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A boundedness result for toric log Del Pezzo surfaces

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Abstract. In this paper we give an upper bound for the Picard number of the rational surfaces which resolve minimally the singularities of toric log Del Pezzo surfaces of given index ℓ. This upper bound turns out to be a quadratic polynomial in the variable ℓ.

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1. Introduction. A normal complex surface X with at worst log terminal singularities, i.e., quotient singularities, is called *log Del Pezzo surface* if its anticanonical divisor $-K_X$ is a $\mathbb{Q}$-Cartier ample divisor. The *index* of such a surface is defined to be the smallest positive integer ℓ for which $-\ell K_X$ is a Cartier divisor. Every log Del Pezzo surface is isomorphic to the *anticanonical model* (in the sense of Sakai [13]) of the rational surface obtained by its minimal desingularization. The following Theorem is due to Nikulin [8] (for related results cf. [1, 9]):

Theorem 1.1. *Let X be a log Del Pezzo surface of index ℓ and $\widetilde{X} \longrightarrow X$ be its minimal desingularization. Then the Picard number $\rho(\widetilde{X})$ of $\widetilde{X}$ (i.e., the rank of its Picard group) is bounded by*

$$\rho(\widetilde{X}) < c \cdot \ell^{\frac{7}{2}}, \tag{1.1}$$

where c is an absolute constant.

The *toric* log Del Pezzo surfaces, i.e., those which are equipped with an algebraic action of a 2-dimensional algebraic torus $\mathbb{T}$, and contain an open dense $\mathbb{T}$-orbit, constitute a special subclass within the entire class of all log Del Pezzo surfaces. (For instance, in the toric case, only *cyclic* quotient singularities can occur.) To indicate how these two classes differ in practice, it would be enough to recall some known results for log Del Pezzo surfaces with Picard number = 1 and index $\ell \leq 2$:

(i) Excluding the "exceptional" $2D_4$-case, there exist, up to isomorphism, exactly 30 surfaces of this kind having index $\ell = 1$ (see [2, Theorem 4.3] or [14, Theorem 1.2]). Among them there are 16 having at worst cyclic quotient singularities. By [4, Theorem 6.10] we see that only 5 out of these 16 surfaces are toric (associated to the 5 *reflexive* triangles).

(ii) Up to isomorphism, there exist exactly 18 surfaces of this kind having index $\ell = 2$ (see [2, Theorem 4.2] or [5, Theorem 1.1 (1)]). Among them there are 14 having only cyclic quotient singularities. By [4, Theorem 6.12] we see that only 7 out of these 14 surfaces are toric.

The purpose of this paper is to prove an analogue of (1.1) for toric log Del Pezzo surfaces of given index.

Theorem 1.2. *Let X_Q be a toric log Del Pezzo surface of index ℓ (associated to the lattice polygon Q) and $\widetilde{X}_Q \longrightarrow X_Q$ be its minimal desingularization. Then $\rho(\widetilde{X}_Q)$ is bounded as follows*:

$$\rho(\widetilde{X}_Q) \leq \begin{cases} 7, & \text{if } \ell = 1, \\ 8\ell^2 - 6\ell + 3, & \text{if } \ell \geq 2. \end{cases} \tag{1.2}$$

Our proof uses tools from toric and discrete geometry.

2. Toric log Del Pezzo surfaces. Let $Q \subset \mathbb{R}^2$ be a (convex) polygon. Denote by $\mathcal{V}(Q)$ and $\mathcal{F}(Q)$ the set of its vertices and the set of its facets (edges), respectively. Q will be called an *LDP-polygon* if it contains the origin in its interior, and its vertices belong to $\mathbb{Z}^2$ and are primitive. If Q is an LDP-polygon, we shall denote by X_Q the compact toric surface constructed by means of the fan

$$\Delta_Q := \{\text{the cones } \sigma_F \text{ together with their faces} \mid F \in \mathcal{F}(Q)\},$$

