^{(r-1)};u,v) &= \sum_{0\leq p,q\leq r-1} (-1)^{p+q}\ h^{p,q}(Y_\ell^{(r-1)})\, u^p v^q \\ &= \sum_{p=0}^{r-1} u^p \left[v^p + (-1)^{r-1}\,\mathfrak{S}\,(r,\, p+1 \mid \ell-1,\, p)\ v^{r-1-p} \right] \end{aligned} \tag{3.15}$$

and Euler number

$$\begin{aligned} e(Y_\ell^{(r-1)}) &= \left[\sum_{p=0}^{r-1} (-1)^{r-1}\,\mathfrak{S}\,(r,\, p+1 \mid \ell-1,\, p) \right] + r \\ &= \frac{1}{\ell}\left((1-\ell)^{r+1} - 1 \right) + r + 1 \end{aligned} \tag{3.16}$$

Proof of Theorem 1.8. (i) If $n+1 \equiv 0 \pmod{\ell}$, then Proposition 3.1 and (1.1) give:

$$\begin{aligned} &E_{\mathrm{str}}(X;u,v) - E(X\smallsetminus\{\mathbf{0}\};u,v) \\ &\quad = \sum_{i=1}^{m} \frac{(uv-1)\,E(D_i^\circ;u,v)}{(uv)^{i(r-\ell)+1}-1} \\ &\qquad + \left[\sum_{i=1}^{m-1} \frac{(uv-1)^2\ E(D_{\{i,i+1\}}^\circ;u,v)}{\left((uv)^{i(r-\ell)+1}-1\right)\left((uv)^{(i+1)(r-\ell)+1}-1\right)} \right] \end{aligned}$$

But $E(D_1^\circ;u,v) = E(Y_\ell^{(r-2)};u,v)\cdot uv$,

$$E(D_i^\circ;u,v) = E(Y_\ell^{(r-2)};u,v)\cdot(uv-1)\,,\ \forall i,\ i\in\{2,\ldots,m-1\},$$

(by (3.14)), and $E(D_m^\circ;u,v) = E(Y_\ell^{(r-1)};u,v) - E(Y_\ell^{(r-2)};u,v)$,

$$E(D_{\{i,i+1\}}^\circ;u,v) = E(Y_\ell^{(r-2)};u,v),\ \forall i,\ i\in\{2,\ldots,m-1\}.$$

Consequently, the difference $E_{\mathrm{str}}(X;u,v)-E(X\smallsetminus\{\mathbf{0}\};u,v)$ equals

$$\begin{aligned}
&(uv-1)E(Y_\ell^{(r-2)};u,v)\\
&\qquad\left[\frac{uv}{(uv)^{r-\ell+1}-1}+\sum_{i=2}^{m-1}\frac{uv-1}{(uv)^{i(r-\ell)+1}-1}-\frac{uv-1}{(uv)^{m(r-\ell)+1}-1}\right]\\
&+\frac{(uv-1)\,E(Y_\ell^{(r-1)};u,v)}{(uv)^{m(r-\ell)+1}-1}+\sum_{i=1}^{m-1}\frac{(uv-1)^2\ \ E(Y_\ell^{(r-2)};u,v)}{\left((uv)^{i(r-\ell)+1}-1\right)\left((uv)^{(i+1)(r-\ell)+1}-1\right)},
\end{aligned}$$

leading to the desired formula via (3.15). Passing to the limit of $E_{\mathrm{str}}(X;u,v)$, for $u,v\to 1$, and taking (1.2) and (3.16) into account, one obtains the corresponding formula for the string-theoretic Euler number $e_{\mathrm{str}}(X)$.

