The finite volume element method in nonconvex polygonal domains

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ABSTRACT. *We consider standard finite volume piecewise linear approximations for second order elliptic boundary value problems on a nonconvex polygonal domain. Based on sharp shift estimates, we derive error estimations in H^1– and L^2–norm, taking into consideration the regularity of the data.*

KEYWORDS: *Finite volume element method, nonconvex polygons, error estimations*

1. Introduction

In this note we study the convergence of the standard finite volume element method for discretization of second order linear elliptic pde's on a non-convex polygonal domain $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary conditions. Namely, for a given function f, we seek u such that

$$Lu = f, \quad \text{in } \Omega, \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

with $Lv \equiv -\operatorname{div}(A\nabla v)$, $A = A(x) = (a_{ij})_{i,j=1}^2$ a given symmetric matrix function with real–valued entries $a_{ij} \in W^{1,\infty}$, $1 \le i, j \le 2$. We assume that the matrix $A(x)$ is uniformly positive definite in Ω, i.e., there exists a positive constant α_0 such that

$$\xi^T A(x)\xi \ge \alpha_0 \xi^T \xi, \quad \forall \xi \in \mathbb{R}^2, \forall x \in \bar{\Omega}. \tag{1.2}$$

Finite volume discretizations for more general convection–diffusion–reaction problems were studied by many authors (for a comprehensive presentation and more references of existing results we refer to [EGH–00]). For convex polygonal domains, H^1 and L^2 norm error estimates were derived in [EWI–02], taking into account the regularity of f.

Our goal in this paper is to study the influence of the corner singularities and insufficient regularity of the right–hand side f, say $f \in L^p(\Omega)$, $p < 2$,

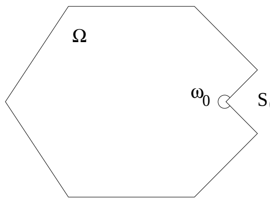

Figure 1. A non-convex domain Ω with a corner S_0 and $\omega_0 > \pi$.

or $f \in H^{-\ell}(\Omega)$, $0 \leq \ell < 1/2$, on the convergence rate of the finite volume element method. We note that we use the conservative version of the method, namely the right–hand side of the scheme is computed by the L^2–inner product of f with the characteristic functions of the finite volumes (or equivalently by the duality between H^ℓ and $H^{-\ell}$ for $0 \leq \ell < 1/2$). For more singular f, i.e. $f \in H^{-\ell}$, $0 \leq \ell \leq 1$, we refer to [DRO–02]. Our analysis of the error estimates in H^1 and L^2 norm follows the approach developed in [CHA–02] and uses known sharp regularity results for the solutions of elliptic boundary value problems, cf. [GRI–85].

2. Preliminaries

In this paper we use standard notation for Sobolev spaces $W^{s,p}$ and $H^s = W^{s,2}$, cf. [ADA–75]. Namely, L^p denotes the space of $p-$integrable real functions over Ω, $|\cdot|_s$ and $\|\cdot\|_s$ the seminorm and norm, respectively, in $H^s = H^s(\Omega)$, $|\cdot|_{W^{s,p}}$ and $\|\cdot\|_{W^{s,p}}$ the seminorm and norm, respectively, in $W^{s,p} = W^{s,p}(\Omega)$, $p \geq 1$, and $s \in \mathbb{R}$. If $s = 0$ we suppress this index.