where $\sigma_F := \{\lambda\mathbf{x} \mid \mathbf{x} \in F \text{ and } \lambda \in \mathbb{R}_{\geq 0}\}$ for all $F \in \mathcal{F}(Q)$. It is known (cf. [4, Remark 6.7]) that every toric log Del Pezzo surface is isomorphic to an X_Q, for a suitable LDP-polygon Q. Moreover, every cone σ_F is lattice-equivalent to the cone $\mathbb{R}_{\geq 0}\binom{1}{0} + \mathbb{R}_{\geq 0}\binom{p_F}{q_F}$, for suitable relatively prime integers p_F, q_F, with $0 \leq p_F < q_F$. (These are uniquely determined, up to replacement of p_F by its *socius* $\widehat{p}_F$, i.e., by the integer $\widehat{p}_F$, $0 \leq \widehat{p}_F < q_F$, satisfying $\gcd(\widehat{p}_F, q_F) = 1$ and $p_F\widehat{p}_F \equiv 1 (\mathrm{mod}\ q_F)$.) The affine toric variety $U_F := \mathrm{Spec}(\mathbb{C}[\sigma_F^\vee \cap (\mathbb{Z}^2)^\vee])$ (where $\sigma_F^\vee$ denotes the dual cone of σ_F and $(\mathbb{Z}^2)^\vee$ the dual lattice of $\mathbb{Z}^2$) is $\cong \mathbb{C}^2$ only if $q_F = 1$. Otherwise, the orbit $\mathrm{orb}(\sigma_F) \in U_F$ of σ_F, i.e., the single point remaining fixed under the canonical action of the algebraic torus $\mathbb{T} := \mathrm{Hom}_{\mathbb{Z}}((\mathbb{Z}^2)^\vee, \mathbb{C}^\star)$ on U_F, is a cyclic quotient singularity. In particular, $U_F \cong \mathbb{C}^2/G_F = \mathrm{Spec}(\mathbb{C}[z_1, z_2]^{G_F})$, with $G_F \subset \mathrm{GL}(2, \mathbb{C})$ denoting the cyclic group of order q_F which is generated by $\mathrm{diag}(\zeta_{q_F}^{-p_F}, \zeta_{q_F})$ (for ζ_{q_F} a q_F-th root of unity). Hence, the singular locus of X_Q equals

$$\mathrm{Sing}(X_Q) = \{\mathrm{orb}(\sigma_F) \mid F \in I_Q\},$$

where $I_Q := \{ F \in \mathcal{F}(Q) | \ q_F > 1 \}$. Its subset $\{ \mathrm{orb}(\sigma_F) | \ F \in \breve{I}_Q \}$, with $\breve{I}_Q$ defined to be $\breve{I}_Q := \{ F \in I_Q \mid p_F = 1 \}$, is the set of the *Gorenstein singularities* of X_Q.

The minimal desingularization of the surface X_Q can be described as follows: Equip the minimal generators of Δ_Q with an order (e.g., anticlockwise), and assume that for every $F \in \mathcal{F}(Q)$ the cone σ_F has $\mathbf{n}^{(F)}, \mathbf{n}'^{(F)} \in \mathbb{Z}^2$ as minimal generators ($\sigma_F = \mathbb{R}_{\geq 0}\, \mathbf{n}^{(F)} + \mathbb{R}_{\geq 0}\, \mathbf{n}'^{(F)}$), with $\mathbf{n}^{(F)}$ coming first w.r.t. this order. Next, for all $F \in I_Q$, consider the negative-regular continued fraction expansion of

$$\frac{q_F}{q_F - p_F} = \left[\!\left[b_1^{(F)}, b_2^{(F)}, \ldots, b_{s_F}^{(F)} \right]\!\right] := b_1^{(F)} - \cfrac{1}{b_2^{(F)} - \cfrac{1}{\ddots \, - \cfrac{1}{b_{s_F}^{(F)}}}}\,, \tag{2.1}$$

and define $\mathbf{u}_0^{(F)} := \mathbf{n}^{(F)}$, $\mathbf{u}_1^{(F)} := \frac{1}{q_F}((q_F - p_F)\mathbf{n}^{(F)} + \mathbf{n}'^{(F)})$, and lattice points $\{\mathbf{u}_j^{(F)} \,|\, 2 \leq j \leq s_F + 1\}$ by the formulae

$$\mathbf{u}_{j+1}^{(F)} := b_j^{(F)} \mathbf{u}_j^{(F)} - \mathbf{u}_{j-1}^{(F)}, \quad \forall j \in \{1, \ldots, s_F\}.$$

It is easy to see that $\mathbf{u}_{s_F+1}^{(F)} = \mathbf{n}'^{(F)}$, and that the integers $b_j^{(F)}$ are ≥ 2, for all $j \in \{1, \ldots, s_F\}$. The singularity $\mathrm{orb}(\sigma_F) \in U_F$ is resolved minimally by the proper birational map induced by the refinement $\{\mathbb{R}_{\geq 0}\, \mathbf{u}_j^{(F)} + \mathbb{R}_{\geq 0}\, \mathbf{u}_{j+1}^{(F)} \mid 0 \leq j \leq s_F \}$ of the fan which is composed of the cone σ_F and its faces. The exceptional divisor is $E^{(F)} := \sum_{j=1}^{s_F} E_j^{(F)}$, having

$$E_j^{(F)} := \overline{\mathrm{orb}(\mathbb{R}_{\geq 0}\, \mathbf{u}_j^{(F)})} \ (\cong \mathbb{P}^1_{\mathbb{C}}), \ \forall j \in \{1, \ldots, s_F\},$$