(ii) If $n\equiv 0\ (\mathrm{mod}\ \ell)$, then 3.1 (ii) and (1.1) give analogously

$$\begin{aligned}
&E_{\mathrm{str}}(X;u,v)-E(X\smallsetminus\{\mathbf{0}\};u,v)\\
&=\sum_{i=1}^{m-1}\frac{(uv-1)\,E(D_i^\circ;u,v)}{(uv)^{i(r-\ell)+1}-1}+\frac{uv-1}{(uv)^{(m-1)\,\ell\,(r-\ell)+r}-1}\,E(D_m^\circ;u,v)\\
&\quad+\left[\sum_{i=1}^{m-2}\frac{(uv-1)^2\ \ E(D_{\{i,i+1\}}^\circ;u,v)}{\left((uv)^{i(r-\ell)+1}-1\right)\left((uv)^{(i+1)(r-\ell)+1}-1\right)}\right]\\
&\quad+\frac{(uv-1)^2\ \ E(D_{\{m-1,m\}}^\circ;u,v)}{\left((uv)^{(m-1)(r-\ell)+1}-1\right)\left((uv)^{(m-1)\,\ell\,(r-\ell)+r}-1\right)}
\end{aligned}$$

Since $D_1^\circ,D_2^\circ,\dots,D_{m-1}^\circ$ are as in (i), and

$$\begin{aligned}
E(D_m^\circ;u,v)&=E(\mathbb{P}_{\mathbb{C}}^{r-1};u,v)-E(Y_\ell^{(r-2)};u,v),\\
E(D_{\{i,i+1\}}^\circ;u,v)&=E(Y_\ell^{(r-2)};u,v),\ \text{ for all }\ i\in\{1,\dots,m-1\},
\end{aligned}$$

we obtain by (3.13) (3.14):

$$\begin{aligned}
&E_{\mathrm{str}}(X;u,v)-E(X\smallsetminus\{\mathbf{0}\};u,v)=(uv-1)E(Y_\ell^{(r-2)};u,v)\\
&\quad\times\left[\frac{uv}{(uv)^{r-\ell+1}-1}+\sum_{i=2}^{m-1}\frac{uv-1}{(uv)^{i(r-\ell)+1}-1}-\frac{uv-1}{(uv)^{(m-1)\ell(r-\ell)+r}-1}\right]\\
&\quad+\frac{(uv-1)\left(\sum_{p=0}^{r-1}(uv)^p\right)}{(uv)^{(m-1)\,\ell\,(r-\ell)+r}-1}\\
&\quad+\left[\sum_{i=1}^{m-2}\frac{(uv-1)^2\ \ E(Y_\ell^{(r-2)};u,v)}{\left((uv)^{i(r-\ell)+1}-1\right)\left((uv)^{(i+1)(r-\ell)+1}-1\right)}\right]\\
&\quad+\frac{(uv-1)^2\ \ E(Y_\ell^{(r-2)};u,v)}{\left((uv)^{(m-1)(r-\ell)+1}-1\right)\left((uv)^{(m-1)\,\ell\,(r-\ell)+r}-1\right)}
\end{aligned}$$

The string-theoretic Euler number is examined as in (i). $\square$

4. Some global geometric examples

The E_{str}-function of a complex r-fold V with only k isolated log-terminal singularities $Q_1, Q_2, .., Q_k$ equals:

$$E_{\mathrm{str}}(V; u, v) = E(V; u, v) + \sum_{i=1}^{k} \left(E_{\mathrm{str}}((V, Q_i); u, v) - 1\right). \tag{4.1}$$

In particular, a simple closed formula for the string-theoretic Euler number e_{str} can be easily built whenever V is a (global) complete intersection in a projective space, equipped with prescribed singularities belonging to the class under consideration.

Proposition 4.1. *Let $V = V_{(d_1, d_2, \ldots, d_{N-r})}$ be an r-dimensional complete intersection of multidegree $(d_1, d_2, \ldots, d_{N-r})$ in $\mathbb{P}^N_{\mathbb{C}}$ having only k isolated singularities $Q_1, \ldots, Q_k$ of types $\mathbf{A}^{(r)}_{n_1,\ell_1}, \ldots, \mathbf{A}^{(r)}_{n_k,\ell_k}$ with either $\ell_i \mid n_i$ or $\ell_i \mid n_i + 1$, for all $i = 1, \ldots, k$. Then its string-theoretic Euler number equals*