Let us first consider the Dirichlet problem for Poisson's equation: Given $f \in L^p$, $p \geq 1$, find a function $u : \Omega \to \mathbb{R}^2$ such that

$$-\Delta u = f, \quad \text{in } \Omega, \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega, \tag{2.1}$$

with Ω a bounded, non-convex, polygonal domain in $\mathbb{R}$. For simplicity we assume that Ω has only one inner angle greater than π, namely $\omega_0 \in (\pi, 2\pi)$, cf. Figure 1. It is known that there exists a unique solution $u \in H_0^1$ of (2.1). Furthermore, u could be represented in the form $u = c_0 w_0 + v$, where $v \in W^{2,p} \cap H_0^1$, c_0 is a constant and $w_0 = r^{\lambda_m} \frac{1}{\sqrt{\omega_0 \lambda_m}} \sin(\lambda_m \theta) \eta(re^{i\theta})$. Here $\lambda_m = \frac{m\pi}{\omega_0}$, $m \in \mathbb{N}$, η is a cutoff function which is one near S_0 and zero away from S_0 and (r, θ) are the polar coordinates with respect to the vertex S_0 with angle ω_0. A crucial role in determining the regularity of u is played by the constant $p_0 \equiv \frac{2}{2-\pi/\omega_0}$.

If $f \in L^p$, $p \geq 1$, in view of [GRI–85, p. 233] and a standard imbedding result, we have that $u \in H^s$ with $s = 2 - s_0 - \delta(p)$, where s_0 and $\delta(p) > 0$ are defined by

$$s_0 = \frac{2}{p_0} - 1 = 1 - \frac{\pi}{\omega_0}, \quad \delta(p) = \begin{cases} \frac{2}{p} - \frac{2}{p_0}, & p < p_0, \\ \text{arbitrarily small}, & p \geq p_0. \end{cases} \tag{2.2}$$

Further, if $f \in H^{-\ell}$, $0 \leq \ell \leq 1$ the solution u of (2.1) satisfies $u \in H^s$, with $s = 2 - s_0 - \delta(\ell)$, where $\delta(\ell) > 0$ is defined by

$$\delta(\ell) = \begin{cases} \ell - s_0, & s_0 < \ell \leq 1, \\ \text{arbitrarily small}, & 0 \leq \ell \leq s_0. \end{cases} \tag{2.3}$$

For the more general problem (1.1) similar results hold. Let $\mathcal{A}$ and $\mathcal{T}$, be matrices such that $\mathcal{A} = (a_{ij}(S_0))_{i,j=1}^2$ and $-\mathcal{T}^T \mathcal{A} \mathcal{T} = I$. Also, let $\omega_0(A)$ be the measure of the angle at $\mathcal{T}S_0$ of $\mathcal{T}\Omega$, with $\mathcal{T}\Omega = \{\mathcal{T}x : x \in \Omega\}$ and $p_0(A) = \frac{2}{2-\pi/\omega_0(A)}$. Then in view of [GRI–85, Theorem 5.2.7], if $f \in L^p$, $p \geq 1$, the solution u of (1.1) is in H^s, with $s = 2 - s_0 - \delta(p)$, where in the definition of s_0 and δ, (2.2), we substitute $p_0(A)$ for p_0.

In the rest of this paper, we will denote by $s = 2 - s_0 - \delta$, where s_0 and δ are defined as above, depending if we are referring to problem (1.1) or (2.1) and whether f is in L^p, $p \geq 1$ or $H^{-\ell}$, $\ell \in [0, 1]$.

3. The finite volume element method

We consider a quasi uniform family $\{T_h\}_{0<h<1}$ of triangulations of Ω, where h denotes the maximum diameter of the triangles of T_h. Let us denote by Z_h^{in} and E_h^{in} the set of interior vertices and edges of T_h, respectively. We construct the control volumes by considering an interior point z_K in each triangle $K \in T_h$ and connecting it with the edge midpoints of K. This partitions K into three subregions K_z, with z a vertex of K, see Figure 2. With each vertex $z \in Z_h^{\mathrm{in}}$ we associate the control volume b_z, which consists of the union of the subregions K_z with common vertex z (see Figure 2). Next, let us consider the finite dimensional spaces $X_h^0 = \{\chi \in C(\Omega) : \ \chi|_K \text{ is linear for all } K \in T_h \text{ and } \chi|_{\partial\Omega} = 0\}$ and $\bar{X}_h^0 = \{\bar{\chi} \in L^2(\Omega) : \ \bar{\chi}|_{b_z} \text{ is constant}, \ z \in Z_h^{\mathrm{in}}, \ \bar{\chi}|_{b_z} = 0, \text{ if } z \in \partial\Omega\}$.

