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RESOLVING 3-DIMENSIONAL TORIC SINGULARITIES

by

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Abstract. — This paper surveys, in the first place, some basic facts from the classification theory of normal complex singularities, including details for the low dimensions 2 and 3. Next, it describes how the toric singularities are located within the class of rational singularities, and recalls their main properties. Finally, it focuses, in particular, on a toric version of Reid's desingularization strategy in dimension three.

1. Introduction

There are certain general *qualitative criteria* available for the rough classification of singularities of complex varieties. The main ones arise:

$$\left\{\begin{array}{l}\bullet\ \text{from the study of the punctual algebraic} \\ \ \text{behaviour of these varieties} \\ \text{(w.r.t. local rings associated to singular points)} \\ [\textit{algebraic classification}] \\ \\ \bullet\ \text{from an intrinsic characterization} \\ \text{for the nature of the possible exceptional} \\ \text{loci w.r.t. any desingularization} \\ [\textit{rational, elliptic, non-elliptic etc.}] \\ \\ \bullet\ \text{from the behaviour of ``discrepancies''} \\ \text{(for } \mathbb{Q}\text{-Gorenstein normal complex varieties)} \\ [\textit{adjunction-theoretic classification}]\end{array}\right.$$

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Algebraic Classification. — At first we recall some fundamental definitions from commutative algebra (cf. [**52**], [**54**]). Let R be a commutative ring with 1. The *height* $\mathrm{ht}(\mathfrak{p})$ of a prime ideal $\mathfrak{p}$ of R is the supremum of the lengths of all prime ideal chains which are contained in $\mathfrak{p}$, and the *dimension* of R is defined to be

$$\dim(R) := \sup\{\mathrm{ht}(\mathfrak{p}) \mid \mathfrak{p} \text{ prime ideal of } R\}.$$

R is *Noetherian* if any ideal of it has a finite system of generators. R is a *local ring* if it is endowed with a *unique* maximal ideal $\mathfrak{m}$. A local ring R is *regular* (resp. *normal*) if $\dim(R) = \dim(\mathfrak{m}/\mathfrak{m}^2)$ (resp. if it is an integral domain and is integrally closed in its field of fractions). A finite sequence $a_1, \ldots, a_\nu$ of elements of a ring R is defined to be a *regular sequence* if a_1 is not a zero-divisor in R and for all i, $i = 2, \ldots, \nu$, a_i is not a zero-divisor of $R/\langle a_1, \ldots, a_{i-1}\rangle$. A Noetherian local ring R (with maximal ideal $\mathfrak{m}$) is called *Cohen-Macaulay* if

$$\mathrm{depth}(R) = \dim(R),$$

where the *depth* of R is defined to be the maximum of the lengths of all regular sequences whose members belong to $\mathfrak{m}$. A Cohen-Macaulay local ring R is called *Gorenstein* if

$$\mathrm{Ext}_R^{\dim(R)}(R/\mathfrak{m}, R) \cong R/\mathfrak{m}.$$

A Noetherian local ring R is said to be a *complete intersection* if there exists a regular local ring R', such that $R \cong R'/(f_1, \ldots, f_q)$ for a finite set $\{f_1, \ldots, f_q\} \subset R'$ whose cardinality equals $q = \dim(R') - \dim(R)$. The hierarchy by inclusion of the above types of Noetherian local rings is known to be described by the following diagram:

$$\begin{array}{ccc} \{\text{Noetherian local rings}\} & \supset & \{\text{normal local rings}\} \\ \cup & & \cup \\ \{\text{Cohen-Macaulay local rings}\} & & \{\text{regular local rings}\} \\ \cup & & \cap \\ \{\text{Gorenstein local rings}\} & \supset & \{\text{complete intersections ("c.i.'s")}\} \end{array} \tag{1.1}$$

An arbitrary Noetherian ring R and its associated affine scheme $\mathrm{Spec}(R)$ are called Cohen-Macaulay, Gorenstein, normal or regular, respectively, iff all the localizations $R_{\mathfrak{m}}$ with respect to all the members $\mathfrak{m} \in \text{Max-Spec}(R)$ of the maximal spectrum of R are of this type. In particular, if the $R_{\mathfrak{m}}$'s for all maximal ideals $\mathfrak{m}$ of R are c.i.'s, then one often says that R is a *locally complete intersection* ("l.c.i.") to distinguish it from the "global" ones. (A *global complete intersection* ("g.c.i.") is defined to be a ring R of finite type over a field $\boldsymbol{k}$ (i.e., an affine $\boldsymbol{k}$-algebra), such that

$$R \cong \boldsymbol{k}[\mathsf{T}_1.., \mathsf{T}_d] \,/\, (\varphi_1(\mathsf{T}_1, .., \mathsf{T}_d), .., \varphi_q(\mathsf{T}_1, .., \mathsf{T}_d))$$

for q polynomials $\varphi_1, \ldots, \varphi_q$ from $\boldsymbol{k}[\mathsf{T}_1, .., \mathsf{T}_d]$ with $q = d - \dim(R)$). Hence, the above inclusion hierarchy can be generalized for all Noetherian rings, just by omitting in (1.1) the word "local" and by substituting l.c.i.'s for c.i.'s.

We shall henceforth consider only *complex varieties* $(X, \mathcal{O}_X)$, i.e., integral separated schemes of finite type over $\boldsymbol{k} = \mathbb{C}$; thus, the punctual algebraic behaviour of X is determined by the stalks $\mathcal{O}_{X,x}$ of its structure sheaf $\mathcal{O}_X$, and X itself is said to have a given *algebraic property* whenever all $\mathcal{O}_{X,x}$'s have the analogous property from (1.1) for all $x \in X$. Furthermore, via the GAGA-correspondence ([**71**], [**30**, § 2]) which preserves the above quoted algebraic properties, we may work within the *analytic category* by using the usual contravariant functor

$$(X, x) \rightsquigarrow \mathcal{O}_{X,x}^{\mathrm{hol}}$$

between the category of isomorphy classes of germs of X and the corresponding category of isomorphy classes of analytic local rings at the marked points x. For a complex variety X and $x \in X$, we denote by $\mathfrak{m}_{X,x}$ the maximal ideal of $\mathcal{O}_{X,x}^{\mathrm{hol}}$ and by

$$\begin{aligned} \operatorname{Sing}(X) &= \left\{x \in X \mid \mathcal{O}_{X,x}^{\mathrm{hol}} \text{ is a non-regular local ring}\right\} \\ &= \left\{x \in X \mid \dim\left(\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2\right) > \dim_x(X)\right\} \end{aligned} \tag{1.2}$$

its *singular locus*. By a *desingularization* (or *resolution of singularities*) $f : \widehat{X} \to X$ of a non-smooth X, we mean a "full" or "overall" desingularization (if not mentioned), i.e., $\operatorname{Sing}(\widehat{X}) = \varnothing$. When we deal with *partial* desingularizations, we mention it explicitly.

Rational and Elliptic Singularities. — We say that X has (at most) *rational singularities* if there exists a desingularization $f : Y \to X$ of X, such that

$$f_* \mathcal{O}_Y = \mathcal{O}_X$$

(equivalently, Y is normal), and

$$R^i f_* \mathcal{O}_Y = 0, \quad \forall i, \quad 1 \leqslant i \leqslant \dim_{\mathbb{C}} X - 1.$$

(The i-th direct image sheaf is defined via

$$U \longmapsto R^i f_* \mathcal{O}_Y(U) := H^i\left(f^{-1}(U), \mathcal{O}_Y|_{f^{-1}(U)}\right)\ ).$$

This definition is independent of the particular choice of the desingularization of X. (Standard example: quotient singularities[(1)] are rational singularities).

We say that a Gorenstein singularity x of X is an *elliptic singularity* if there exists a desingularization $f : Y \to X$ of $x \in X$, such that

$$R^i f_* \mathcal{O}_Y = 0, \quad \forall i, \quad 1 \leqslant i \leqslant \dim_{\mathbb{C}} X - 2,$$

[(1)] The *quotient singularities* are of the form $(\mathbb{C}^r/G, [\mathbf{0}])$, where G is a finite subgroup of $\mathrm{GL}(r, \mathbb{C})$ (without pseudoreflections) acting linearly on $\mathbb{C}^r$, $p : \mathbb{C}^r \to \mathbb{C}^r/G = \operatorname{Spec}(\mathbb{C}[\boldsymbol{z}]^G)$ the quotient map, and $[\mathbf{0}] = p(\mathbf{0})$. Note that

$$\operatorname{Sing}(\mathbb{C}^r/G) = p(\{\boldsymbol{z} \in \mathbb{C}^r \mid G_{\boldsymbol{z}} \neq \{\mathrm{Id}\}\})$$

(cf. (1.2)), where $G_{\boldsymbol{z}} := \{g \in G \mid g \cdot \boldsymbol{z} = \boldsymbol{z}\}$ is the isotropy group of $\boldsymbol{z} \in \mathbb{C}^r$.

and

$$R^{\dim_{\mathbb{C}} X-1} f_* \mathcal{O}_Y \cong \mathbb{C}.$$

(The definition is again independent of the particular choice of the desingularization).

Adjunction-Theoretic Classification. — If X is a *normal* complex variety, then its Weil divisors can be described by means of "divisorial" sheaves as follows:

Lemma 1.1 **([34,** 1.6]**).** — *For a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules the following conditions are equivalent*:
(i) $\mathcal{F}$ *is* reflexive (*i.e.*, $\mathcal{F} \cong \mathcal{F}^{\vee\vee}$, *with* $\mathcal{F}^{\vee} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ *denoting the dual of* $\mathcal{F}$) *and has rank one.*
(ii) *If X^0 is a non-singular open subvariety of X with* $\operatorname{codim}_X \left(X \smallsetminus X^0\right) \geqslant 2$, *then* $\mathcal{F}|_{X^0}$ *is invertible and*

$$\mathcal{F} \cong \iota_* \left(\mathcal{F}|_{X^0}\right) \cong \iota_* \iota^* \left(\mathcal{F}\right),$$

where $\iota : X^0 \hookrightarrow X$ denotes the inclusion map.

The *divisorial sheaves* are exactly those satisfying one of the above conditions. Since a divisorial sheaf is torsion free, there is a non-zero section $\gamma \in H^0\left(X, Rat_X \otimes_{\mathcal{O}_X} \mathcal{F}\right)$, with

$$H^0\left(X, Rat_X \otimes_{\mathcal{O}_X} \mathcal{F}\right) \cong \mathbb{C}(X) \cdot \gamma,$$

and $\mathcal{F}$ can be considered as a subsheaf of the constant sheaf Rat_X of rational functions of X, i.e., as a special *fractional ideal sheaf.*

Proposition 1.2 **([63,** App. of § 1]**).** — *The correspondence*

$$Cl(X) \ni \{D\} \xrightarrow{\ \delta\ } \{\mathcal{O}_X(D)\} \in \left\{ \begin{array}{c} \textit{divisorial coherent} \\ \textit{subsheaves of } Rat_X \end{array} \right\} \Big/ \ H^0\left(X, \mathcal{O}_X^*\right)$$

with $\mathcal{O}_X(D)$ defined by sending every non-empty open set U of X onto

$$U \longmapsto \mathcal{O}_X(D)(U) := \left\{ \varphi \in \mathbb{C}(X)^* \mid \left(\operatorname{div}(\varphi) + D\right)|_U \geqslant 0 \right\} \cup \{0\},$$

is a bijection, and induces a $\mathbb{Z}$-module isomorphism. In fact, to avoid torsion, one defines this $\mathbb{Z}$-module structure by setting

$$\delta\left(D_1 + D_2\right) := \left(\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)\right)^{\vee\vee} \text{and } \delta(\kappa D) := \mathcal{O}_X(D)^{[\kappa]} = \mathcal{O}_X(\kappa D)^{\vee\vee},$$

for any Weil divisors D, D_1, D_2 and $\kappa \in \mathbb{Z}$.