$$\begin{aligned} e_{\mathrm{str}}(V) = {} & \left[\binom{N+1}{r} + \sum_{\nu=1}^{r} (-1)^{\nu} \binom{N+1}{r-\nu} \left(\sum_{1 \le j_1 \le \cdots \le j_\nu \le N-r} d_{j_1} \cdots d_{j_\nu} \right)\right] \left(\prod_{j=1}^{N-r} d_j \right) \\ & + \sum_{i=1}^{k} \left[e_{\mathrm{str}}(V, Q_i) + (-1)^{r+1} n_i (\ell_i - 1)^r - 1 \right], \end{aligned} \tag{4.2}$$

where $e_{\mathrm{str}}(Y, Q_i)$, $i = 1, \ldots, k$, are computable via Theorem 1.8.

Proof. By a small deformation of V one can always obtain a non-singular complete intersection V' in $\mathbb{P}^N_{\mathbb{C}}$ having multidegree $(d_1, d_2, \ldots, d_{N-r})$. Using a standard technique which involves the Mayer–Vietoris sequence (cf. [11, Ch. 5, Cor. 4.4 (ii)]) one shows easily that

$$e(V) = e\left(V'\right) + (-1)^{r+1} \sum_{i=1}^{k} [\text{Milnor number of } (V, Q_i)].$$

The Euler number of V' can be computed again by evaluating the highest Chern class of V' at its fundamental cycle (cf. Chen–Ogiue [6, Thm. 2.1]), and is expressible by the closed formula:

$$e\left(V'\right) = \left[\binom{N+1}{r} + \sum_{\nu=1}^{r} (-1)^{\nu} \binom{N+1}{r-\nu} \left(\sum_{1 \le j_1 \le \cdots \le j_\nu \le N-r} d_{j_1} \cdots d_{j_\nu} \right)\right] \left(\prod_{j=1}^{N-r} d_j \right).$$

(4.2) follows clearly from (4.1). $\square$

Examples 4.2. Let us now apply (4.2) for some well-known hypersurfaces and complete intersections.

(i) Generalizing Hirzebruch's method of constructing a singular quintic with 126 nodes ([21, p. 762]), Werner defines in [28, pp. 216–217] a hypersurface $V \subset \mathbb{P}^4_{\mathbb{C}}$ of degree 5 by homogenizing a three-dimensional affine complex variety of the form

$$\left\{ (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \;\middle|\; \lambda\, g_1(z_1, z_2) - g_2(z_3, z_4) = 0 \right\},\ \lambda \in \mathbb{C}^*,$$

where $\{g_i = 0\}$, $i = 1, 2$, are plane quintic curves having the three axes and a circumscribed conic (about the corresponding coordinate triangle) as their irreducible components (see Fig. 3). Since each of these curves has 3 $\mathbf{D}_4$-singularities, V (after homogenization) will have $3^2 = 9$ singularities of type $\mathbf{A}^{(3)}_{2,3}$. This means that

$$e_{\mathrm{str}}(V) = -200 + 9 \cdot \left(9 + 2^4 - 1\right) = 16.$$

In fact, $e_{\mathrm{str}}(V) = e(\widetilde{V}) = 16$, where $\widetilde{V} \to V$ is the crepant desingularization of V arising from a single simultaneous blow-up of the 9 singularities (cf. 3.2 (i)). $\widetilde{V}$ is obviously a 3-dimensional Calabi–Yau manifold.

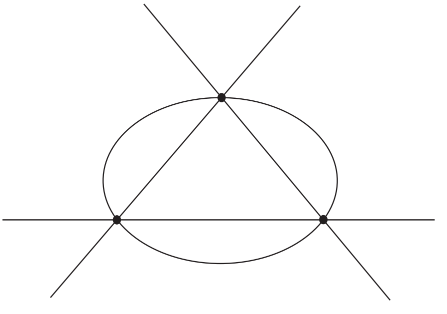

Fig. 3.