We consider then the following finite volume method for (1.1): Find $u_h \in X_h^0$ such that for every $\chi \in X_h^0$

$$a_h(u_h, \chi) \equiv - \sum_{z \in Z_h^{\mathrm{in}}} \chi(z) \int_{\partial b_z} (A \nabla u_h) \cdot n \, ds = \sum_{z \in Z_h^{\mathrm{in}}} \chi(z) \int_{b_z} f \, dx. \tag{3.1}$$

We now introduce the interpolation operator $\bar{I}_h : C(\Omega) \to \bar{X}_h^0$, defined by

$$\bar{I}_h v = \sum_{z \in Z_h^{\mathrm{in}}} v(z) \bar{\varphi}_z,$$

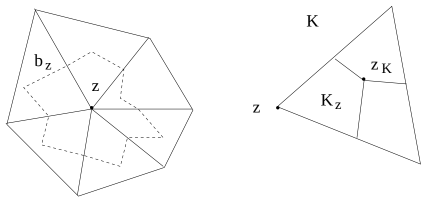

Figure 2. *Left:* A sample region; with dotted lines the corresponding box b_z. *Right:* A triangle K partitioned into three subregions K_z.

where $\bar{\varphi}_z$ is the characteristic function of b_z. Note that,

$$(f, \bar{I}_h \chi) = \sum_{z \in Z_h} \chi(z) \int_\Omega f \bar{\varphi}_z \, dx = \sum_{z \in Z_h} \chi(z) \int_{b_z} f \, dx, \quad \forall f \in L^2, \ \chi \in X_h^0.$$

Thus (3.1) can be written equivalently in the form

$$a_h(u_h, \chi) = (f, \bar{I}_h \chi), \quad \forall \chi \in X_h^0. \tag{3.2}$$

In addition, if $A = I$, we have

$$a_h(\chi, \psi) = a(\chi, \psi) = \int_\Omega \nabla \chi \cdot \nabla \psi \, dx, \quad \forall \chi, \psi \in X_h^0, \tag{3.3}$$

cf., e.g., [BAN–87]. Thus, (3.2) takes the form

$$a(u_h, \chi) = (f, \bar{I}_h \chi), \quad \forall \chi \in X_h^0.$$

In the general case of problem (1.1), the identity (3.3) is not valid. However, following [CHA–02], we are able to rewrite a_h in a form similar to a. Indeed, take the integral of (1.1) over K_z, for $z \in Z_h^{\text{in}}$ and $K \in T_h$, so that after integration by parts we obtain

$$\int_{K_z} L\chi \, dx + \int_{\partial K_z \cap \partial K} A \nabla \chi \cdot n \, ds = - \int_{\partial K_z \cap \partial b_z} A \nabla \chi \cdot n \, ds, \quad \forall \chi \in X_h^0.$$

Multiplying this by $\psi(z)$, $\psi \in X_h^0$, and summing over the triangles that have z as a common vertex and the vertices $z \in Z_h^{\text{in}}$, we get

$$a_h(\chi, \psi) = \sum_K \{ \int_K L\chi \, \bar{I}_h \psi \, dx + \int_{\partial K} A \nabla \chi \cdot n \, \bar{I}_h \psi \, ds \}, \quad \forall \chi, \psi \in X_h^0. \tag{3.4}$$

This is similar to

$$a(\chi, \psi) \equiv (A \nabla \chi, \nabla \psi) = \sum_K \{ \int_K L\chi \, \psi \, dx + \int_{\partial K} A \nabla \chi \cdot n \, \psi \, ds \}, \ \forall \chi, \psi \in X_h^0.$$