Let now $\Omega_{\operatorname{Reg}(X)/\mathbb{C}}$ be the sheaf of regular 1-forms, or Kähler differentials, on

$$\operatorname{Reg}(X) = X \smallsetminus \operatorname{Sing}(X) \overset{\iota}{\hookrightarrow} X,$$

(cf. [**36**, § 5.3]) and for $i \geqslant 1$, let us set

$$\Omega^i_{\operatorname{Reg}(X)/\mathbb{C}} := \bigwedge^i \Omega_{\operatorname{Reg}(X)/\mathbb{C}}\ .$$

The unique (up to rational equivalence) Weil divisor K_X, which maps under δ to the *canonical divisorial sheaf*

$$\omega_X := \iota_* \left(\Omega^{\dim_{\mathbb{C}}(X)}_{\mathrm{Reg}(X)/\mathbb{C}}\right),$$

is called the *canonical divisor* of X. Another equivalent interpretation of ω_X, when X is Cohen-Macaulay, can be given by means of the Duality Theory (see [**32**], [**29**]). If $\mathbb{D}^+_c(\mathcal{O}_X)$ denotes the derived category of below bounded complexes whose cohomology sheaves are coherent, then there exists a *dualizing complex*[(2)] $\omega^\bullet_X \in \mathbb{D}^+_c(\mathcal{O}_X)$ over X. If X is Cohen-Macaulay, then the i-th cohomology sheaf $\mathcal{H}^i(\omega^\bullet_X)$ vanishes for all $i \in \mathbb{Z}\smallsetminus\{-\dim_{\mathbb{C}}(X)\}$, and $\omega_X \cong \mathcal{H}^{-\dim_{\mathbb{C}}(X)}(\omega^\bullet_X)$. This leads to the following:

Proposition 1.3. — *A normal complex variety X is Gorenstein if and only if it is Cohen-Macaulay and ω_X is invertible.*

Proof. — If X is Gorenstein, then $\mathcal{O}_{X,x}$ satisfies the equivalent conditions of [**54**, Thm. 18.1], for all $x \in X$. This means that $\mathcal{O}_{X,x}$ (as Noetherian local ring) is a dualizing complex for itself (cf. [**32**, Ch. V, Thm. 9.1, p. 293]). Since dualizing complexes are unique up to tensoring with an invertible sheaf, say L, over X, shifted by an integer n (cf. [**32**, Ch. V, Cor. 2.3, p. 259]), we shall have $\omega^\bullet_X \cong \mathcal{O}^\bullet_{X,x} \otimes L[n]$. Hence, ω_X itself will be also invertible. The converse follows from the isomorphisms $\omega_X \cong \mathcal{H}^{-\dim_{\mathbb{C}}(X)}(\omega^\bullet_X)$ and $\omega_{X,x} \cong \mathcal{O}_{X,x}$, for all $x \in X$. (Alternatively, one may use the fact that $x \in X$ is Gorenstein iff $\mathcal{O}_{X,x}$ is Cohen-Macaulay and $H^{\dim_{\mathbb{C}}(X)}_{\mathfrak{m}_{X,x}}(\mathcal{O}_{X,x})$ is a dualizing module for it, cf. [**29**, Prop. 4.14, p. 65]. The classical duality [**29**, Thm. 6.3, p. 85], [**32**, Ch. V, Cor. 6.5, p. 280], combined with the above uniqueness argument, gives again the required equivalence). ☐

***Theorem 1.4* (Kempf [43, p. 50], Elkik [23], [24], Bingener-Storch [5])**
Let X a normal complex variety of dimension $\geqslant 2$. Then

$$\begin{pmatrix} X \text{ has at most} \\ \text{rational singularities}\end{pmatrix} \Longleftrightarrow \begin{pmatrix} X \text{ is Cohen-Macaulay} \\ \text{and } \omega_X \cong f_*\omega_Y\end{pmatrix},$$

where $f : Y \longrightarrow X$ is any desingularization of X.

(Note that, if $E = f^{-1}(\mathrm{Sing}(X))$ and $\iota : \mathrm{Reg}(X) \hookrightarrow X$, $j : Y\smallsetminus E \hookrightarrow Y$ are the natural inclusions, then by the commutative diagram

$$\begin{tikzcd} Y\smallsetminus E \arrow[r, hook, "j"] \arrow[d, "\wr"] & Y \arrow[d, "f"] \\ \mathrm{Reg}(X) \arrow[r, hook, "\iota"] & X \end{tikzcd}$$

[(2)] There is a canonical morphism $\omega_X[\dim_{\mathbb{C}}(X)] \to \omega^\bullet_X$ which is a quasi-isomorphism iff X is Cohen-Macaulay.

we have in general $f_*\omega_Y \hookrightarrow f_*j_*\left(\omega_Y\,|_{Y\smallsetminus E}\right) = \iota_* f_*\left(\omega_Y\,|_{Y\smallsetminus E}\right) = \iota_*\left(\omega_{\mathrm{Reg}(X)}\right) \cong \omega_X$. In fact, $f_*\omega_Y$ does not depend on the particular choice of the desingularization.)

Sketch of proof. — Let $G_X := \mathrm{Coker}\left(f_*\omega_Y \hookrightarrow \omega_X\right)$, and

$$\mathcal{L}_X^\bullet = \{\mathcal{L}_X^i\} \in \mathbb{D}_c^+\left(\mathcal{O}_X\right), \quad \mathcal{M}_X^\bullet = \{\mathcal{M}_X^i\} \in \mathbb{D}_c^+\left(\mathcal{O}_X\right),$$

the map cones of the canonical homomorphisms

$$\mathcal{O}_X \longrightarrow R^\bullet f_*\mathcal{O}_Y \quad \text{and} \quad f_*\left(\omega_Y[\dim_{\mathbb{C}}(X)]\right) \longrightarrow \omega_X^\bullet,$$

respectively. By Grauert-Riemenschneider vanishing theorem, $R^i f_*\omega_Y = 0$ for all $i \in \mathbb{Z}_{\geqslant 1}$, which means that the canonical morphism

$$f_*\left(\omega_Y[\dim_{\mathbb{C}}(X)]\right) \longrightarrow \mathbf{R}f_*\left(\omega_Y[\dim_{\mathbb{C}}(X)]\right)$$

is an isomorphism. Moreover, $\mathcal{H}^i\left(\mathcal{L}_X^\bullet\right) = \mathcal{L}_X^i$, for all $i \in \mathbb{Z}$,and

$$\mathcal{H}^i\left(\mathcal{M}_X^\bullet\right) = \begin{cases} \mathcal{H}^i\left(\omega_X^\bullet\right), & \text{for } i \in [-\dim_{\mathbb{C}}(X)+1, -t\left(X\right)], \\ G_X & \text{for } i = -\dim_{\mathbb{C}}(X), \\ 0 & \text{otherwise,} \end{cases}$$

where $t(X) = \inf\{\,\mathrm{depth}\left(\mathcal{O}_{X,x}\right) \mid x \in X\}$. By the Duality Theorem for proper maps (cf. [**32**, Ch. III, Thm. 11.1, p. 210]) we obtain a canonical isomorphism

$$f_*\left(\omega_Y[\dim_{\mathbb{C}}(X)]\right) \cong \mathbf{R}f_*\left(\omega_Y[\dim_{\mathbb{C}}(X)]\right) \;\cong \mathbf{R}\mathit{Hom}_{\mathcal{O}_X}\left(R^\bullet f_*\mathcal{O}_Y, \omega_X^\bullet\right)$$

By dualization this reads as

$$\mathbf{R}\mathit{Hom}_{\mathcal{O}_X}\left(\, f_*\left(\omega_Y[\dim_{\mathbb{C}}(X)]\right), \omega_X^\bullet\right) \cong R^\bullet f_*\mathcal{O}_Y.$$

From this isomorphism we deduce that

$$\mathcal{H}^i\left(\mathcal{M}_X^\bullet\right) = \mathit{Ext}_{\mathcal{O}_X}^{i+1}\left(\mathcal{M}_X^\bullet, \omega_X^\bullet\right) \quad \text{and} \quad \mathcal{H}^i\left(\mathcal{L}_X^\bullet\right) = \mathit{Ext}_{\mathcal{O}_X}^{i+1}\left(\mathcal{L}_X^\bullet, \omega_X^\bullet\right)$$

Hence, the assertion follows from the equivalence: $\mathcal{L}_X^\bullet = 0 \Longleftrightarrow \mathcal{M}_X^\bullet = 0$. □

Remark 1.5. — For another approach, see Flenner [**27**, Satz 1.3]. For a proof which avoids Duality Theory, cf. [**45**, Cor. 11.9, p. 281] or [**47**, Lemma 5.12, p. 156].

Definition 1.6. — A normal complex variety X is called $\mathbb{Q}$-*Gorenstein* if

$$\omega_X = \mathcal{O}_X\left(K_X\right)$$

with K_X a $\mathbb{Q}$-Cartier divisor. (The smallest positive integer ℓ, for which ℓK_X is Cartier, is called *the index* of X.) Let X be a singular $\mathbb{Q}$-Gorenstein complex variety of dimension $\geqslant 2$. Take a desingularization $f : Y \to X$ of X, such that the exceptional locus of f is a divisor $\bigcup_i D_i$ with only simple normal crossings, and define *the discrepancy*[3]

$$K_Y - f^*\left(K_X\right) = \sum_i a_i D_i\ .$$

[3] We may formally define the *pull-back* $f^*\left(K_X\right)$ as the $\mathbb{Q}$-Cartier divisor $\frac{1}{\ell}f^*\left(\ell K_X\right)$, where ℓ is the index of X.

We say that X has *terminal* (resp. *canonical*, resp. *log-terminal*, resp. *log-canonical*) *singularities* if all a_i's are > 0 (resp. $\geqslant 0$ / > -1 / $\geqslant -1$). This definition is independent of the particular choice of the desingularization.

Remark 1.7
(i) If all a_i's are $= 0$, then $f : Y \to X$ is called a *crepant* desingularization of X. In fact, the number of crepant divisors $\#\{i \mid a_i = 0\}$ remains invariant w.r.t. all f's as long as X has at most canonical singularities.
(ii) Terminal singularities constitute the smallest class of singularities to run the MMP (= minimal model program) for smooth varieties. The canonical singularities are precisely the singularities which appear on the canonical models of varieties of general type. Finally, the log-singularities are those singularities for which the *discrepancy function* (assigned to $\mathbb{Q}$-Gorenstein complex varieties X) still makes sense[(4)]. For details about the general MMP, see [**42**], [**13**, 6.3, p. 39], [**46**] and [**47**].

Theorem 1.8. — *Log-terminal singularities are rational.*

Proof. — This follows from [**42**, Thm. 1-3-6, p. 311], [**45**, Cor. 11.14, p. 283] or [**47**, Thm. 5.22, p. 161]. □

Corollary 1.9. — *A singularity $x \in X$ is canonical of index* 1 *if and only if it is rational and Gorenstein.*

Proof. — "$\Rightarrow$" is obvious by Thm. 1.8 and Proposition 1.3. (The rationality of canonical singularities was first shown by Elkik [**24**]). "$\Leftarrow$" follows from the fact that ω_X is locally free and from $\omega_X \cong f_*\omega_Y$ (via the other direction of Thm. 1.8), as this is equivalent for $x \in X$ to be canonical of index 1 (cf. [**63**, (1.1), p. 276]). □

Definition 1.10. — Let $(\mathcal{O}_{X,x}, \mathfrak{m}_{X,x})$ be the local ring of a point x of a normal quasiprojective complex variety X and $V_x \subset \mathcal{O}_{X,x}$ a finite-dimensional $\mathbb{C}$-vector space generating $\mathfrak{m}_{X,x}$. A *general hyperplane section through* x is a $\mathbb{C}$-algebraic subscheme $\mathbb{H} \subset U_x$ determined in a suitable Zariski-open neighbourhood U_x of x by the ideal sheaf $\mathcal{O}_X \cdot v$, where $v \in V_x$ is sufficiently general. (*Sufficiently general* means that v can be regarded as a $\mathbb{C}$-point of a whole Zariski-open dense subset of V_x.)