(i.e., the closures of the $\mathbb{T}$-orbits of the "new" rays) as its components, with self-intersection number $(E_j^{(F)})^2 = -b_j^{(F)}$ (see [12, Corollary 1.18 and Proposition 1.19, pp. 23–25]).

Note 2.1. (i) If $F \in \mathcal{F}(Q)$, and $\boldsymbol{\eta}_F \in (\mathbb{Z}^2)^\vee$ is its unique primitive outer normal vector, we define its *local index* to be the positive integer $l_F := \langle \boldsymbol{\eta}_F, F \rangle$, where

$$\langle \cdot, \cdot \rangle : \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R}) \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

is the usual inner product. For $F \in \mathcal{F}(Q) \smallsetminus I_Q$ we have obviously $l_F = 1$. For $F \in I_Q$, let $K(E^{(F)})$ be the *local canonical divisor* of the minimal resolution of $\mathrm{orb}(\sigma_F) \in U_F$ (in the sense of [4, p. 75]). $K(E^{(F)})$ is a $\mathbb{Q}$-Cartier divisor (a rational linear combination of $E_j^{(F)}$'s), and

$$l_F = \ \min \left\{ \xi \in \mathbb{N} \mid \ \xi K(E^{(F)}) \text{ is a Cartier divisor} \right\} = \tfrac{q_F}{\gcd(q_F, p_F - 1)}. \tag{2.2}$$

(ii) If $F \in I_Q$, denoting by $\mathfrak{m}_{X_Q,\mathrm{orb}(\sigma_F)}$ the maximal ideal of the local ring $\mathcal{O}_{X_Q,\mathrm{orb}(\sigma_F)}$ of the singularity $\mathrm{orb}(\sigma_F)$, and by

$$m_F := \dim_{\mathbb{C}}((\mathfrak{m}_{X_Q,\mathrm{orb}(\sigma_F)})/(\mathfrak{m}^2_{X_Q,\mathrm{orb}(\sigma_F)})) - 1$$

its *multiplicity*, it is known (cf. [3, Satz 2.11]) that

$$m_F = 2 + \sum_{j=1}^{s_F} (b_j^{(F)} - 2). \tag{2.3}$$

Lemma 2.2. *For all $F \in I_Q$ we have*

$$m_F \leq 2l_F.$$

Proof. See [7, Lemma 1.1 (iii)]. □

Lemma 2.3. *For all $F \in I_Q$ the self-intersection number of $K(E^{(F)})$ equals*

$$K(E^{(F)})^2 = -\left(\frac{2-(p_F+\widehat{p}_F)}{q_F} + (m_F - 2)\right).$$

Proof. Follows from [4, Corollary 4.6] and formula (2.3). □

The minimal desingularization $\varphi : \widetilde{X}_Q \longrightarrow X_Q$ of X_Q is constructed by means of the smooth compact toric surface $\widetilde{X}_Q$ which is defined by the fan

$$\widetilde{\Delta}_Q := \left\{ \begin{array}{c} \text{the cones } \{\sigma_F \mid F \in \mathcal{F}(Q) \smallsetminus I_Q\} \text{ and} \\ \left\{ \mathbb{R}_{\geq 0}\, \mathbf{u}_j^{(F)} + \mathbb{R}_{\geq 0}\, \mathbf{u}_{j+1}^{(F)} \;\middle|\; F \in I_Q,\ j \in \{0,1,\ldots,s_F\} \right\}, \\ \text{together with their faces} \end{array} \right\}$$

(refining each of the cones $\{\sigma_F \mid F \in I_Q\}$ of Δ_Q as mentioned above). Furthermore, the corresponding *discrepancy divisor* equals

$$K_{\widetilde{X}_Q} - \varphi^{\star} K_{X_Q} = \sum_{F \in I_Q} K(E^{(F)}). \tag{2.4}$$

(By $K_{X_Q}, K_{\widetilde{X}_Q}$ we denote the canonical divisors of X_Q and $\widetilde{X}_Q$, respectively.)