4. Error estimates

For the analysis of the finite volume method (3.2) we shall need to estimate the errors ε_h and ε_a defined by

$$\varepsilon_h(f,\chi) = (f,\chi) - (f,\bar{I}_h\chi), \quad \forall f \in L^p,\ \chi \in X_h^0,$$
$$\varepsilon_a(\chi,\psi) = a(\chi,\psi) - a_h(\chi,\psi), \quad \forall \chi,\psi \in X_h^0.$$

We can easily see that $\bar{I}_h$ satisfies the following property:

$$\|\chi - \bar{I}_h\chi\|_{L^q(K)}^q = \sum_{z\in Z_h(K)} \int_{K_z} (\chi - \chi(z))^q\,dx \le h^q|\chi|_{W^{1,q}(K)}^q,\ \forall \chi \in X_h^0, \tag{4.1}$$

with $1 \le q < \infty$ and $Z_h(K)$ the set of the vertices of K. Also, if in the construction of the control volumes we choose z_K to be the barycenter of K, then

$$\int_K \chi\,dx = \int_K \bar{I}_h\chi\,dx, \quad \forall K \in T_h,\ \forall \chi \in X_h^0. \tag{4.2}$$

In addition, using known interpolation results, cf., e.g., [BRE–94, p. 285], we get

$$\inf_{\chi\in X_h^0}(\|v-\chi\| + h\|v-\chi\|_1) \le Ch^s\|v\|_s, \quad \forall v \in H^s \cap H_0^1,\ 1 \le s \le 2. \tag{4.3}$$

In the sequel we shall give some auxiliary lemmas which are used for estimating ε_h and ε_a.

Lemma 4.1 *There exists a constant C, such that for every $\chi \in X_h^0$*

$$|\varepsilon_h(f,\chi)| \le Ch^{\min(1,2-2/p)}\|f\|_{L^p}\,|\chi|_1, \quad \forall f \in L^p, \tag{4.4}$$

$$|\varepsilon_h(f,\chi)| \le Ch^2|f|_{W^{1,p}}\,|\chi|_{W^{1,q}}, \quad \forall f \in W^{1,p},\ \frac{1}{p}+\frac{1}{q} = 1, \tag{4.5}$$

$$|\varepsilon_h(f,\chi)| \le Ch^{1-\ell}\|f\|_{-\ell}\,|\chi|_1, \quad \forall f \in H^{-\ell},\ 0 < \ell < 1/2,\ . \tag{4.6}$$

Proof: The estimate (4.4) is based on (4.1) and the inverse inequality $|\chi|_{W^{1,q}} \le Ch^{2/q-1}|\chi|_1$, $q \ge 2$, for $\chi \in X_h^0$. The estimate (4.5) is obtained similarly, by taking into consideration (4.2). Finally, (4.6) is based on

$$|\chi - \bar{I}_h\chi|_\ell \le Ch^{1-\ell}|\chi|_1, \quad 0 < \ell < \frac{1}{2}, \quad \forall \chi \in X_h^0. \qquad \Box$$

Lemma 4.2 *There exists a positive constant C such that*

$$|\varepsilon_a(\psi,\chi)| \le Ch|\psi|_1\,|\chi|_1, \quad \forall \chi,\psi \in X_h^0, \tag{4.7}$$

$$|\varepsilon_a(u_h,\chi)| \le Ch(|u_h - u|_1 + h|u|_{W^{2,p}})|\chi|_{W^{1,q}},\ \chi \in X_h^0,\ A \in W^{2,\infty}. \tag{4.8}$$

Proof: We first note that in view of (3.4),

$$\varepsilon_a(\psi,\chi) = \sum_K \int_K L\psi\,(\chi - \bar{I}_h\chi)\,dx + \sum_K \int_{\partial K} A\nabla\psi \cdot n\,(\chi - \bar{I}_h\chi)\,ds = I + II.$$