***Theorem 1.11* (M. Reid, [63, 2.6], [66, 3.10], [47, 5.30-5.31, p. 164])**
Let X be a normal quasiprojective complex variety of dimension $r \geqslant 3$ and $x \in \mathrm{Sing}(X)$. If (X, x) is a rational Gorenstein singularity, then, for a general hyperplane section $\mathbb{H}$ through x, $(\mathbb{H}, x)$ is either a rational or an elliptic $(r-1)$-dimensional singularity.

(4) Cf. [**13**, 6.3, p. 39], [**46**, Prop. 1.9, p. 14] and [**47**, p. 57].

2. Basic facts about two- and three-dimensional normal singularities

In dimension 2, the definition of rational and elliptic singularities fits quite well our intuition of what "rational" and "elliptic" ought to be. "Terminal" points are the smooth ones and the canonical singularities turn out to be the traditional RDP's (see below Theorem 2.5). Moreover, terminal and canonical points have always index 1. On the other hand, the existence of a unique *minimal*[(5)] (and *good minimal*[(6)]) desingularization makes the study of normal surface singularities easier that in higher dimensions.

Definition 2.1. — Let X be a normal singular surface, $x \in \mathrm{Sing}(X)$, and $f : X' \to X$ a good resolution of X. To the support $f^{-1}(x) = \cup_{i=1}^{k} C_i$ of the exceptional divisor w.r.t. f (resolving the singularity at x) we can associate a *weighted dual graph* by assigning a weighted vertex to each C_i, with the weight being the self intersection number C_i^2, and linking two vertices corresponding to C_i and C_j by an edge of weight $(C_i \cdot C_j)$. The *fundamental cycle*

$$Z_{\text{fund}} = \sum_{i=1}^{k} n_i C_i, \ n_i > 0, \quad \forall i, \quad 1 \leqslant i \leqslant k,$$

of f w.r.t. (X, x) is the unique, smallest positive cycle for which $(Z_{\text{fund}} \cdot C_i) \leqslant 0$, for all i, $1 \leqslant i \leqslant k$.

***Theorem 2.2* (Artin [3, Thm. 3])**. — *The following statements are equivalent*:
(i) (X, x) *is a rational surface singularity.*
(ii) $p_a(Z_{\text{fund}}) = 0$. (p_a *denotes here the arithmetic genus*).

***Corollary 2.3* (Brieskorn [12, Lemma 1.3])**. — *For (X, x) a rational surface singularity, $\cup_{i=1}^{k} C_i$ has the following properties*:

(i) *all C_i's are smooth rational curves.*

(ii) $C_i \cap C_j \cap C_l = \varnothing$ *for pairwise distinct* i, j, l.

(iii) $(C_i \cdot C_j) \in \{0, 1\}$, *for* $i \neq j$.

(iv) *The weighted dual graph contains no cycles.*

***Corollary 2.4* (Artin [3, Cor. 6])**. — *If (X, x) is a rational surface singularity,*

$$\mathrm{mult}_x(X) = \mathrm{mult}(\mathcal{O}_{X,x})$$

[(5)] A desingularization $f : X' \to X$ of a normal surface X is minimal if $\mathrm{Exc}(f)$ does not contain any curve with self-intersection number -1 or, equivalently, if for an arbitrary desingularization $g : X'' \to X$ of X, there exists a unique morphism $h : X'' \to X'$ with $g = f \circ h$.

[(6)] A desingularization of a normal surface is good if (i) the irreducible components of the exceptional locus are smooth curves, and (ii) the support of the inverse image of each singular point is a divisor with simple normal crossings. For the proof of the uniqueness (up to a biregular isomorphism) of both minimal and good minimal desingularizations, see Brieskorn [**11**, Lemma 1.6] and Laufer [**50**, Thm. 5.12].

the multiplicity *of* X *at* x *and*

$$\operatorname{edim}(X,x) = \dim_{\mathbb{C}}\left(\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}\right)$$

its minimal embedding dimension*, then we have*:

$$-Z^2_{\text{fund}} = \operatorname{mult}_x(X) = \operatorname{edim}(X,x) - 1.$$

Theorem 2.5. — *The following conditions for a normal surface singularity* (X,x) *are equivalent*:

(i) (X,x) *is a canonical singularity.*

(ii) (X,x) *is a rational Gorenstein singularity.*

(iii) (X,x) *is a* rational double point (RDP) *or a* Kleinian *or* Du Val singularity, *i.e., it is analytically equivalent to the hypersurface singularity*

$$\left(\left\{(z_1,z_2,z_3)\in\mathbb{C}^3 \;\middle|\; \varphi(z_1,z_2,z_3)=0\right\},(0,0,0)\right)$$

which is determined by one of the quasihomogeneous polynomials of type A-D-E of the table:

Type	$\varphi(z_1,z_2,z_3)$
$\mathbf{A}_n \quad (n\geqslant 1)$	$z_1^{n+1}+z_2^2+z_3^2$
$\mathbf{D}_n \quad (n\geqslant 4)$	$z_1^{n-1}+z_1z_2^2+z_3^2$
$\mathbf{E}_6$	$z_1^4+z_2^3+z_3^2$
$\mathbf{E}_7$	$z_1^3z_2+z_2^3+z_3^2$
$\mathbf{E}_8$	$z_1^5+z_2^3+z_3^2$

(iv) (X,x) *is analytically equivalent to a quotient singularity* $\left(\mathbb{C}^2/G,[\mathbf{0}]\right)$, *where* G *denotes a finite subgroup of* $\mathrm{SL}(2,\mathbb{C})$. *More precisely, taking into account the classification* (*up to conjugacy*) *of these groups* (see [**21**], [**22**], [**49**, p. 35], [**72**, § 4.4]), *we get the correspondence*:

Type (as above)	$\mathbf{A}_n$	$\mathbf{D}_n$	$\mathbf{E}_6$	$\mathbf{E}_7$	$\mathbf{E}_8$
Type of the acting group G	$\mathbf{C}_n$	$\mathbf{D}_n$	$\mathbf{T}$	$\mathbf{O}$	$\mathbf{I}$

(*By* $\mathbf{C}_n$ *we denote a cyclic group of order* n*, and by* $\mathbf{D}_n, \mathbf{T}, \mathbf{O}$ *and* $\mathbf{I}$ *the binary dihedral, tetrahedral, octahedral and icosahedral subgroups of* $\mathrm{SL}(2,\mathbb{C})$*, having orders* $4(n-2)$, $24, 48$ *and* 120*, respectively*).

(v) [Inductive criterion] (X,x) *is an* absolutely isolated double point*, in the sense, that for any finite sequence*

$$\{\pi_{j-1} : X_j = \mathbf{Bl}^{\mathrm{red}}_{\{x_{j-1}\}} \longrightarrow X_{j-1} \ \mid 1 \leqslant j \leqslant l\}$$

of blow-ups with closed (*reduced*) *points as centers and* $X_0 = X$*, the only singular points of* X_l *are isolated double points.* (*In particular,* (X,x) *is a hypersurface double point whose normal cone is either a* (*not necessarily irreducible*) *plane conic or a double line*).

Definition 2.6. — Let (X,x) be a normal surface singularity. Assume that (X,x) is elliptic[(7)] and Gorenstein. We define the Laufer-Reid invariant $\mathrm{LRI}(X,x)$ of x in X to be the self-intersection number of Artin's fundamental cycle with opposite sign:

$$\mathrm{LRI}\,(X,x) = -Z^2_{\mathrm{fund}}$$

***Theorem 2.7* (Laufer [51], Reid [62])**. — *Let* (X,x) *be a normal surface singularity. Assume that* (X,x) *is elliptic and Gorenstein. Then* $\mathrm{LRI}(X,x) \geqslant 1$ *and has the following properties*:

(i) *If* $\mathrm{LRI}(X,x) = 1$*, then*

$$(X,x) \cong (\{(z_1,z_2,z_3) \in \mathbb{C}^3 \ \mid z_1^2 + z_2^3 + \varphi\,(z_2,z_3) = 0\}, (0,0,0)),$$

with $\varphi\,(z_2,z_3)$ *a sum of monomials of the form* $z_2 z_3^{\kappa}$, $\kappa \in \mathbb{Z}_{\geqslant 4}$*, and* z_3^{κ}, $\kappa \in \mathbb{Z}_{\geqslant 6}$*. Performing the monoidal transformation*

$$\mathbf{Bl}^{(z_1^3, z_2^2, z_3)}_{\{x\}} \longrightarrow X$$

we get a normal surface having at most one Du Val point.

(ii) *If* $\mathrm{LRI}(X,x) = 2$*, then*

$$(X,x) \cong (\{(z_1,z_2,z_3) \in \mathbb{C}^3 \ \mid z_1^2 + \varphi\,(z_2,z_3) = 0\}, (0,0,0)),$$

(7) Laufer [**51**] calls an elliptic singularity (in the above sense) "minimally elliptic".

with $\varphi(z_2, z_3)$ *a sum of monomials of the form* $z_2^{\kappa} z_3^{\lambda}, \kappa + \lambda \in \mathbb{Z}_{\geqslant 4}$. *In this case, the normalized blow-up* X *at* x

$$\mathrm{Norm}\left[\mathbf{Bl}_{\{x\}}^{\mathrm{red}}\right] = \mathbf{Bl}_{\{x\}}^{\left(z_1^2, z_2, z_3\right)} \longrightarrow X$$

has only Du Val singular points.

(iii) *If* $\mathrm{LRI}(X, x) \geqslant 2$, *then*

$$\mathrm{LRI}\,(X, x) = \mathrm{mult}_x\,(X)\,.$$

(iv) *If* $\mathrm{LRI}(X, x) \geqslant 3$, *then*

$$\mathrm{LRI}\,(X, x) = \mathrm{edim}(X, x)$$

and the overlying space of the ordinary blow-up

$$\mathbf{Bl}_{\{x\}}^{\mathrm{red}} = \mathbf{Bl}_{\{x\}}^{\mathfrak{m}_{X,x}} \longrightarrow X$$

of x *is a normal surface with at most Du Val singularities.*

Theorem 2.8. — *Let* (X, x) *be a normal surface singularity. Then*(8)

$$\left\{\begin{array}{lcc} \text{(i)} \quad x \textit{ is } \text{terminal} & \Longleftrightarrow & x \in \mathrm{Reg}\,(X) \\[2ex] \text{(ii)} \quad x \textit{ is } \text{canonical} & \Longleftrightarrow & \left(\begin{array}{c} (X, x) \cong \left(\mathbb{C}^2/G, [\mathbf{0}]\right) \text{ with} \\ G \text{ a finite subgroup of } \mathrm{SL}(2, \mathbb{C}) \end{array}\right) \\[2ex] \text{(iii)} \ x \textit{ is } \text{log-terminal} & \Longleftrightarrow & \left(\begin{array}{c} (X, x) \cong \left(\mathbb{C}^2/G, [\mathbf{0}]\right) \text{ with} \\ G \text{ a finite subgroup of } \mathrm{GL}(2, \mathbb{C}) \end{array}\right) \\[2ex] \text{(iv)} \ x \textit{ is } \text{log-canonical} & \Longleftrightarrow & \left(\begin{array}{c} x \text{ is simple-elliptic, a cusp} \\ \text{or a regular point} \\ \text{or a quotient thereof} \end{array}\right) \end{array}\right.$$

Log-terminal surface singularities are rational (by Theorem 1.8). This is, of course, not the case for log-canonical surface singularities which are not log-terminal. For the fine classification of log-terminal and log-canonical surface singularities, the reader is referred to the papers of Brieskorn(9) [**12**], Iliev [**37**], Kawamata [**41**], Alexeev [**2**] and Blache [**6**].

• Next, let us recall some basic facts from the theory of 3-dimensional terminal and canonical singularities.

(8) Explanation of terminology: Let $f : X' \to X$ be the good minimal resolution of X. Then x is called *simple-elliptic* (resp. *a cusp*) if the support of the exceptional divisor w.r.t. f lying over x consists of a smooth elliptic curve (resp. a cycle of $\mathbb{P}^1_{\mathbb{C}}$'s).