(ii) The $(N-1)$-dimensional Goryunov's quartics [13]:

$$V_\kappa := \left\{ [z_1 : .. : z_{N+1}] \in \mathbb{P}^N_{\mathbb{C}} \;\middle|\; 2\,(\kappa+1) \sum_{1 \le i < j \le N+1} z_i^2 z_j^2 + \kappa \left(\sum_{1 \le j \le N+1} z_j^2 \right)^2 = 0 \right\}$$

$(N \ge \kappa,\ N \ge 3,\ \kappa \ge 0)$ have $2^\kappa \binom{N+1}{\kappa+1}$ $\mathbf{A}_1$-singularities ($\mathbf{A}^{(N-1)}_{1,2}$-singularities, in our notation), and string-theoretic Euler number

$$\begin{aligned} e_{\mathrm{str}}(V_\kappa) = {} & \frac{1}{4}\left((-3)^{N+1} - 1\right) + N + 1 \\ & + 2^\kappa \binom{N+1}{\kappa+1} \left[\left(\frac{1}{N-2} \left(\frac{1}{2} \left((-1)^N - 1 \right) + N \right) + (-1)^N - 1 \right) \right]. \end{aligned}$$

Note that, e.g., for $N = 5$, the string-theoretic index of the underlying space of each of the singularities is $3 > 1$, whereas the string-theoretic index $\mathrm{ind}_{\mathrm{str}}(V_\kappa)$ of V_κ can be equal to 1 (for $\kappa \in \{0, 1, 3, 4\}$).

Table 1.

n	4	5	6	7	8	9	10	11	12	13	14	15
$e_{\mathrm{str}}(V)$	8	6	$\frac{27}{4}$	$\frac{96}{13}$	$\frac{160}{19}$	$\frac{120}{13}$	$\frac{175}{17}$	$\frac{480}{43}$	$\frac{648}{53}$	$\frac{105}{8}$	$\frac{539}{38}$	$\frac{1344}{89}$

(iii) The $(n-2)$-dimensional Segre–Knörrer complete intersection of two quadrics

$$V := \left\{\mathbf{z} = \left[z_1 : z_2 : \ldots : z_{n+1}\right] \in \mathbb{P}^n_{\mathbb{C}} \;\middle|\; {}^t\mathbf{z}\, M\, \mathbf{z} = {}^t\mathbf{z}\, M'\, \mathbf{z} \,=\, 0\right\}, \quad n \geq 4,$$

where M and M' are the $(n+1) \times (n+1)$-matrices:

$$M = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & 0 & \cdots & \cdots & 1 & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 \\ 1 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}, \quad M' = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \cdots & \cdots & 1 & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

has $Q = [1 : 0 : \cdots : 0 : 0]$ as single isolated point which is of type $\mathbf{A}_n$ (i.e., $\mathbf{A}^{(n-2)}_{n,2}$ in our notation, see [23, p. 48]). According to (4.2), the string-theoretic Euler number of V equals

$$\begin{aligned} e_{\mathrm{str}}(V) &= \sum_{\nu=0}^{n-2} (-1)^{\nu}\, 2^{\nu+2 \binom{n+1}{\nu+3}}\, (\nu+1) + e_{\mathrm{str}}(V, Q) + (-1)^{n-1}\, n - 1 \\ &= n - 1 + e_{\mathrm{str}}(V, Q), \end{aligned}$$

with

$$e_{\mathrm{str}}(V, Q) = \begin{cases} \frac{(n-1)^2}{n^2-3n-2}, & \text{if } n \text{ odd} \\ \frac{(n-2)(n+1)}{n(n-4)+(n-2)}, & \text{if } n \text{ even} \end{cases}$$

For $n \leq 15$, $e_{\mathrm{str}}(V)$ takes the values shown in Table 1.