Using (4.1) and (4.2), to estimate I, we obtain the desired bounds of (4.7) and (4.8), respectively. The contribution of II can be calculated using the estimate

$$\Big|\int_e \varphi(\chi - \bar{I}_h\chi)\,ds\Big| \le Ch|\varphi|_{W^{1,p}(K)}\,|\chi|_{W^{1,q}(K)},$$

and the fact that, if $\bar{A}_e = A(m_e)$ with m_e the midpoint of the edge e, then

$$\sum_K \int_{\partial K} \bar{A}_e \nabla(u_h - u)\cdot n\,(\chi - \bar{I}_h\chi)\,ds = 0. \qquad \square$$

Theorem 4.1 *Let* u *and* u_h *be the solutions of* (1.1) *and* (3.2)*, respectively, with* $f \in L^p$*,* $p > 1$*. Then, there exists a constant* C*, independent of* h*, such that*

$$\|u - u_h\|_1 \le C\big(h^{s-1}\|u\|_s + h^{\min(1,2-2/p)}\|f\|_{L^p}\big), \tag{4.9}$$

$$\|u - u_h\| \le C\big(h^{2(s-1)}\|u\|_s + h^{\min(1,2-2/p)}\|f\|_{L^p}\big). \tag{4.10}$$

Proof: The proof is similar to the corresponding proof in the finite element method. Hence, for (4.9) it suffices to estimate $a(u-u_h,\psi)$, for every $\psi \in X_h^0$. Due to (3.2), we easily see that

$$a(u - u_h, \psi) = \varepsilon_h(f,\psi) - \varepsilon_a(u_h,\psi). \tag{4.11}$$

Combining now Lemmas 4.1 and 4.2 and choosing $\psi = u_h - \chi$, for h sufficiently small, we obtain (4.9). To show (4.10) we use a standard duality argument by introducing the auxiliary problem: Find $\varphi \in H^s \cap H_0^1$ such that: $L\varphi = u - u_h$, in Ω. Then, it suffices to estimate the terms I and II in

$$\|u - u_h\|^2 = a(u - u_h, \varphi) = a(u - u_h, \varphi - \chi) + a(u - u_h, \chi) = I + II. \tag{4.12}$$

The first term, I, is estimated similarly as in the finite element method. For the second term II we use (4.11) and Lemmas 4.1 and 4.2. $\square$

Remark 4.1 Since $2(s-1) > 1$, $\|u - u_h\| = O(h^{\min(1,2-2/p)})$. If $f \in L^p$, with $p < p_0$, then $s - 1 = 2 - 2/p < 1$, cf. [GRI–85, p. 233], thus $\|u - u_h\|_1 = O(h^{2-2/p})$. Also, if $p \ge p_0$ then $s - 1 < 2 - 2/p$, therefore $\|u - u_h\|_1 = O(h^{1-s_0-\delta})$.

Theorem 4.2 *Let u and u_h be the solutions of* (1.1) *and* (3.2)*, respectively, with $A = I$, $f \in H^{-\ell}$, $0 \leq \ell < 1/2$. Then there exists a constant C independent of h such that*

$$\|u - u_h\|_1 \leq C\big(h^{s-1}\|u\|_s + h^{1-\ell}\|f\|_{-\ell}\big), \tag{4.13}$$

$$\|u - u_h\| \leq C\big(h^{2(s-1)}\|u\|_s + h^{1-\ell}\|f\|_{-\ell}\big). \tag{4.14}$$

Proof: The proof is similar to the proof of Theorem 4.1. The desired estimates are based on bounding $a(u - u_h, \chi)$ for every $\chi \in X_h^0$. In view of Lemma 4.1 we have

$$|a(u - u_h, u_h - \chi)| \leq Ch^{1-\ell}\|f\|_{-\ell}\|\chi\|_1, \ \forall \chi \in X_h^0. \qquad \square$$

Remark 4.2 If $f \in H^{-\ell}$, with $s_0 < \ell < 1/2$, then in view of (2.3) $s - 1 = 1 - \ell$, thus $\|u - u_h\|_1 = O(h^{1-\ell})$. If $0 \leq \ell \leq s_0$ then $s - 1 < 1 - \ell$, thus $\|u - u_h\|_1 = O(h^{1-s_0-\delta})$.