(9) Brieskorn classified (up to conjugation) *all* finite subgroups of $\mathrm{GL}(2, \mathbb{C})$ in [**12**, 2.10 and 2.11].

Definition 2.9. — A normal threefold singularity (X,x) is called *compound Du Val singularity* (abbreviated: *cDV singularity*) if for some general hyperplane section $\mathbb{H}$ through x, $(\mathbb{H},x)$ is a Du Val singularity, or equivalently, if

$$(X,x) \cong (\{(z_1,z_2,z_3,z_4) \in \mathbb{C}^4 \mid \varphi(z_1,z_2,z_3) + z_4 \cdot g(z_1,z_2,z_3,z_4) = 0\}, (0,0,0,0)),$$

where $\varphi(z_1,z_2,z_3)$ is one of the quasihomogeneous polynomials listed in the Thm. 2.5 (iii) and $g(z_1,z_2,z_3,z_4)$ an arbitrary polynomial in $\mathbb{C}[z_1,z_2,z_3,z_4]$. According to the type of $\varphi(z_1,z_2,z_3)$, (X,x) is called $c\mathbf{A}_n, c\mathbf{D}_n, c\mathbf{E}_6, c\mathbf{E}_7$ and $c\mathbf{E}_8$-point, respectively. Compound Du Val singularities are not necessarily isolated.

***Theorem 2.10* (Reid [64, 0.6 (I), 1.1, 1.11])**. — *Let (X,x) be a normal threefold singularity. Then*

$$\begin{cases} \text{(i)} \quad x \text{ is terminal} & \Longrightarrow \quad x \text{ is isolated} \\ \text{(ii)} \quad x \text{ is terminal of index } 1 & \Longleftrightarrow \quad x \text{ is an isolated cDV point} \\ \text{(iii)} \quad x \text{ is terminal of index } \geqslant 1 & \Longleftrightarrow \left(\begin{array}{c} x \text{ is a quotient of an isolated} \\ \text{cDV point by a finite cyclic group} \end{array} \right) \\ \text{(iv)} \quad x \text{ is a cDV point} & \Longrightarrow \quad x \text{ is canonical} \end{cases}$$

For the extended lists of the fine classification of 3-dimensional terminal singularities of arbitrary index, see Mori [**55**], Reid [**66**] and Kollár & Shepherd-Barron [**48**]. The normal forms of the defining equations of cDV points have been studied by Markushevich in [**53**]. On the other hand, 3-dimensional terminal cyclic quotient singularities, which play a crucial role in the above cited investigations, are quite simple.

***Theorem 2.11* (Danilov [20], Morrison-Stevens [58])**. — *Let (X,x) be a terminal threefold singularity. Then*

$$\left(\begin{array}{c} (X,x) \cong (\mathbb{C}^3/G, [\mathbf{0}]) \text{ with } G \text{ a linearly acting finite} \\ \text{cyclic subgroup of } \mathrm{GL}(3,\mathbb{C}) \text{ without pseudoreflections} \end{array} \right)$$

$$\Updownarrow$$

$$\left(\begin{array}{c} \text{the action of } G \text{ is given (up to permutations} \\ \text{of } (z_1,z_2,z_3) \text{ and group symmetries) by} \\ (z_1,z_2,z_3) \longmapsto (\zeta_\mu^\lambda\ z_1, \zeta_\mu^{-\lambda}\ z_2, \zeta_\mu\ z_3), \text{ where } \mu := |G|, \\ \gcd(\lambda,\mu) = 1, \text{ and } \zeta_\mu \text{ denotes a } \mu\text{-th root of unity} \end{array} \right)$$

Reduction of 3-dimensional canonical singularities. — The singularities of a quasiprojective threefold X can be reduced by a "canonical modification" $X^{\mathrm{can}} \xrightarrow{g} X$, so that $K_{X^{\mathrm{can}}}$ is g-ample. X^{can} can be also modified by a "terminal modification"

$X^{\mathrm{ter}} \xrightarrow{f} X^{\mathrm{can}}$, so that X^{ter} has at most terminal singularities, where f is projective and crepant. Finally, X^{ter} can be modified by another modification $X^{\mathbb{Q}\text{-f-ter}} \xrightarrow{h} X^{\mathrm{ter}}$, so that $X^{\mathbb{Q}\text{-f-ter}}$ has at most $\mathbb{Q}$-factorial terminal singularities(10), and h is projective and an isomorphism in codimension 1. (See [**63**], [**64**], [**66**], [**41**], [**56**], and [**47**, section 6.3]). The main steps of the intrinsic construction of f, due to Miles Reid, will be explained in broad outline and will be applied in the framework of toric geometry in section 4.

Step 1. Reduction to index 1 *canonical singularities by index cover*

If $x \in X := X^{\mathrm{can}}$ is a canonical singularity of index $\ell > 1$, then one considers the finite Galois cover

$$\phi : Y = \mathrm{Spec}\left(\bigoplus_{i=0}^{\ell-1} \mathcal{O}_X\left(\ell K_X\right)\right) \longrightarrow X.$$

The preimage $\phi^{-1}(x)$ constists of just one point, say y, and if $y \in Y$ is terminal, then the same is also valid for $x \in X$. Moreover, if $\psi : Y' \to Y$ is a crepant resolution of Y (as those ones which will be constructed in the next steps), then we get a commutative diagram

$$\begin{tikzcd} Y' \arrow[r, "\psi"] \arrow[d, "\phi'"'] & Y \arrow[d, "\phi"] \\ Y'/\mathbb{Z}_\ell \arrow[r, "\psi'"] & X \end{tikzcd}$$

extending the action of $\mathbb{Z}_\ell$ on (X, x) to an action on Y' where ψ' is crepant with at least one exceptional prime divisor and ϕ' is etale in codimension 1.

Step 2. Weighted blow-ups of non-cDv singularities. — From now on we may assume that X contains at most canonical singularities of index 1 (i.e., rational Gorenstein singularities). If X contains non-cDV points $x \in X$, then for a general hyperplane section $\mathbb{H}$ through x, $(\mathbb{H}, x)$ is an elliptic surface singularity. Using Theorem 2.7 one obtains the following:

Proposition 2.12
(i) *If* $\mathrm{LRI}(\mathbb{H}, x) = 1$, *then*

$$(X, x) \cong (\left\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \;\middle|\; z_1^2 + z_2^3 + \varphi(z_2, z_3, z_4) = 0\right\}, (0,0,0,0)),$$

with $\varphi(z_2, z_3, z_4) = z_2 F_1(z_3, z_4) + F_2(z_3, z_4)$, *where* F_1 (*resp.* F_2) *is a sum of monomials* $z_3^{\kappa} z_4^{\lambda}$ *of degree* $\kappa + \lambda \geqslant 4$ (*resp.* $\geqslant 6$).

(10) The morphism h can be constructed by taking successively birational morphisms of the form

$$\mathbf{Proj}(\bigoplus_{\nu \geqslant 0} \mathcal{O}_{X^{\mathrm{ter}}}(\nu D)) \longrightarrow X^{\mathrm{ter}},$$

where D's are Weil divisors which are not $\mathbb{Q}$-Cartier divisors (cf. [**47**, p. 201]).

(ii) *If* $\mathrm{LRI}(\mathbb{H}, x) = 2$, *then*

$$(X, x) \cong (\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid z_1^2 + \varphi(z_2, z_3, z_4) = 0\}, (0,0,0,0)),$$

with $\varphi(z_2, z_3, z_4)$ *a sum of monomials of degree* $\geqslant 4$.

(iii) *If* $\mathrm{LRI}(\mathbb{H}, x) \geqslant 3$, *then* $\mathrm{LRI}(\mathbb{H}, x) = \mathrm{edim}(\mathbb{H}, x) = \mathrm{edim}(X, x) - 1$.

Blowing up $x \in X$ with respect to the weights $(2,1,1,1), (3,2,1,1)$ and $(1,1,1,1)$ for $\mathrm{LRI}(\mathbb{H}, x) = 1, 2$ and $\geqslant 3$, respectively, we get a projective crepant partial desingularization of X. Repeating this procedure for all the non-cDV points of X, we reduce our singularities to cDV singularities.

Step 3. Simultaneous blow-up of one-dimensional singular loci. — From now on we may assume that X contains at most cDV singularities. If $\mathrm{Sing}(X)$ contains one-dimensional components, then we blow their union up (by endowing it with the reduced subscheme structure). This blow-up is realized by a projective, crepant birational morphism. Repeating this procedure finitely many times we reduce our singularities to isolated cDV singularities, i.e., to terminal singularities of index 1.

Remark 2.13. — After step 3, one may use the above projective birational morphism h to get only $\mathbb{Q}$-factorial terminal singularities. Sometimes, it is also useful to desingularize overall our threefold by resolving the remaining non-$\mathbb{Q}$-factorial terminal singularities.

3. Toric singularities

Toric singularities occupy a distinguished position within the class of rational singularities, as they can be described by binomial-type equations. In this section we shall introduce the brief toric glossary (a)-(k) and the notation which will be used in the sequel, and we shall summarize their main properties. For further details on toric geometry the reader is referred to the textbooks of Oda [**61**], Fulton [**28**] and Ewald [**25**], and to the lecture notes [**43**].

(a) The *linear hull,* the *affine hull*, the *positive hull* and *the convex hull* of a set B of vectors of $\mathbb{R}^r$, $r \geqslant 1$, will be denoted by $\mathrm{lin}(B)$, $\mathrm{aff}(B)$, $\mathrm{pos}(B)$ (or $\mathbb{R}_{\geqslant 0} B$) and $\mathrm{conv}(B)$, respectively. The *dimension* $\dim(B)$ of a $B \subset \mathbb{R}^r$ is defined to be the dimension of $\mathrm{aff}(B)$.

(b) Let N be a free $\mathbb{Z}$-module of rank $r \geqslant 1$. N can be regarded as a *lattice* in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r$. The *lattice determinant* $\det(N)$ of N is the r-volume of the parallelepiped spanned by any $\mathbb{Z}$-basis of it. An $n \in N$ is called *primitive* if $\mathrm{conv}(\{\mathbf{0}, n\}) \cap N$ contains no other points except $\mathbf{0}$ and n.

Let N be as above, $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual lattice, $N_{\mathbb{R}}, M_{\mathbb{R}}$ their real scalar extensions, and $\langle .,. \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ the natural $\mathbb{R}$-bilinear pairing. A subset σ of

$N_{\mathbb{R}}$ is called *convex polyhedral cone* (*c.p.c.*, for short) if there exist $n_1, \ldots, n_k \in N_{\mathbb{R}}$, such that

$$\sigma = \operatorname{pos}\left(\{n_1, \ldots, n_k\}\right) .$$

Its *relative interior* $\operatorname{int}(\sigma)$ is the usual topological interior of it, considered as subset of $\operatorname{lin}(\sigma) = \sigma + (-\sigma)$. The *dual cone* $\sigma^\vee$ of a c.p.c. σ is a c.p. cone defined by

$$\sigma^\vee := \{\boldsymbol{y} \in M_{\mathbb{R}} \mid \langle \boldsymbol{y}, \boldsymbol{x} \rangle \geqslant 0,\ \forall \boldsymbol{x},\ \boldsymbol{x} \in \sigma\} .$$

Note that $(\sigma^\vee)^\vee = \sigma$ and

$$\dim\left(\sigma \cap (-\sigma)\right) + \dim\left(\sigma^\vee\right) = \dim\left(\sigma^\vee \cap (-\sigma^\vee)\right) + \dim\left(\sigma\right) = r.$$

A subset τ of a c.p.c. σ is called a *face* of σ (notation: $\tau \prec \sigma$), if

$$\tau = \{\boldsymbol{x} \in \sigma \mid \langle m_0, \boldsymbol{x} \rangle = 0\},$$

for some $m_0 \in \sigma^\vee$. A c.p.c. $\sigma = \operatorname{pos}(\{n_1, \ldots, n_k\})$ is called *simplicial* (resp. *rational*) if $n_1, \ldots, n_k$ are $\mathbb{R}$-linearly independent (resp. if $n_1, \ldots, n_k \in N_{\mathbb{Q}}$, where $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$). A *strongly convex polyhedral cone* (*s.c.p.c.*, for short) is a c.p.c. σ for which $\sigma \cap (-\sigma) = \{\mathbf{0}\}$, i.e., for which $\dim(\sigma^\vee) = r$. The s.c.p. cones are alternatively called *pointed cones* (having $\mathbf{0}$ as their apex).