Note 2.4. By virtue of (2.2) and (2.4) the index ℓ of X_Q (as defined in §1) equals

$$\ell = \operatorname{lcm}\{\, l_F \mid F \in \mathcal{F}(Q)\}. \tag{2.5}$$

(For simplicity, sometimes ℓ is referred as *index* of Q.) In fact, if we denote by

$$Q^* := \left\{ \mathbf{y} \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R}) \;\middle|\; \langle \mathbf{y}, \mathbf{x} \rangle \leq 1,\ \forall\, \mathbf{x} \in Q \right\}$$

the *polar* of the polygon Q, the index ℓ is nothing but $\min\left\{ k \in \mathbb{N} \mid \mathcal{V}(kQ^*) \subset \mathbb{Z}^2 \right\}$, where $kQ^* := \{ k\mathbf{y} | \, \mathbf{y} \in Q^* \}$. In other words, ℓ equals the least common multiple of the (smallest) denominators of the (rational) coordinates of the vertices of Q^*.

3. Proof of main theorem. The proof follows from suitable combination of the two upper bounds given in Lemmas 3.1 and 3.2. (Henceforth we use freely the notation introduced in §2.)

Lemma 3.1. *Let X_Q be a toric log Del Pezzo surface of index $\ell \geq 1$. Then*

$$\sharp(\mathcal{V}(Q)) \leq 4 \max\{\, l_H |\ H \in \mathcal{F}(Q)\} + 2 \leq 4\ell + 2. \tag{3.1}$$

Moreover, $\sharp(\mathcal{V}(Q)) = 4 \max\{\, l_H |\ H \in \mathcal{F}(Q)\} + 2$, *if and only if* $\ell = 1$, *and* Q *is the unique hexagon* (*up to lattice-equivalence*) *with one interior lattice point. This means, in particular, that for indices* $\ell \geq 2$ *we have*

$$\sharp(\mathcal{V}(Q)) \leq 4\ell + 1. \tag{3.2}$$

Proof. Obviously, there exists a facet $F \in \mathcal{F}(Q)$ such that $\sum_{\mathbf{v}\in\mathcal{V}(Q)} \mathbf{v} \in \sigma_F$ (this is a *special facet*, in the sense of [11, Section 3]). In addition, since Q is two-dimensional, we have for all integers j:

$$\sharp\{\, \mathbf{v} \in \mathcal{V}(Q) |\ \langle \boldsymbol{\eta}_F, \mathbf{v}\rangle = j\} \leq 2.$$

Writing $\mathcal{V}(Q)$ as disjoint union $\mathcal{V}(Q) = \mathcal{V}^{(F)}_{\geq 0}(Q) \bigsqcup \mathcal{V}^{(F)}_{<0}(Q)$, where

$$\mathcal{V}^{(F)}_{\geq 0}(Q) := \{\, \mathbf{v} \in \mathcal{V}(Q) |\ \langle \boldsymbol{\eta}_F, \mathbf{v}\rangle \geq 0\} \quad \text{and} \quad \mathcal{V}^{(F)}_{<0}(Q) := \{\, \mathbf{v} \in \mathcal{V}(Q) |\ \langle \boldsymbol{\eta}_F, \mathbf{v}\rangle < 0\},$$

we observe that

$$\sharp(\mathcal{V}^{(F)}_{\geq 0}(Q)) \leq 2\,(l_F + 1)\,,$$

because $\langle \boldsymbol{\eta}_F, \mathbf{v}\rangle \in \{0, 1, \ldots, l_F\}$ for all $\mathbf{v} \in \mathcal{V}^{(F)}_{\geq 0}(Q)$. On the other hand,

$$\begin{aligned} 0 \leq \left\langle \boldsymbol{\eta}_F, \sum_{\mathbf{v}\in\mathcal{V}(Q)} \mathbf{v} \right\rangle &= \sum_{\mathbf{v}\in\mathcal{V}^{(F)}_{\geq 0}(Q)} \langle \boldsymbol{\eta}_F, \mathbf{v}\rangle + \sum_{\mathbf{v}\in\mathcal{V}^{(F)}_{<0}(Q)} \langle \boldsymbol{\eta}_F, \mathbf{v}\rangle \\ &= \sum_{j=0}^{l_F} \sum_{\{\mathbf{v}\in\mathcal{V}^{(F)}_{\geq 0}(Q) |\ \langle \boldsymbol{\eta}_F, \mathbf{v}\rangle = j\}} \langle \boldsymbol{\eta}_F, \mathbf{v}\rangle + \sum_{\mathbf{v}\in\mathcal{V}^{(F)}_{<0}(Q)} \langle \boldsymbol{\eta}_F, \mathbf{v}\rangle \\ &\leq \sum_{j=0}^{l_F} 2j + \sum_{\mathbf{v}\in\mathcal{V}^{(F)}_{<0}(Q)} \langle \boldsymbol{\eta}_F, \mathbf{v}\rangle\,. \end{aligned}$$