(iv) Werner's 3-dimensional complete intersection of a cubic and two quadrics

$$V := \left\{ [z_1 : z_2 : \ldots : z_7] \in \mathbb{P}^6_{\mathbb{C}} \;\middle|\; \begin{aligned} &\sum_{i=1}^{4} z_i^3 = \sum_{j=2}^{7} z_j^2 \\ &\qquad = \sum_{i=1}^{3} i\, z_{i+1}^2 + \sum_{j=4}^{6} (j-3)\, z_{j+1}^2 = 0 \end{aligned} \right\}$$

has 4 singularities of type $\mathbf{A}_{2,3}^{(3)}$ at the points $\left[0:0:0:0:\pm 1:\pm\sqrt{-2}:1\right]$ and 18 singularities of type $\mathbf{A}_{1,2}^{(3)}$ (i.e., nodes) at the points

$$\left[-\zeta_3^j:1:0:0:\pm\sqrt{-1}:0:0\right],\ \left[-\zeta_3^j:0:1:0:0:\pm\sqrt{-1}:0\right],$$
$$\left[-\zeta_3^j:0:0:1:0:0:\pm\sqrt{-1}\right],$$

$j=1,2,3,$ where ζ_3 is a primitive third root of unity (see [28, pp. 221–222]). Its string-theoretic Euler number equals

$$e_{\mathrm{str}}(V)=-144+4\cdot\left(9+2^4-1\right)+18\cdot 2=-12=e(\widetilde{V}),$$

where $\widetilde{V}$ is a Calabi–Yau threefold which arises after a crepant desingularization of V coming from the simultaneous (usual) blow-up of the 9 $\mathbf{A}_{2,3}^{(3)}$-singularities and an appropriate *small, projective* resolution of the 18 nodes.

(v) Let $V=V_1\cap V_2\cap\cdots\cap V_{N-r}\subset\mathbb{P}_{\mathbb{C}}^N$ be a complete intersection of Fermat hypersurfaces

$$V_i=\left\{\left[z_1:\ldots:z_{N+1}\right]\in\mathbb{P}_{\mathbb{C}}^N\ \middle|\ \sum_{j=1}^{N+1}b_{ij}\,z_j^d=0\right\},\quad 1\le i\le N-r,$$

of degree d, $2\le d\le r$, and assume that V is r-dimensional, i.e.,

$$\operatorname{rank}\left((b_{ij})_{1\le i\le N-r,1\le j\le N+1}\right)=N-r.$$

Further, consider the map

$$\Phi_d:\mathbb{P}_{\mathbb{C}}^N\longrightarrow\mathbb{P}_{\mathbb{C}}^N,\ \left[z_1:\ldots:z_{N+1}\right]\longmapsto\left[z_1^d:\ldots:z_{N+1}^d\right]=\left[\xi_1:\ldots:\xi_{N+1}\right].$$

Φ_d displays $\mathbb{P}_{\mathbb{C}}^N$ as a d^N-sheeted ramified covering of itself, branched along the coordinate axes $\{\xi_j=0\}$. On the other hand,

$$\Phi_d\left(V_i\right)=\left\{\left[\xi_1:..:\xi_{N+1}\right]\in\mathbb{P}_{\mathbb{C}}^N\ \middle|\ \sum_{j=1}^{N+1}b_{ij}\,\xi_j=0\right\},\quad 1\le i\le N-r,$$

and $\Phi_d\left(V\right)\cong\mathbb{P}_{\mathbb{C}}^r\subset\mathbb{P}_{\mathbb{C}}^N$. Now if

$$\mathcal{L}_j:=\{\xi_j=0\}\cap\Phi_d\left(V\right)\subset\mathbb{P}_{\mathbb{C}}^r,\quad 1\le j\le N+1,$$

denote by $\mathcal{M}\left(\mathbb{P}_{\mathbb{C}}^N\right)=\mathbb{C}\left(z_2/z_1,\ldots,z_{N+1}/z_1\right)$ the rational function field of $\mathbb{P}_{\mathbb{C}}^N$, and let

$$\mathcal{M}\left(\mathbb{P}_{\mathbb{C}}^N\right)\left(\sqrt[d]{\frac{\psi_2}{\psi_1}},\ldots,\sqrt[d]{\frac{\psi_{N+1}}{\psi_1}}\right)$$