(c) If $\sigma \subset N_{\mathbb{R}}$ is a c.p. cone, then the subsemigroup $\sigma \cap N$ of N is a monoid. The following proposition is due to Gordan, Hilbert and van der Corput and describes its fundamental properties.

***Proposition 3.1* (Minimal generating system).** — *If $\sigma \subset N_{\mathbb{R}}$ is a c.p. rational cone, then $\sigma \cap N$ is finitely generated as additive semigroup. Moreover, if σ is strongly convex, then among all the systems of generators of $\sigma \cap N$, there is a system $\mathbf{Hilb}_N(\sigma)$ of* minimal cardinality, *which is uniquely determined* (*up to the ordering of its elements*) *by the following characterization*:

$$\mathbf{Hilb}_N(\sigma) = \left\{ n \in \sigma \cap (N \smallsetminus \{\mathbf{0}\}) \;\middle|\; \begin{array}{l} n \text{ cannot be expressed} \\ \text{as the sum of two other vectors} \\ \text{belonging to } \ \sigma \cap (N \smallsetminus \{\mathbf{0}\}) \end{array} \right\} \tag{3.1}$$

$\mathbf{Hilb}_N(\sigma)$ is called *the Hilbert basis of σ w.r.t. N.*

(d) For a lattice N of rank r having M as its dual, we define an r-dimensional *algebraic torus* $T_N \cong (\mathbb{C}^*)^r$ by setting $T_N := \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^*$. Every $m \in M$ assigns a character $\boldsymbol{e}(m) : T_N \to \mathbb{C}^*$. Moreover, each $n \in N$ determines a 1-parameter subgroup

$$\vartheta_n : \mathbb{C}^* \to T_N \quad \text{with} \quad \vartheta_n(\lambda)(m) := \lambda^{\langle m, n\rangle}, \quad \text{for} \quad \lambda \in \mathbb{C}^*,\ m \in M .$$

We can therefore identify M with the character group of T_N and N with the group of 1-parameter subgroups of T_N. On the other hand, for a rational s.c.p.c. σ with

$M \cap \sigma^{\vee} = \mathbb{Z}_{\geqslant 0}\ m_1 + \cdots + \mathbb{Z}_{\geqslant 0}\ m_\nu$, we associate to the finitely generated monoidal subalgebra

$$\mathbb{C}\,[M \cap \sigma^{\vee}] = \bigoplus_{m \in M \cap \sigma^{\vee}} \boldsymbol{e}\,(m)$$

of the $\mathbb{C}$-algebra $\mathbb{C}\,[M] = \oplus_{m \in M} \boldsymbol{e}\,(m)$ an affine complex variety[11]

$$U_\sigma := \operatorname{Spec}\left(\mathbb{C}\,[M \cap \sigma^{\vee}]\right),$$

which can be identified with the set of semigroup homomorphisms :

$$U_\sigma = \left\{ u : M \cap \sigma^{\vee} \ \to \mathbb{C} \ \middle| \ \begin{array}{c} u\,(\mathbf{0}) = 1,\ u\,(m+m') = u\,(m) \cdot u\,(m')\,, \\ \text{for all} \quad m, m' \in M \cap \sigma^{\vee} \end{array} \right\},$$

where $\boldsymbol{e}\,(m)\,(u) := u\,(m)\,,\ \forall m,\ m \in M \cap \sigma^{\vee}$ and $\forall u,\ u \in U_\sigma$.

***Proposition 3.2* (Embedding by binomials)**. — *In the analytic category, U_σ, identified with its image under the injective map*

$$(\boldsymbol{e}\,(m_1)\,, \ldots, \boldsymbol{e}\,(m_\nu)) : U_\sigma \hookrightarrow \mathbb{C}^\nu,$$

can be regarded as an analytic set determined by a system of equations of the form: (monomial) = (monomial). *This analytic structure induced on U_σ is independent of the semigroup generators $\{m_1, \ldots, m_\nu\}$ and each map $\boldsymbol{e}\,(m)$ on U_σ is holomorphic w.r.t. it. In particular, for $\tau \prec \sigma$, U_τ is an open subset of U_σ. Moreover, if $\#\,(\mathbf{Hilb}_M\,(\sigma^{\vee})) = k$ $(\leqslant \nu)$, then, by* (3.1), *k is nothing but the* embedding dimension *of U_σ, i.e., the* minimal *number of generators of the maximal ideal of the local $\mathbb{C}$-algebra $\mathcal{O}^{\mathrm{hol}}_{U_\sigma,\ \mathbf{0}}$.*

Proof. See Oda [**61**, Prop. 1.2 and 1.3., pp. 4-7]. □

(e) A *fan* w.r.t. a free $\mathbb{Z}$-module N is a finite collection Δ of rational s.c.p. cones in $N_{\mathbb{R}}$, such that :

(i) any face τ of $\sigma \in \Delta$ belongs to Δ, and

(ii) for $\sigma_1, \sigma_2 \in \Delta$, the intersection $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

By $|\Delta| := \cup \{\sigma \mid \sigma \in \Delta\}$ one denotes the *support* and by $\Delta\,(i)$ the set of all i-dimensional cones of a fan Δ for $0 \leqslant i \leqslant r$. If $\varrho \in \Delta\,(1)$ is a ray, then there exists a unique primitive vector $n\,(\varrho) \in N \cap \varrho$ with $\varrho = \mathbb{R}_{\geqslant 0}\ n\,(\varrho)$ and each cone $\sigma \in \Delta$ can be therefore written as

$$\sigma = \sum_{\varrho \in \Delta(1),\ \varrho \prec \sigma} \mathbb{R}_{\geqslant 0}\ n\,(\varrho) \quad .$$

The set

$$\operatorname{Gen}\,(\sigma) := \{n\,(\varrho) \mid \varrho \in \Delta\,(1)\,, \varrho \prec \sigma\}$$

is called the *set of minimal generators* (within the pure first skeleton) of σ. For Δ itself one defines analogously $\operatorname{Gen}(\Delta) := \bigcup_{\sigma \in \Delta} \operatorname{Gen}(\sigma)$.

[11] As point-set U_σ is actually the "maximal spectrum" Max-Spec($\mathbb{C}\,[M \cap \sigma^{\vee}]$).

(f) The *toric variety* $X(N,\Delta)$ associated to a fan Δ w.r.t. the lattice N is by definition the identification space

$$X(N,\Delta) := \Big(\Big(\coprod_{\sigma\in\Delta} U_\sigma\Big) \;/\; \sim\Big) \tag{3.2}$$

with $U_{\sigma_1} \ni u_1 \sim u_2 \in U_{\sigma_2}$ if and only if there is a $\tau \in \Delta$, such that $\tau \prec \sigma_1 \cap \sigma_2$ and $u_1 = u_2$ within U_τ. $X(N,\Delta)$ is called *simplicial* if all the cones of Δ are simplicial. $X(N,\Delta)$ is compact iff $|\Delta| = N_\mathbb{R}$ ([**61**], thm. 1.11, p. 16). Moreover, $X(N,\Delta)$ admits a canonical T_N-action which extends the group multiplication of $T_N = U_{\{\mathbf{0}\}}$:

$$T_N \times X(N,\Delta) \ni (t,u) \longmapsto t\cdot u \in X(N,\Delta) \tag{3.3}$$

where, for $u \in U_\sigma \subset X(N,\Delta)$,

$$(t\cdot u)(m) := t(m)\cdot u(m)\,,\ \forall m,\ m\in M\cap\sigma^\vee.$$

The orbits w.r.t. the action (3.3) are parametrized by the set of all the cones belonging to Δ. For a $\tau \in \Delta$, we denote by $\mathrm{orb}(\tau)$ (resp. by $V(\tau)$) the orbit (resp. the closure of the orbit) which is associated to τ.

(g) The group of T_N-invariant Weil divisors of a toric variety $X(N,\Delta)$ has the set $\{V(\varrho) \mid \varrho \in \Delta(1)\}$ as $\mathbb{Z}$-basis. In fact, such a divisor D is of the form $D = D_\psi$, where

$$D_\psi := -\sum_{\varrho\in\Delta(1)} \psi(n(\varrho))V(\varrho)$$

and $\psi : |\Delta| \to \mathbb{R}$ a *PL-Δ-support function*, i.e., an $\mathbb{R}$-valued, positively homogeneous function on $|\Delta|$ with $\psi(N\cap|\Delta|) \subset \mathbb{Z}$ which is piecewise linear and upper convex on each $\sigma\in\Delta$. (*Upper convex* on a $\sigma\in\Delta$ means that $\psi|_\sigma(\boldsymbol{x}+\boldsymbol{x}') \geqslant \psi|_\sigma(\boldsymbol{x}) + \psi|_\sigma(\boldsymbol{x}')$, for all $\boldsymbol{x},\boldsymbol{x}'\in\sigma$). For example, the canonical divisor $K_{X(N,\Delta)}$ of $X(N,\Delta)$ equals D_ψ for ψ a PL-Δ-support function with $\psi(n(\varrho)) = 1$, for all rays $\varrho\in\Delta(1)$. A divisor $D = D_\psi$ is Cartier iff ψ is a *linear* Δ-support function (i.e., $\psi|_\sigma$ is overall linear on each $\sigma\in\Delta$). Obviously, D_ψ is $\mathbb{Q}$-Cartier iff $k\cdot\psi$ is a linear Δ-support function for some $k\in\mathbb{N}$.

***Theorem 3.3* (Ampleness criterion).** — *A T_N-invariant $\mathbb{Q}$-Cartier divisor $D = D_\psi$ of a toric variety $X(N,\Delta)$ of dimension r is ample if and only if there exists a $\kappa\in\mathbb{N}$, such that $\kappa\cdot\psi$ is a* strictly upper convex *linear Δ-support function, i.e., iff for every $\sigma\in\Delta(r)$ there is a unique $m_\sigma \in M = \mathrm{Hom}_\mathbb{Z}(N,\mathbb{Z})$, such that*

$$\kappa\cdot\psi(\boldsymbol{x}) \leqslant \langle m_\sigma, \boldsymbol{x}\rangle\,,\quad \textit{for all } \boldsymbol{x}\in|\Delta|\,,$$

with equality being valid iff $\boldsymbol{x}\in\sigma$.

Proof. — It follows from [**43**, Thm. 13, p. 48]. □

(h) The behaviour of toric varieties with regard to the algebraic properties (1.1) is as follows.

***Theorem 3.4* (Normality and CM-property).** — *All toric varieties are normal and Cohen-Macaulay.*

Proof. — For a proof of the normality property see [**61**, Thm. 1.4, p. 7]. The CM-property for toric varieties was first shown by Hochster in [**35**]. See also Kempf [**43**, Thm. 14, p. 52], and Oda [**61**, 3.9, p. 125]. □

In fact, by the definition (3.2) of $X(N, \Delta)$, all the algebraic properties of this kind are *local* with respect to its affine covering, i.e., it is enough to be checked for the affine toric varieties U_σ for all (maximal) cones σ of the fan Δ.

***Definition 3.5* (Multiplicities and basic cones).** — Let N be a free $\mathbb{Z}$-module of rank r and $\sigma \subset N_{\mathbb{R}}$ a simplicial, rational s.c.p.c. of dimension $d \leqslant r$. σ can be obviously written as $\sigma = \varrho_1 + \cdots + \varrho_d$, for distinct rays $\varrho_1, \ldots, \varrho_d$. The *multiplicity* $\mathrm{mult}(\sigma; N)$ of σ with respect to N is defined as

$$\mathrm{mult}\,(\sigma; N) := \frac{\det\left(\mathbb{Z}\, n\,(\varrho_1) \oplus \cdots \oplus \mathbb{Z}\, n\,(\varrho_d)\right)}{\det\,(N_\sigma)},$$

where N_σ is the lattice in $\mathrm{lin}(\sigma)$ induced by N. If $\mathrm{mult}(\sigma; N) = 1$, then σ is called a *basic cone* w.r.t. N.