This implies

$$a := -\sum_{\mathbf{v}\in\mathcal{V}^{(F)}_{<0}(Q)} \langle \boldsymbol{\eta}_F, \mathbf{v}\rangle \leq 2\binom{l_F + 1}{2}.$$

Setting $\mu := \sharp(\mathcal{V}^{(F)}_{<0}(Q))$ we examine two cases: (i) If $\mu = 2\lambda$, for a $\lambda \in \mathbb{N}$, then

$$\sum_{j=0}^{\lambda} 2j \leq a \Longrightarrow 2\binom{\lambda + 1}{2} \leq 2\binom{l_F + 1}{2} \Longrightarrow \lambda \leq l_F \text{ and } \mu \leq 2l_F.$$

(ii) If $\mu = 2\lambda + 1$, for a $\lambda \in \mathbb{Z}_{\geq 0}$, then $\sum_{j=0}^{\lambda} 2j + (\lambda + 1) \leq a$, i.e.,

$$2\binom{\lambda+1}{2} + (\lambda+1) \leq 2\binom{l_F+1}{2} \Longrightarrow \lambda \leq l_F - 1 \text{ and } \mu \leq 2l_F - 1.$$

Hence,

$$\sharp(\mathcal{V}(Q)) = \sharp(\mathcal{V}^{(F)}_{\geq 0}(Q)) + \sharp(\mathcal{V}^{(F)}_{<0}(Q)) \leq 2\,(l_F+1) + \mu$$

$$\leq 2\,(l_F+1) + 2l_F = 4l_F + 2 \leq 4\max\{\, l_H \mid H \in \mathcal{F}(Q)\} + 2, \tag{3.3}$$

with the latter upper bound $\leq 4\ell + 2$ (by (2.5)), giving the inequality (3.1). Finally, we deal with the case of equality: Suppose that $\sharp(\mathcal{V}(Q)) = 4\ell' + 2$, where

$$\ell' := \max\{\, l_H \mid H \in \mathcal{F}(Q)\}\,.$$

From (3.3) we see that $\mu = 2l_F$, and $\lambda = l_F = \ell'$. Therefore, by the equalities in (i) we have for the integers $j = -\ell', \ldots, 0, \ldots, \ell'$:

$$\sharp\{\, \mathbf{v} \in \mathcal{V}(Q) \mid \langle \boldsymbol{\eta}_F, \mathbf{v}\rangle = j\} = 2. \tag{3.4}$$

In particular, $0 = \langle \boldsymbol{\eta}_F, \sum_{\mathbf{v}\in\mathcal{V}(Q)} \mathbf{v}\rangle$, i.e., $\sum_{v\in\mathcal{V}(Q)} \mathbf{v} = \mathbf{0}$. Hence, the previous argument holds for *any* facet. Now let F' be another facet of Q having a common vertex, say $\mathbf{v}$, with F. If $\mathcal{V}(F) = \{\mathbf{u}, \mathbf{v}\}$ and $\mathcal{V}(F') = \{\mathbf{v}, \mathbf{w}\}$, then applying (3.4) for *both* F and F' we get $\langle \boldsymbol{\eta}_F, \mathbf{w}\rangle = \ell' - 1$ and $\langle \boldsymbol{\eta}_{F'}, \mathbf{u}\rangle = \ell' - 1$. This implies $\ell' = 1 = \ell$, since otherwise the primitive vertex $\mathbf{v}$ equals $(\ell'/(\ell'-1))(\mathbf{w} + \mathbf{u} - \mathbf{v})$, a contradiction. Consequently, Q has to be the unique hexagon (up to lattice-equivalence) with just one interior lattice point (see [10, Proposition 2.1]). □

Lemma 3.2. *If X_Q is a toric log Del Pezzo surface of index $\ell \geq 2$ and $\widetilde{X}_Q \xrightarrow{\varphi} X_Q$ its minimal desingularization, then*

$$\rho(\widetilde{X}_Q) < 2\,\sharp(I_Q \smallsetminus \check{I}_Q)(\ell - 1) - \frac{1}{\ell}\,\sharp(\mathcal{V}(Q)) + 10. \tag{3.5}$$