Next, we will give an improved error estimate for $u - u_h$ in the L^2-norm.

Theorem 4.3 *Let u and u_h be the solutions of* (1.1) *and* (3.2)*, respectively, with $f \in W^{1,p}$, $p > 1$, then $u \in H^s$, with $s = 2 - s_0 - \delta$ and $\delta > 0$ arbitrary small. Also, if in the construction of the control volumes b_z, z_K is the barycenter of the triangle K then, there exists a constant C, independent of h, such that*

$$\|u - u_h\| \leq C\big(h^{2(s-1)}\|u\|_s + h^s\|f\| + h^{\min(2, s+2-2/p)}\|f\|_{W^{1,p}}\big) = O(h^{2(s-1)}).$$

Proof: The proof is similar to that of Theorem 4.1. It is obvious that it suffices to estimate the term II in (4.12). Since $f \in W^{1,p}$, we have $f \in L^2$, and therefore $u \in H^s$, with $s = 2 - s_0 - \delta$. Due to (4.11) and Lemmas 4.1 and 4.2, we obtain

$$|II| \leq C\big(h^2(|f|_{W^{1,p}} + |u|_s) + h\|u - u_h\|_1\big)\,|\chi|_{W^{1,q}}, \quad \forall \chi \in X_h^0,$$

with $1/p + 1/q = 1$. Choosing now χ to be an appropriate interpolant of φ, with appropriate stability properties, and using standard imbedding arguments and an inverse inequality we obtain

$$\|u - u_h\|^2 \leq C\big(h^{2(s-1)}\|u\|_s + h^s\|f\| + h^{\min(2, s+2-2/p)}\|f\|_{W^{1,p}}\big)\|u - u_h\|.$$

We can easily see that since $3/2 < s < 2$ we have $2(s - 1) \leq s$. Also, the fact that $s \leq 2 \leq 4 - 2/p$ suggests $2(s - 1) < \min(2, 2 + s - 2/p)$. Combining now these with the above error estimation we obtain the desired result. $\square$

Remark 4.3 Our L^2–norm error estimates are in contrast to known estimates for the finite element method. For example, the finite element approximation

Table 1. *Convergence rate for exact solution* $u = (r^{2/3} + r^{\beta})sin(2\theta/3)$

β	1/2	2/3	3/4	4/5
H^1-norm	0.54 (0.50)	0.66 (0.66)	0.69 (0.66)	0.69 (0.66)
L_2-norm	1.20 (1.00)	1.34 (1.34)	1.37 (1.34)	1.37 (1.34)

u_h^{FE}, defined by $a(u_h^{\mathrm{FE}}, \chi) = (f, \chi)$, $\forall \chi \in X_h^0$, is known to satisfy, cf., e.g., [BRE–94, Chapter 12],

$$\|u - u_h^{\mathrm{FE}}\|_1 \le Ch^{s-1}\|u\|_s, \quad \|u - u_h^{\mathrm{FE}}\| \le Ch^{2(s-1)}\|u\|_s.$$

In Table 4 we present the computed rates of convergence of the finite volume method which illustrate the results of Theorem 4.1. We considered the Dirichlet boundary value problem for the Poisson equations in an L-shaped domain with an exact solution $u = (r^{2/3}+r^{\beta})sin(2\theta/3)$. One can see that u is almost in $H^{3/2}$ if $\beta = 1/2$ and u is almost in $H^{5/3}$ if $\beta \ge 2/3$. The numerical experiments show that the finite volume scheme recovers the solution with the expected rates in H^1-norm. The convergence rates in L_2-norm in some cases are slightly higher than the ones predicted by the theory.

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