***Theorem 3.6* (Smoothness criterion).** — *The affine toric variety U_σ is smooth iff σ is basic w.r.t. N. (Correspondingly, an arbitrary toric variety $X(N, \Delta)$ is smooth if and only if it is simplicial and each s.c.p. cone $\sigma \in \Delta$ is basic w.r.t. N.)*

Proof. — See [**43**, Ch. I, Thm. 4, p. 14] and [**61**, Thm. 1.10, p. 15]. □

***Theorem 3.7* ($\mathbb{Q}$-factoriality).** — *A toric variety $X(N, \Delta)$ is $\mathbb{Q}$-factorial if and only if Δ is simplicial, i.e., if and only if $X(N, \Delta)$ has at most abelian quotient singularities.*

Proof. — Since this is a local property, it is enough to consider the case in which $X(N, \Delta) = U_\sigma$, where the cone $\sigma = \mathbb{R}_{\geqslant 0} v_1 + \cdots + \mathbb{R}_{\geqslant 0} v_r$ is of maximal dimension, $\mathrm{Gen}(\sigma) = \{v_1, ..., v_r\}$, and $\Delta = \{\tau \mid \tau \preceq \sigma\}$. U_σ is $\mathbb{Q}$-factorial if and only if all the T_N-invariant prime divisors D_{v_i} are $\mathbb{Q}$-Cartier. This is equivalent to the existence of elements $m_i \in M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$, with $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, for which $\langle m_i, v_j \rangle = \delta_{ij}$ (the Kronecker delta). But this means that σ is simplicial. □

Next theorem is due to Stanley ([**73**, § 6]), who worked directly with the monoidal $\mathbb{C}$-algebra $\mathbb{C}[M \cap \sigma^\vee]$, as well as to Ishida ([**38**, § 7]), Danilov and Reid ([**63**, p. 294]), who provided a purely algebraic-geometric characterization of the Gorenstein property.

***Theorem 3.8* (Gorenstein property).** —*Let N be a free $\mathbb{Z}$-module of rank r, and $\sigma \subset N_{\mathbb{R}}$ a s.c.p. cone of dimension $d \leqslant r$. Then the following conditions are equivalent*:

(i) *U_σ is Gorenstein.*

(ii) *There exists an element m_σ of M, such that*

$$M \cap (\mathrm{int}\,(\sigma^\vee)) = m_\sigma + M \cap \sigma^\vee.$$

(iii) $\mathrm{Gen}(\sigma) \subset \mathbf{H}$, *where* $\mathbf{H}$ *denotes a primitive affine hyperplane of* $(N_\sigma)_{\mathbb{R}}$.

Moreover, if $d = r$, *then* m_σ *in* (ii) *is a* uniquely determined *primitive element of* $M \cap (\mathrm{int}\,(\sigma^\vee))$ *and* $\mathbf{H}$ *in* (iii) *equals*

$$\mathbf{H} = \{\boldsymbol{x} \in N_{\mathbb{R}} \mid \langle m_\sigma, \boldsymbol{x}\rangle = 1\}\,.$$

Definition 3.9. — If N_1 and N_2 are two free $\mathbb{Z}$-modules (not necessarily of the same rank) and $P_1 \subset (N_1)_{\mathbb{R}}$, $P_2 \subset (N_2)_{\mathbb{R}}$ two lattice polytopes w.r.t. them, we shall say that P_1 and P_2 are *lattice equivalent* to each other, if P_1 is affinely equivalent to P_2 via an affine map $\varpi : (N_1)_{\mathbb{R}} \to (N_2)_{\mathbb{R}}$, such that $\varpi\,\big|_{\mathrm{aff}(P)} : \mathrm{aff}(P) \to \mathrm{aff}(P')$ is a bijection mapping P_1 onto the (necessarily equidimensional) polytope P_2, every i-dimensional face of P_1 onto an i-dimensional face of P_2, for all i, $0 \leqslant i \leqslant \dim(P_1) = \dim(P_2)$, and, in addition, N_{P_1} onto the lattice N_{P_2}, where by N_{P_j} is meant the sublattice of N_j generated (as subgroup) by $\mathrm{aff}(P_j) \cap N_j$, $j = 1, 2$.

Definition 3.10. — *Nakajima polytopes* $P \subset \mathbb{R}^r$ are lattice polytopes w.r.t. the usual rectangular lattice $\mathbb{Z}^r$, defined inductively as follows: In dimension 0, P is an (arbitrary) point of $\mathbb{R}^r$, while in dimension $d \leqslant r$, P is of the form

$$P = \{\, \boldsymbol{x} = (\boldsymbol{x}', x_r) \in F \times \mathbb{R} \subset \mathbb{R}^r \mid 0 \leqslant x_r \leqslant \langle \boldsymbol{m}, \boldsymbol{x}'\rangle\}\,,$$

where the facet $F \subset \mathbb{R}^{r-1}$ is a Nakajima polytope of dimension $d-1$, and $\boldsymbol{m} \in (\mathbb{Z}^r)^\vee$ is a functional taking non-negative values on F.

***Theorem 3.11* (Toric L.C.I.'s)**. — *Let N be a free $\mathbb{Z}$-module of rank r, and $\sigma \subset N_{\mathbb{R}}$ a s.c.p. cone of dimension $d \leqslant r$, such that U_σ is Gorenstein. Writing σ as $\sigma = \sigma' \oplus \{\mathbf{0}\}$ with σ' a d-dimensional cone in $(N_\sigma)_{\mathbb{R}}$, we obtain an analytic isomorphism:*

$$U_\sigma \cong U_{\sigma'} \times (\mathbb{C}^*)^{r-d}\,.$$

Let $m_{\sigma'}$ be the unique primitive element of $M_\sigma \cap \left(\mathrm{int}\left((\sigma')^\vee\right)\right)$, as it is defined in Theorem 3.8. *Then U_σ is a* local complete intersection[12] *if and only if the lattice polytope*

$$P_{\sigma'} := \sigma' \cap \{\boldsymbol{x} \in (N_\sigma)_{\mathbb{R}} \mid \langle m_{\sigma'}, \boldsymbol{x}\rangle = 1\}$$

is lattice equivalent to a Nakajima polytope (*cf.* 3.9, 3.10)

This was proved in [**38**] for dimension 3 and in [**59**] for arbitrary dimensions.

(i) A *map of fans* $\varpi : (N', \Delta') \to (N, \Delta)$ is a $\mathbb{Z}$-linear homomorphism $\varpi : N' \to N$ whose scalar extension $\varpi \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{R}} : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ satisfies the property:

$$\forall \sigma',\ \sigma' \in \Delta' \quad \exists\, \sigma,\ \sigma \in \Delta \quad \text{with} \quad \varpi\,(\sigma') \subset \sigma\,.$$

$\varpi \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{C}^*} : T_{N'} = N' \otimes_{\mathbb{Z}} \mathbb{C}^* \to T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ is a homomorphism from $T_{N'}$ to T_N and the scalar extension $\varpi^\vee \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{R}} : M_{\mathbb{R}} \to M'_{\mathbb{R}}$ of the dual $\mathbb{Z}$-linear map $\varpi^\vee : M \to M'$ induces canonically an *equivariant holomorphic map*

$$\varpi_* : X\,(N', \Delta') \to X\,(N, \Delta)\,.$$

[12] Obviously, for $d = r$, U_σ is a "g.c.i." in the sense of § 1.

This map is *proper* if and only if $\varpi^{-1}(|\Delta|) = |\Delta'|$. In particular, if $N = N'$ and Δ' is a refinement of Δ, then $\mathrm{id}_* : X(N, \Delta') \to X(N, \Delta)$ is *proper* and *birational* (cf. [**61**, Thm. 1.15 and Cor. 1.18]).

(j) By Carathéodory's Theorem concerning convex polyhedral cones (cf. [**25**, III 2.6 and V 4.2]) one can choose a refinement Δ' of any given fan Δ, so that Δ' becomes simplicial. Since further subdivisions of Δ' reduce the multiplicities of its cones, we may arrive (after finitely many subdivisions) at a fan $\widetilde{\Delta}$ having only basic cones. Hence, for every toric variety $X(N, \Delta)$ there exists a refinement $\widetilde{\Delta}$ of Δ consisting of exclusively basic cones w.r.t. N, i.e., such that

$$f = \mathrm{id}_* : X(N, \widetilde{\Delta}) \longrightarrow X(N, \Delta)$$

is a T_N-equivariant (full) desingularization.

Theorem 3.12. — *All $\mathbb{Q}$-Gorenstein toric varieties have at most log-terminal singularities.*

Proof. — We may again assume that $X = X(N, \Delta) = U_\sigma$ (where

$$\sigma = \mathbb{R}_{\geqslant 0} v_1 + \cdots + \mathbb{R}_{\geqslant 0} v_r$$

is a cone of maximal dimension, with Gen $(\sigma) = \{v_1, ..., v_r\}$ and

$$\Delta = \{\tau \mid \tau \preceq \sigma\}).$$

Since $\omega_X = \mathcal{O}_X(K_X)$ with $K_X = -\sum D_{v_i} \leqslant 0$, for K_X to be $\mathbb{Q}$-Cartier means that K_X is a (T_N-invariant) Cartier divisor after multiplication by an integer. This multiple of K_X has to be a divisor of the form $\mathrm{div}(\boldsymbol{e}(u))$, for some $u \in M$ (cf. [**61**, Prop. 2.1, pp. 68-69] or [**28**, Lemma of p. 61]). Hence, there must be an $m_\sigma \in M_{\mathbb{Q}}$, such that $\langle m_\sigma, \sigma \rangle \geqslant 0$ and $\langle m_\sigma, v_j \rangle = 1$ (with m_σ regarded as a *linear* support function on σ, cf. (g)). Let now

$$f : Y = X(N, \Delta') \to X = X(N, \Delta) = U_\sigma$$

be a desingularization of X obtained by a subdivision of σ into smaller basic strongly convex rational polyhedral cones. Suppose that the primitive lattice points of the *new* introduced rays in Δ' are $v'_1, v'_2, \ldots, v'_s$. Then

$$f^* K_X = -\sum_{v \in \mathrm{Gen}(\Delta')} \langle m_\sigma, v \rangle \, D_v.$$

Since

$$\mathcal{O}_Y(K_Y)\big|_{\mathrm{Reg}(Y)} = \mathcal{O}_Y(f^* K_X)\big|_{\mathrm{Reg}(Y)}$$

we have

$$K_Y - f^* K_X = \sum_{j=1}^{s} \left(\langle m_\sigma, v'_j \rangle - 1 \right) D_{v'_j} \tag{3.4}$$

Since the discrepancy is > -1, $(U_\sigma, \mathrm{orb}(\sigma))$ is a log-terminal singularity whose index equals $\ell = \min\{\kappa \in \mathbb{Z}_{\geqslant 1} \mid m_\sigma \kappa \in M\}$. □

Theorem 3.13. — *All toric singularities are rational singularities.*

Proof. — For $\mathbb{Q}$-Gorenstein toric singularities this follows from Theorems 1.8 and 3.12. For the general case see Ishida [**38**] and Oda [**61**, Cor. 3.9, p. 125]. □

Formula (3.4) gives us the following purely combinatorial characterization of terminal (resp. canonical) toric singularities.

Corollary 3.14. — *A toric singularity* $(U_\sigma, \mathrm{orb}(\sigma))$ *is terminal* (*resp. canonical*) *of index* ℓ *if and only if*

$$\exists\ m_\sigma \in M : \langle m_\sigma, \mathrm{Gen}(\sigma)\rangle = \ell \text{ and } \langle m_\sigma, n\rangle > \ell \ \text{ for all } \ n \in \sigma \cap N \smallsetminus (\{\mathbf{0}\} \cup \mathrm{Gen}(\sigma)),$$

$$(\textit{resp. } \exists\ m_\sigma \in M : \langle m_\sigma, \mathrm{Gen}(\sigma)\rangle = \ell \text{ and } \ \langle m_\sigma, n\rangle \geqslant \ell \ \text{ for all } \ n \in \sigma \cap N \smallsetminus \{\mathbf{0}\}).$$

(m_σ is uniquely determined whenever σ is of maximal dimension in the fan).