be the Kummer extension of $\mathcal{M}\left(\mathbb{P}_{\mathbb{C}}^N\right)$ determined by adjoining "d-th roots of ratios", where ψ_j is the linear form defining the hyperplane $\mathcal{L}_j$. This is an abelian extension with Galois group $(\mathbb{Z}/d\mathbb{Z})^N$. The variety V can be thought of as the normalization of $\mathbb{P}^N_{\mathbb{C}}$ w.r.t. this field, as being the total space of the d^N-sheeted covering

$$\Phi_d\,|_V\;:\;V\longrightarrow\mathbb{P}^r_{\mathbb{C}}$$

of $\mathbb{P}^r_{\mathbb{C}}$, branched along the $\mathcal{L}_j$'s. The hyperplane arrangement

$$\mathfrak{L}:=\bigcup_{j=1}^{N+1}\mathcal{L}_j=\left\{\prod_{j=1}^{N+1}\psi_j=0\right\}\subset\mathbb{P}^r_{\mathbb{C}}$$

admits a natural stratification

$$\mathfrak{L}=\mathfrak{L}^{(1)}\supset\mathfrak{L}^{(2)}=\bigcup_{1\le j_1<j_2\le N+1}\mathcal{L}_{j_1,j_2}\supset\cdots\supset\mathfrak{L}^{(r)}=\bigcup_{1\le j_1<j_2<\cdots<j_r\le N+1}\mathcal{L}_{j_1,j_2,\ldots,j_r}$$

where

$$\mathcal{L}_{j_1,j_2,\ldots,j_k}:=\mathcal{L}_{j_1}\cap\mathcal{L}_{j_2}\cap\cdots\cap\mathcal{L}_{j_k}\cong\mathbb{P}^{r-k}_{\mathbb{C}}\subset\mathbb{P}^r_{\mathbb{C}},\quad 1\le k\le r.$$

($\mathfrak{L}^{(r)}$ consists of the *points* of $\mathfrak{L}$, $\mathfrak{L}^{(r-1)}$ consists of the *lines* of $\mathfrak{L}$, etc). Let us now define

$$t_i:=t_i\,(0):=\#\left\{\begin{array}{l}\text{elements of }\mathfrak{L}^{(r)}\text{ (i.e., points of }\mathfrak{L})\\ \text{contained in exactly } i \text{ hyperplanes of }\mathfrak{L}\end{array}\right\}$$

and, in general,

$$t_i\,(\kappa):=\#\left\{\begin{array}{l}\text{elements of }\mathfrak{L}^{(r-\kappa)}\text{ contained}\\ \text{in exactly } i \text{ hyperplanes of }\mathfrak{L}\end{array}\right\},\quad 0\le\kappa\le r.$$

$\mathfrak{L}$ is called a *point arrangement* if

$$t_i\,(\kappa)=0,\quad\text{for all } i>r-\kappa \text{ and for all } \kappa\in\{1,\ldots,r-2\}.$$

The V's defined by means of point arrangements have at most isolated singularities; more precisely, by analogy with the two-dimensional case (cf. [20]), V inherits exactly d^{N-i} isolated singularities over each point of $\mathfrak{L}$ contained in $i\ge r+1$ hyperplanes. In particular, for point arrangements $\mathfrak{L}$ within $\mathbb{P}^r_{\mathbb{C}}$, for which

$$t_i=0,\quad\forall\, i,\quad i\ge r+2,$$

all singularities of V have to be $\mathbf{A}^{(r)}_{d-1,d}$ -singularities. In this case, formula (4.2) reads as follows:

$$\begin{aligned}e_{\mathrm{str}}(V)=&\left[\sum_{\nu=0}^{r}(-1)^\nu\binom{N+1}{r-\nu}\binom{N-r+\nu-1}{\nu}d^{\nu+N-r}\right]+\\ &+\;t_{r+1}\cdot d^{N-r-1}\cdot\Bigg(\frac{1}{r-d+1}\left[\frac{1}{d}((1-d)^{r+1}-1)+r+1\right]\\ &+(-1)^{r+1}\,(d-1)^{r+1}-1\Bigg).\end{aligned}\qquad(4.3)$$