Proof. By Noether's formula and (2.4) we deduce

$$\rho(\widetilde{X}_Q) = 10 - K^2_{\widetilde{X}_Q} = 10 - K^2_{X_Q} - \sum_{F\in I_Q} K(E^{(F)})^2.$$

Since $-\ell K_{X_Q}$ is an ample Cartier divisor on X_Q, we can compute by [12, Proposition 2.10, p. 79] its self-intersection number:

$$(-\ell K_{X_Q})^2 = 2\,\mathrm{area}(\ell Q^*) \Longrightarrow K^2_{X_Q} = \frac{2}{\ell^2}\,\mathrm{area}(\ell Q^*) = 2\,\mathrm{area}(Q^*).$$

For any facet H of ℓQ^* the primitive outer normal vector is given by some vertex of Q, i.e., the lattice distance of H from $\mathbf{0}$ equals ℓ. This implies

$$\mathrm{area}(\ell Q^*) \geq \frac{1}{2}\ell\;\sharp(\mathcal{F}(\ell Q^*)) = \frac{1}{2}\ell\;\sharp(\mathcal{V}(Q)).$$

Hence,

$$-K^2_{X_Q} = -\frac{2}{\ell^2}\,\mathrm{area}(\ell Q^*) \leq -\frac{1}{\ell}\,\sharp(\mathcal{V}(Q)).$$

On the other hand, by Lemma 2.3 we infer that

$$-\sum_{F\in I_Q} K(E^{(F)})^2 = \sum_{F\in I_Q}\left(\frac{2-(p_F+\widehat{p}_F)}{q_F}+(m_F-2)\right).$$

Taking into account that $m_F = 2$ for all $F \in \breve{I}_Q$, and that $p_F + \widehat{p}_F \geq 2$ for all $F \in I_Q$, which is valid as equality only for $p_F = \widehat{p}_F = 1$, i.e., whenever $F \in \breve{I}_Q$, we obtain

$$-\sum_{F\in I_Q} K(E^{(F)})^2 = -\sum_{F\in I_Q\smallsetminus\breve{I}_Q} K(E^{(F)})^2 < \sum_{F\in I_Q\smallsetminus\breve{I}_Q}(m_F-2)$$

$$\leq \sharp(I_Q\smallsetminus\breve{I}_Q)\ \max\left\{m_F-2 \mid F\in I_Q\smallsetminus\breve{I}_Q\right\}$$
$$\leq \sharp(I_Q\smallsetminus\breve{I}_Q)\ \max\left\{2(l_F-1) \mid F\in I_Q\smallsetminus\breve{I}_Q\right\} \leq 2\,\sharp(I_Q\smallsetminus\breve{I}_Q)(\ell-1),$$

where the last but one inequality follows from Lemma 2.2. Thus, $\rho(\widetilde{X}_Q)$ is strictly smaller than the sum $10 - \sharp(\mathcal{V}(Q))/\ell + 2\,\sharp(I_Q\smallsetminus\breve{I}_Q)(\ell-1)$. $\square$

Proof of Theorem 1.2. If $\ell = 1$, then $\rho(\widetilde{X}_Q) \leq 7$ by the known classification of the reflexive polygons (see [6] or [10, Proposition 2.1]). If $\ell \geq 2$, applying (3.2) and (3.5), and the inequality $\sharp(I_Q\smallsetminus\breve{I}_Q) \leq \sharp(\mathcal{V}(Q))$, we get

$$\rho(\widetilde{X}_Q) < 2\,\sharp(I_Q\smallsetminus\breve{I}_Q)(\ell-1) - \frac{1}{\ell}\,\sharp(\mathcal{V}(Q)) + 10$$

$$\leq \sharp(\mathcal{V}(Q))\left(2(\ell-1)-\frac{1}{\ell}\right)+10 \leq (4\ell+1)\left(2(\ell-1)-\frac{1}{\ell}\right)+10,$$

i.e., $\rho(\widetilde{X}_Q) < 8\ell^2 - 6\ell + 4 - \frac{1}{\ell}$, which yields the bound for $\ell \geq 2$. $\square$

4. Discussion, improvements and examples. First, let us note that from the proof of Theorem 1.2 we derive a *linear* upper bound on $\rho(\widetilde{X}_Q)$, if the number of vertices of Q is *fixed*. It is therefore natural to ask for an example of an infinite family $\{Q_i\}$ of LDP-polygons with increasing number of vertices, for which $\rho(\widetilde{X}_{Q_i})$ exhibits a non-linear growth with respect to the indices of its members. To the best knowledge of the authors, this seems to be an open question.