(k) **Recapitulation**. This is divided into two parts. The first one contains the standard dictionary:

Discrete Geometry	**Algebraic Geometry**
strongly convex rational pol.cone $\sigma \subset N_{\mathbb{R}} \cong \mathbb{R}^r$	affine toric variety $U_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$
Fans Δ	toric varieties $X(N, \Delta)$ (after glueing)
τ face of σ	U_τ open subset of U_σ
$\lvert\Delta\rvert = \mathbb{R}^r$	$X(N, \Delta)$ is complete
All cones of Δ are basic	$X(N, \Delta)$ is non-singular
All cones of Δ are simplicial	$X(N, \Delta)$ has at most abelian quotient singularities
Δ admits a strictly convex upper support function	$X(N, \Delta)$ is a quasi-projective variety
Δ' a cone subdivision of Δ	$X(N, \Delta') \longrightarrow X(N, \Delta)$ proper birational morphism

while the second one describes the main properties of toric singularities:

Discrete Geometry	**Algebraic Geometry**
strongly convex rational pol.cone $\sigma \subset \mathbb{R}^r$	U_σ is always normal, and Cohen-Macaulay
$\mathrm{orb}\,(\sigma) \in U_\sigma$, σ non-basic cone of maximal dimension r (keep this assumption in what follows)	$(U_\sigma, \mathrm{orb}\,(\sigma))$ is a rational singularity
$\exists!\ m_\sigma \in M_{\mathbb{Q}}$: $\mathrm{Gen}\,(\sigma) \subset \{y \in N_{\mathbb{R}} \mid \langle m_\sigma, y\rangle = 1\}$	U_σ is $\mathbb{Q}$-Gorenstein and i.p. $(U_\sigma, \mathrm{orb}\,(\sigma))$ is a log-terminal singularity of index $\min\{\kappa \in \mathbb{Z}_{\geqslant 1} \mid m_\sigma \kappa \in M\}$
$\left\{\begin{array}{l} \exists!\ m_\sigma \in M_{\mathbb{Q}} : \\ \mathrm{Gen}\,(\sigma) \subset \{y \in N_{\mathbb{R}} \mid \langle m_\sigma, y\rangle = 1\} \\ \text{and} \quad N \cap \sigma \cap \{y \in N_{\mathbb{R}} \mid \langle m_\sigma, y\rangle < 1\} = \{\mathbf{0}\} \end{array}\right.$	$(U_\sigma, \mathrm{orb}\,(\sigma))$ is a canonical singularity of index $\min\{\kappa \in \mathbb{Z}_{\geqslant 1} \mid m_\sigma \kappa \in M\}$
$\left\{\begin{array}{l} \exists!\ m_\sigma \in M_{\mathbb{Q}} : \\ \mathrm{Gen}\,(\sigma) \subset \{\boldsymbol{x} \in N_{\mathbb{R}} \mid \langle m_\sigma, \boldsymbol{x}\rangle = 1\} \\ \text{and} \quad N \cap \sigma \cap \{\boldsymbol{x} \in N_{\mathbb{R}} \mid \langle m_\sigma, \boldsymbol{x}\rangle \leqslant 1\} \\ = \{\mathbf{0}\} \cup \mathrm{Gen}\,(\sigma) \end{array}\right.$	$(U_\sigma, \mathrm{orb}\,(\sigma))$ is a terminal singularity of index $\min\{\kappa \in \mathbb{Z}_{\geqslant 1} \mid m_\sigma \kappa \in M\}$
$\left\{\begin{array}{l} \exists!\ m_\sigma \in M : \\ \mathrm{Gen}\,(\sigma) \subset \{\boldsymbol{x} \in N_{\mathbb{R}} \mid \langle m_\sigma, \boldsymbol{x}\rangle = 1\} \\ \text{ with } \sigma \text{ supporting the lattice polytope} \\ P_\sigma := \sigma \cap \{\boldsymbol{x} \in N_{\mathbb{R}} \mid \langle m_\sigma, \boldsymbol{x}\rangle = 1\} \text{ and} \\ N \cap \mathrm{conv}\,(\{\mathbf{0}\} \cup P_\sigma) = \{\mathbf{0}\} \cup (N \cap P_\sigma) \end{array}\right.$	$(U_\sigma, \mathrm{orb}\,(\sigma))$ is a Gorenstein singularity (i.e., canonical of index 1)
P_σ (as above) is lattice-equivalent to a Nakajima polytope	$(U_\sigma, \mathrm{orb}\,(\sigma))$ is a locally complete intersection singularity

Remark 3.15

(i) A lattice polytope P is called *elementary* if the lattice points belonging to it are exactly its vertices. A lattice simplex is said to be *basic* (or *unimodular*) if its vertices constitute a part of a $\mathbb{Z}$-basis of the reference lattice (or equivalently, if its relative, normalized volume equals 1).

(ii) It is clear by the fifth row of the above table that the classification of terminal (resp. $\mathbb{Q}$-factorial terminal) r-dimensional Gorenstein toric singularities is *equivalent* to the classification of all $(r-1)$-dimensional elementary lattice polytopes (resp. elementary lattice simplices). The readers, who would like to learn more about partial classification results (on both "sides") in dimensions $\geqslant 3$, are referred (in chronological order) to [**20**], [**58**], [**57**], [**39**], [**61**, pp. 34-36], [**69**], [**7**], and [**8**], for investigations from the point of view of computational algebraic geometry, as well as to [**40**], [**4**], [**31**], for a study of "width-functions" (which are closely related to the index of the corresponding singularities) from the point of view of number theory, combinatorics and integer programming(13).

(iii) On the other extreme, all toric l.c.i.-singularities admit *crepant* resolutions in *all* dimensions. (This was proved recently in [**18**] by showing inductively that all Nakajima polytopes admit lattice triangulations consisting exclusively of basic simplices; cf. Theorem 3.6). Nevetheless, regarding non-l.c.i., Gorenstein toric singularities, the determination of necessary and sufficient "intrinsic" conditions, under which they possess resolutions of this kind, remains an unsolved problem. Finally, let us stress that one can always reduce log-terminal non-canonical toric singularities to canonical ones by a torus-equivariant, natural "canonical modification" which is uniquely determined (cf. below footnote to 4.1 (i)).

4. Toric two- and three-dimensional singularities, and their resolutions

Now we focus on the desingularization methods of low-dimensional toric singularities.

• All *2-dimensional* toric singularities $(U_\sigma, \mathrm{orb}(\sigma))$ are abelian (in fact, cyclic) quotient singularities and can be treated by means of the finite continued fractions (see [**61**, § 1.6]). For them there exists always a uniquely determined "minimal" resolution(14). In fact, this resolution has the nice property that

$$\mathbf{Hilb}_N(\sigma) = \mathrm{Gen}(\Delta'),$$

where Δ' is the fan which refines σ into basic cones and $\mathbf{Hilb}_N(\sigma)$ the Hilbert basis of σ w.r.t. N. Figure 1 shows this minimal resolution $U_\sigma \xleftarrow{f} X(N, \Delta')$ of U_σ for the cone

$$\sigma = \mathrm{pos}(\{(1,0), (4,5)\}) \subset \mathbb{R}^2$$

(w.r.t. the standard lattice $N = \mathbb{Z}^2$) constructed by its subdivision into two basic subcones.

(13) Note that in the literature, instead of "elementary" polytopes, there are also in use *different names* like "fundamental", "lattice-point-free", "hollow", or even "empty" polytopes.
(14) This is actually a "good minimal" resolution in the sense of § 2.

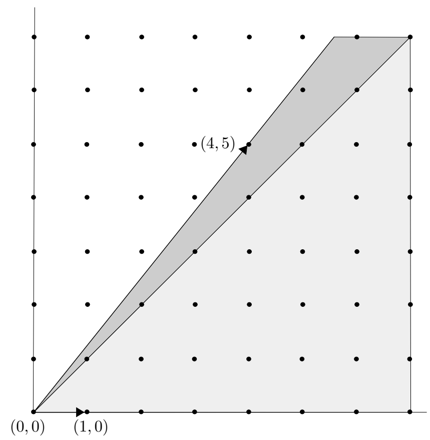

FIGURE 1

Note that $U_\sigma = \mathbb{C}^2/G$, where $G \subset \mathrm{GL}(2,\mathbb{C})$ is the finite cyclic group generated by the matrix

$$\begin{pmatrix} e^{\frac{2\pi\sqrt{-1}}{5}} & 0 \\ 0 & e^{\frac{2\pi\sqrt{-1}}{5}} \end{pmatrix},$$

and

$$\sigma^\vee = \mathrm{pos}\left(\{(0,1),(5,-4)\}\right) \subset \left(\mathbb{R}^2\right)^\vee.$$

In particular, $\mathbb{C}[\left(\mathbb{R}^2\right)^\vee \cap \sigma^\vee] \cong \mathbb{C}\,[u,w]^G$ is generated by the monomials

$$\{\, u^i w^j \;\big|\; i+j \equiv 0(\mathrm{mod} 5)\}.$$

The *minimal* generating system of $(\mathbb{R}^2)^\vee \cap \sigma^\vee$ equals

$$\mathbf{Hilb}_{(\mathbb{R}^2)^\vee}(\sigma^\vee) = \{k_0, k_1, k_2, k_3, k_4, k_5\},$$

with $k_0 = (0,1)$, $k_5 = (5,-4)$ the primitive vectors spanning $\sigma^\vee$. The remaining elements are determined by the vectorial matrix multiplication

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} k_0 \\ (0,0) \\ (0,0) \\ k_5 \end{pmatrix}$$

where the diagonal entries of the first matrix of the right-hand side are exactly those arising in the continued fraction development

$$\frac{5}{4} = 2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2}}}.$$

(cf. [**61**, Prop. 1.21, p. 26]). U_σ has embedding dimension 6, and taking into account the linear dependencies between the members of this Hilbert basis, it can be described as the zero-locus of $\binom{5}{2} = 10$ square-free binomials (by applying Proposition 3.2). More precisely (cf. [**68**]), setting $z_i := \boldsymbol{e}(k_i)$, $0 \leqslant i \leqslant 5$, we obtain:

$$U_\sigma \cong \left\{ (z_0, z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^6 \;\middle|\; \operatorname{rank} \begin{pmatrix} z_0 & z_1 & z_2 & z_3 & z_4 \\ z_1 & z_2 & z_3 & z_4 & z_5 \end{pmatrix} \leqslant 1 \right\}.$$

• In *dimension* 3 things are more complicated. The singularities can be resolved by more or less "canonical" procedures (and by *projective* birational morphisms) but the "uniqueness" is mostly lost (even if one requires "minimality" in the sense of "MMP", i.e., $\mathbb{Q}$-terminalizations), though "not completely" (of course, as usual, meant "up to isomorphisms in codimension 1"). In the literature you may find several different approaches:

(a) **Hilb**-*desingularizations.* — Bouvier and Gonzalez-Sprinberg used in [**9**], [**10**] "**Hilb-**desingularizations" (i.e., again with $\mathbf{Hilb}_N(\sigma) = \mathrm{Gen}(\Delta')$) to resolve three-dimensional toric singularities $(U_\sigma, \mathrm{orb}(\sigma))$. Their exposition is well-structured and fits together with more general concepts of MMP (see [**20**], [**41**], [**56**]). Nevertheless, any kind of "uniqueness" is (in general) lost already from the "second" step by producing their "minimal terminal subdivisions".

(b) **Hilb**-*desingularizations with extra lexicographic ordering.* — Similar constructive method, due to Aguzzoli and Mundici [**1**] (again by making use of "indispensable" exceptional divisors), but with a fixed ordering for performing the starring operations. The "uniqueness" is lost as long as one uses other orderings. Moreover, they also assume $\mathbb{Q}$-factorialization from the very beginning.