For $r = 3$, several combinatorial properties of hyperplane arrangements in $\mathbb{P}^3_{\mathbb{C}}$, as well as properties of birational geometry of the resulting coverings, have been studied by Hunt [22]. As far as point arrangements are concerned (with $t_4 \geq 1$, $t_5 = t_6 = 0$) there are some interesting and aesthetically pleasing examples, given by the facet planes of certain regular (platonic) and semiregular (archimedean) solids (see Fig. 4). For these point arrangements, formula (4.3) gives the results in Table 2.

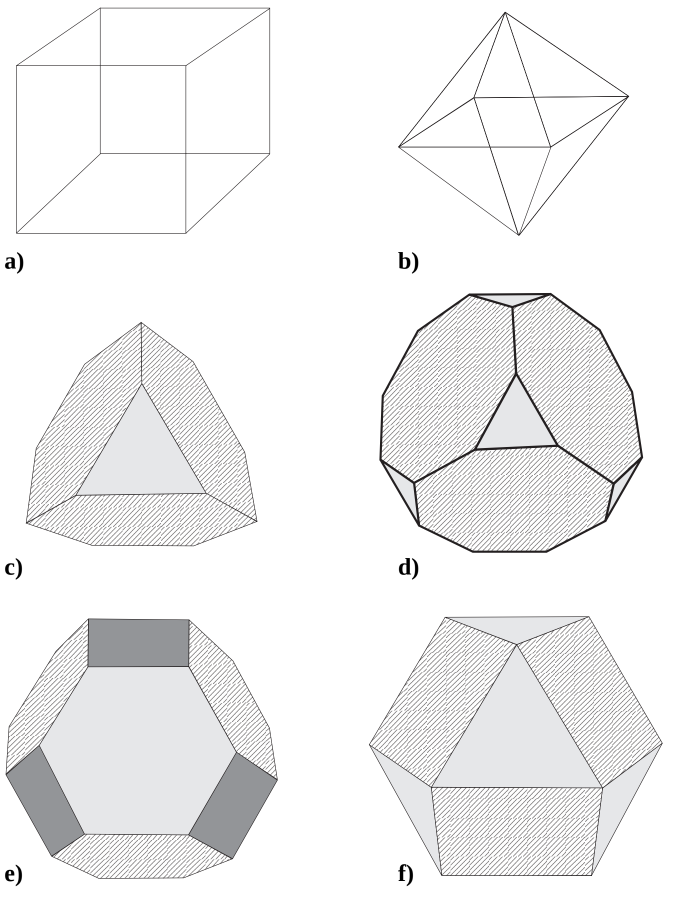

Fig. 4. a) Cube; **b)** Octahedron;
c) Trunctated Tetrahedron ((3, 6, 6)-solid); **d)** Trunctated Cube ((3, 8, 8)-solid);
e) Trunctated Octahedron ((4, 6, 6)-solid); **f)** Trunctated Cuboctahedron ((3, 3, 4)-solid)

Examples **a)** (with $d = 3$) and **b)** (with $d = 2$) were first mentioned by Hirzebruch [21, pp. 764–765], who used them to construct 3-dimensional Calabi-Yau manifolds $\widetilde{V}$ with Euler number 72 (resp., 64) by a “big” (resp. “small”, projective)

Table 2.

Solids	N	t_3	$t_4 = \frac{1}{4}\left[\binom{N+1}{3} - t_3\right]$	$e_{\mathrm{str}}(V)$ (for $d = 2$)	$e_{\mathrm{str}}(V)$ (for $d = 3$)
a)	5	8	3	12	72
b)	7	8	12	64	-324
c)	7	32	6	-32	-4212
d), **e)**	13	256	27	$-111\,616$	$-68\,496\,840$
f)	13	208	39	$-99\,328$	$-62\,828\,136$

crepant resolution of the 9 (resp., 96) singularities of V (cf. the remarks in [28, p. 219]).

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