Now, in some specific cases we can further improve the bound (1.2). If Q is an LDP-polygon and $F \in I_Q$, then, according to (2.2), there is a positive integer β_F such that

$$p_F - 1 = \beta_F\cdot\frac{q_F}{l_F} \Longrightarrow l_F\,(p_F-1) = \beta_F\,q_F.$$

Since $l_F(p_F-1) < l_F(q_F-1) < l_F q_F$, we have $\beta_F \in \{1,\ldots,l_F-1\}$. In Proposition 4.1 we construct a better upper bound for $\rho(\widetilde{X}_Q)$ provided that β_F takes one of the extreme values $1, l_F-1$, and $l_F^2 \mid q_F$ for all $F \in I_Q\smallsetminus\breve{I}_Q$.

Proposition 4.1. *Let Q be an LDP-polygon such that X_Q has index $\ell \geq 2$. Suppose that for all $F \in I_Q \smallsetminus \check{I}_Q$ the following conditions are satisfied*:

(i) $\beta_F \in \{1, l_F - 1\}$, *and*
(ii) $l_F^2 \mid q_F$. *Then*

$$\rho(\widetilde{X}_Q) \leq 4\ell^2 - 3\ell + 4. \tag{4.1}$$

Proof. For $F \in I_Q \smallsetminus \check{I}_Q$ define $\xi_F := \frac{q_F}{\ell^2}$. If $\beta_F = 1$, then $\frac{q_F}{q_F - p_F}$ equals

$$\begin{cases} 1 + \frac{1}{(l_F-2)+\frac{1}{l_F+1}} = [\![\underbrace{2,...,2}_{(l_F-2)\text{-times}}, l_F+2]\!], & \text{if } \xi_F = 1, \\ 1 + \frac{1}{l_F-2+\frac{1}{1+\frac{1}{\xi_F-1+\frac{1}{l_F}}}} = [\![\underbrace{2,..,2}_{(l_F-2)\text{-times}}, 3, \underbrace{2,..,2}_{(\xi_F-2)\text{-times}}, \ell+1]\!], & \text{if } \xi_F \geq 2, \end{cases}$$

(cf. [4, Proposition 3.1]), $\widehat{p}_F = q_F - l_F \xi_F + 1$, and

$$m_F - 2 = \sum_{j=1}^{s_F} (b_j^{(F)} - 2) = l_F, \ \forall F \in I_Q \smallsetminus \check{I}_Q.$$

Correspondingly, if $\beta_F = l_F - 1$, then $\frac{q_F}{q_F - p_F}$ equals

$$\begin{cases} (l_F+1) + \frac{1}{l_F-1} = [\![l_F+2, \underbrace{2,...,2}_{(l_F-2)\text{-times}}]\!], & \text{if } \xi_F = 1, \\ l_F + \frac{1}{(\xi_F-1)+\frac{1}{1+\frac{1}{l_F-1}}} = [\![\ell+1, \underbrace{2,...,2}_{(\xi_F-2)\text{-times}}, 3, \underbrace{2,...,2}_{(l_F-2)\text{-times}}]\!], & \text{if } \xi_F \geq 2, \end{cases}$$

$\widehat{p}_F = l_F \xi_F + 1$, and $m_F - 2 = \sum_{j=1}^{s_F} (b_j^{(F)} - 2) = l_F$, $\forall F \in I_Q \smallsetminus \check{I}_Q$. Thus,

$$\begin{aligned} -\sum_{F \in I_Q \smallsetminus \check{I}_Q} K(E^{(F)})^2 &= \sum_{F \in I_Q \smallsetminus \check{I}_Q} \left(\tfrac{2-(p_F+\widehat{p}_F)}{q_F} + (m_F - 2) \right) \\ &= \sum_{F \in I_Q \smallsetminus \check{I}_Q} (l_F - 1) \leq \sharp(I_Q \smallsetminus \check{I}_Q)(\ell - 1). \end{aligned}$$

Since $\sharp(I_Q \smallsetminus \check{I}_Q) \leq \sharp(\mathcal{V}(Q))$, applying Lemma 3.1 and the reasoning used in the proof of Lemma 3.2, we get

$$\begin{aligned} \rho(\widetilde{X}_Q) &< \sharp(I_Q \smallsetminus \check{I}_Q)(\ell-1) - \frac{1}{\ell}\, \sharp(\mathcal{V}(Q)) + 10 \\ &\leq \sharp(\mathcal{V}(Q)) \left(\ell - 1 - \frac{1}{\ell} \right) + 10 \leq (4\ell + 1) \left(\ell - 1 - \frac{1}{l} \right) + 10. \end{aligned}$$