(c) *Distinguished crepant desingularization (via the Hilbert scheme of G-clusters) for the case in which σ is simplicial and $U_\sigma = \mathbb{C}^3/G$ Gorenstein.* — Motivated by the so-called "McKay correspondence" in dimension 3 (see [**67**] for a recent exposition), Nakamura [**60**] and Craw & Reid [**14**], [**15**], [**16**] have constructed a *distinguished* crepant resolution of $U_\sigma = \mathbb{C}^3/G$ (G an abelian finite subgroup of $\mathrm{SL}(3,\mathbb{C})$), by expressing the G-orbit Hilbert scheme of $\mathbb{C}^3$ as a fan of basic cones supporting certain "regular tesselations" of the corresponding junior simplex. Unfortunately, it is not known if there is an analogue of this method for the case in which σ is not simplicial, and it is known that it does not work (at least as crepant resolution) in higher dimensions.

(d) *Desingularizations via the "initial strategy" of M. Reid* [**63**], [**64**], [**66**]. — This method was already described in broad outline in section 2. Applying it for arbitrary three-dimensional toric singularities (cf. [**19**]), we win more explicit information about the resolution (compared with the general case[15]) by obtaining the following:

Theorem 4.1. — *Let* $(U_\sigma, \operatorname{orb}(\sigma))$ *be an arbitrary 3-dimensional toric singularity, where* $U_\sigma = X(N, \Delta)$.

(i) *There exists a* uniquely determined *partial desingularization*

$$U_\sigma = X(N, \Delta) \longleftarrow X(N, \Delta_{\text{can}}) \tag{4.1}$$

such that $X(N, \Delta_{\text{can}})$ *has only canonical singularities.* (*This is actually a general fact valid for all dimensions*[16]).

(ii) *The singularities of* $X(N, \Delta_{\text{can}})$ *which have index* > 1 *can be reduced to canonical singularities of index* 1 *up to cyclic coverings* (*via lattice dilations*). *Moreover, one may treat them in a special manner* (*either by the Ishida-Iwashita classification* [**39**] *or by Bouvier, Gonzalez-Sprinberg's* [**10**] **Hilb**-*desingularizations*[17]).

(iii) *By* (ii) *we may restrict ourselves to the case in which* $X(N, \Delta_{\text{can}})$ *has only canonical singularities of index* 1 *and, in particular, to cones* $\tau \in \Delta_{\text{can}}(3)$, *for which* $\operatorname{orb}(\tau) \in \operatorname{Sing}(X(N, \Delta_{\text{can}}))$. *Each of these* τ*'s has* (*up to lattice automorphism*) *the form*

$$\tau_P = \{(\lambda x_1, \lambda x_2, \lambda) \mid \lambda \in \mathbb{R}_{\geqslant 0}\,, \quad (x_1, x_2) \in P\}$$

(*w.r.t. the standard rectangular lattice* $\mathbb{Z}^3$) *for some lattice polygon* P. *There exists a composite of torus-equivariant, crepant, projective, partial desingularizations*

$$U_{\tau_P} \xleftarrow{f_1} X_{\mathcal{S}_1} \xleftarrow{f_2} X_{\mathcal{S}_2} \xleftarrow{f_3} X_{\mathcal{S}_3} \longleftarrow \cdots \xleftarrow{f_{\kappa-1}} X_{\mathcal{S}_{\kappa-1}} \xleftarrow{f_\kappa} X_{\mathcal{S}_\kappa} = Y_{\tau_P} \tag{4.2}$$

where $\mathcal{S}_1, \ldots, \mathcal{S}_\kappa$ *are successive polygonal subdivisions of* P (*with* $X_{\mathcal{S}_i}$ *the toric variety associated to the cone supporting* $\mathcal{S}_i$), *such that each* f_i *in* (4.2) *is the usual toric blow-up of the arising singular point and* Y_{τ_P} *is* unique *w.r.t. this property, possessing at most compound Du Val singularities whose types can be written in a very short*

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[15] Cf. [**63**, Rem. 6.10, p. 308] and [**64**, Rem. 0.8 (d), p. 135, and Ex. 2.7, p. 145].

[16] To construct the "canonical modification" (4.1) one has just to pass to the fan-subdivision

$$\Delta_{\text{can}} = \{\{\operatorname{pos}(F) \mid\ F \text{ faces of } \operatorname{conv}(\sigma \cap N \smallsetminus \{\mathbf{0}\})\}\,,\ \sigma \in \Delta\}$$

[17] For the "uniqueness" of this reduction procedure to the index 1 case (from the point of view of toric geometry) see, in particular, [**10**, Théorème 2.23, p. 144].

list[18]. (*This is in fact an intrinsic procedure involving in each step the auxiliary 2-dimensional subcones and the finite continued fraction expansions for each vertex of the new central lattice polygon.*)

(iv) *There exists a composite of torus-equivariant, crepant, projective, partial desingularizations*

$$Y_{\tau_P} \xleftarrow{g_1} X_{\widetilde{\mathcal{S}}_1} \xleftarrow{g_2} X_{\widetilde{\mathcal{S}}_2} \xleftarrow{g_3} X_{\widetilde{\mathcal{S}}_3} \longleftarrow \cdots \xleftarrow{g_{\nu-1}} X_{\widetilde{\mathcal{S}}_{\nu-1}} \xleftarrow{g_\nu} X_{\widetilde{\mathcal{S}}_\nu} = Z_{\tau_P} \tag{4.3}$$

of Y_{τ_P}, *such that each* g_j *in* (4.3) *is the usual toric blow-up of the ideal of the* 1-*dimensional locus of* $X_{\widetilde{\mathcal{S}}_{j-1}}$ *and* Z_{τ_P} *is* unique *w.r.t. this property possessing at most* ordinary double points. (Z_{τ_P} *is a terminalization of* Y_{τ_P}).

(v) *All the* $2^{\#(\text{ordinary double points})}$ *remaining choices to obtain "full" crepant desingularizations* (*just by filling up our lattice triangulations by box diagonals*) *are realized by projective birational morphisms.*

The proof involves explicit (constructive) techniques from discrete geometry of cones, together with certain results of combinatorial nature from [**26**], [**38**], [**39**] and [**44**]. (The projectivity of birational morphisms in (v) can be checked by means of the ampleness criterion 3.3). Moreover, in [**19**], it will be shown how toric geometry enables us to keep under control (in each step) the Fano surfaces arising as exceptional prime divisors.

Example 4.2. — As a simple example of how Theorem 4.1 (iii)-(v) works, take the cone

$$\sigma = \mathrm{pos}(P) \subset \mathbb{R}^2 \times \{1\} \subset \mathbb{R}^3$$

supporting the lattice tringle $P = \mathrm{conv}\{(-3,3,1)\,,(3,1,1)\,,(0,-3,1)\}$ of Fig. **2** (w.r.t. the standard rectangular lattice $\mathbb{Z}^3$).

[18] This list is the following:

Cases	**Possible cDu Val singularities**	**Types**
(i)	$\mathrm{Spec}\,(\mathbb{C}[z_1,z_2,z_3]\,/\,(z_1z_2 - z_3^{\kappa})) \times \mathrm{Spec}\,(\mathbb{C}[z_4])\,,\quad \kappa \in \mathbb{Z}_{\geqslant 2}$	$\mathbf{A}_{\kappa-1} \times \mathbb{C}$
(ii)	$\mathrm{Spec}\left(\mathbb{C}[z_1,z_2,z_3,z_4]\,/\,\left(z_1z_2 - z_3^{\kappa}z_4^{\kappa+\lambda}\right)\right),\quad \kappa \in \mathbb{Z}_{\geqslant 1},\ \lambda \in \mathbb{Z}_{\geqslant 0}$	$\mathbf{cA}_{2\kappa+\lambda-1}$
(iii)	$\mathrm{Spec}\,(\mathbb{C}[z_1,z_2,z_3,z_4]\,/\,(z_1z_2z_3 - z_4^2)) \cong \mathrm{Spec}(\mathbb{C}[t_1,t_2,t_3]^G)$ (where G is obtained by the linear representation of the Kleinian four-group into $\mathrm{SL}(3,\mathbb{C})$)	$\mathbf{cD}_4$

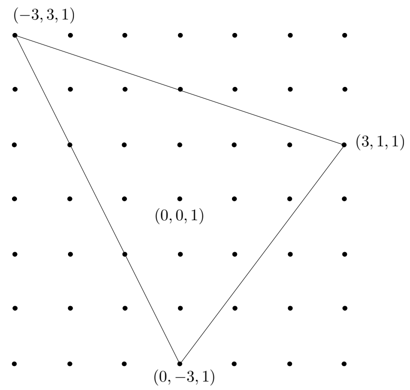

FIGURE 2

The singularity $(U_\sigma, \mathrm{orb}\,(\sigma))$ is Gorenstein, and Proposition 3.2 gives[19]:

$$\mathrm{edim}\,(U_\sigma, \mathrm{orb}\,(\sigma)) = \#(\mathbf{Hilb}_{(\mathbb{Z}^3)^\vee}\,(\sigma^\vee)) = 14.$$

Hence, by Theorem 2.7 (iv) and by Proposition 2.12 (iii), the Laufer-Reid invariant of a general hyperplane section $\mathbb{H}$ through $\mathrm{orb}(\sigma)$ equals

$$\mathrm{LRI}\,(\mathbb{H}, \mathrm{orb}\,(\sigma)) = \mathrm{edim}\,(\mathbb{H}, \mathrm{orb}\,(\sigma)) = \mathrm{edim}\,(U_\sigma, \mathrm{orb}\,(\sigma)) - 1 = 13.$$

Figure 3 shows the result of blowing up $\mathrm{orb}(\sigma) \in U_\sigma$ (equipped with the reduced subscheme structure).

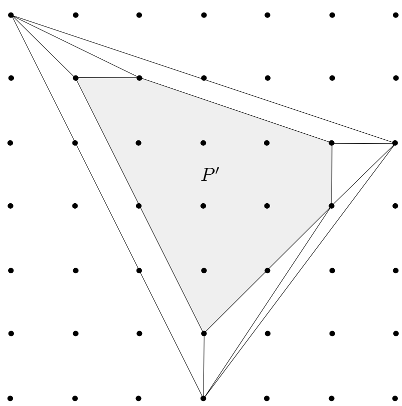

FIGURE 3

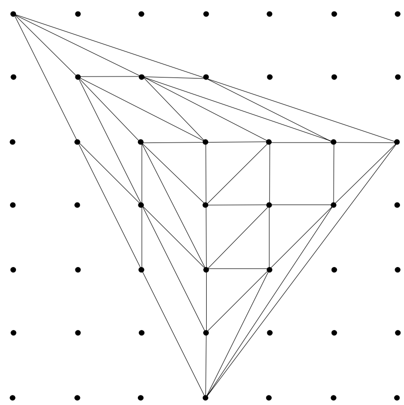

FIGURE 4

[19] This can be computed directly. A general formula, expressing the embedding dimension in terms of the vertex coordinates of arbitrary initial lattice polygons, will appear in [**19**].

It is worth mentioning that the "central" new subcone $\sigma' = \mathrm{pos}(P')$ of σ supports the lattice pentagon $P' = \mathrm{conv}(\{(-2,2,1), (-1,2,1), (2,1,1), (2,0,1), (0,-2,1)\})$ which is nothing but the polygon defined as the convex hull of the inner points of P. Next, we perform 4.1 (iii)-(iv) again and again until we arrive at the lattice polygonal subdivision of P shown in Figure 4.

In the last step (4.1 (v)), all $2^3 = 8$ possible choices of completing the polygonal subdivision in Fig. **4** to a triangulation (by filling up box diagonals) lead equally well to crepant, projective, full resolutions of $(U_\sigma, \mathrm{orb}(\sigma))$.

Remark 4.3. — A method of how one may achieve "$\mathbb{Q}$-factorialization" of toric singularities (after M. Reid [**65**] and S. Mori) *in arbitrary dimensions* was partially discussed in Wiśniewski's lectures [**74**].

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