The upper bound (4.1) follows from this inequality. $\square$

By [4, Lemma 6.9] we see that the conditions (i), (ii) in Proposition 4.1 are automatically satisfied for all toric log Del Pezzo surfaces of index $\ell = 2$. Hence, the upper bound 14 improves noticeably (1.2) (which equals 23 in this case). In fact, for $\ell = 2$, it can be shown (though, at the cost of passing through ad hoc classification results for the corresponding LDP-polygons) that the *sharp* upper bound equals 10.

Finally, in Proposition 4.3 we classify those LDP-triangles of *arbitrary* index, whose toric log Del Pezzo surfaces have exactly one singularity. Somehow surprisingly, the Picard number of their minimal desingularizations is bounded; moreover, it takes always the *smallest* possible value, namely 2. Note that from Lemma 3.2 one only derives that the Picard numbers behave *at most linearly* with respect to the index, once the number of non-Gorenstein singularities $\sharp(I_Q \smallsetminus \breve{I}_Q)$ is fixed.

Lemma 4.2. *If X_Q is a toric log Del Pezzo surface with Picard number $\rho(X_Q) = 1$ (i.e, if Q is an LDP-*triangle) *and $\sharp(I_Q) = 1$, then Q is lattice-equivalent to the triangle Q_p having $\binom{1}{0}, \binom{p}{p+1}$ and $\binom{-1}{-1}$ as its vertices, for some positive integer p.*

Proof. If $I_Q = \{F\}$, setting $p := p_F$ and $q := q_F$, there is a unimodular transformation mapping $\mathbf{n}^{(F)}$ onto $\mathbf{n}_1 := \binom{1}{0}$, $\mathbf{n}'^{(F)}$ onto $\mathbf{n}_2 := \binom{p}{q}$, and the third vertex of Q onto an $\mathbf{n}_3 = \binom{x_1}{x_2}$ which belongs necessarily to the set $\left\{ \binom{x_1}{x_2} \in \mathbb{Z}^2 \;\middle|\; \frac{q}{p} x_1 < x_2 < 0 \right\}$. Since $|\det(\mathbf{n}_2, \mathbf{n}_3)| = |\det(\mathbf{n}_3, \mathbf{n}_1)| = 1$, we have $x_2 = -1$ and $x_1 = -\frac{p+1}{q}$. Hence, $q \mid p+1$, which implies $q = p+1$ (because $p < q$). □

Proposition 4.3. *Let X_Q be a toric log Del Pezzo surface which has Picard number $\rho(X_Q) = 1$, arbitrary index $\ell \geq 1$, and $\sharp(I_Q) = 1$. For ℓ odd ≥ 3 we have either $X_Q \cong X_{Q_{\ell-1}}$ or $X_Q \cong X_{Q_{2\ell-1}}$, whereas for $\ell \in \{1\} \cup 2\mathbb{Z}$ we have $X_Q \cong X_{Q_{2\ell-1}}$. Furthermore, for all $\ell \geq 1$, the Picard number of the rational surface $\widetilde{X}_Q$ obtained by the minimal resolution of the singularity of X_Q equals*

$$\rho(\widetilde{X}_Q) = 2.$$

Proof. By Lemma 4.2, the LDP-triangle Q is lattice-equivalent to Q_p, for some positive integer p. Since $q = p+1$ and $\gcd(p+1, p-1) \in \{1, 2\}$, the index ℓ of $X_Q \cong X_{Q_p}$ equals $\frac{p+1}{2}$ whenever p is odd and $p+1$ whenever p is even (see (2.5)). This bears out our first assertion. On the other hand, since $\widetilde{\Delta}_{Q_p}$ is obtained from Δ_{Q_p} by adding just one new ray (namely $\mathbb{R}_{\geq 0}\binom{1}{1}$), we have

$$\rho(\widetilde{X}_Q) = \rho(\widetilde{X}_{Q_p}) = \sharp\{\text{rays of } \widetilde{\Delta}_{Q_p}\} - 2 = 4 - 2 = 2,$$

(cf. [12, Corollary 2.5, p. 74]). Thus, the second assertion is also true. □

It would be interesting to generalize this result by regarding LDP-*polygons* of arbitrary index, whose toric log Del Pezzo surfaces have exactly one singularity.

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